K3 Surfaces, Mathieu Moonshine, and Quantum Codes

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a project with Jeff Harvey

….. still in progress ….

1. Brief Background On Moonshine
2. Quantum Mukai Theorem & GTVW Model
3. Supercurrents & Codes
4. RR States: MOG Construction Of The Golay Code
5. Conway Group Moonshine
6. Conclusion

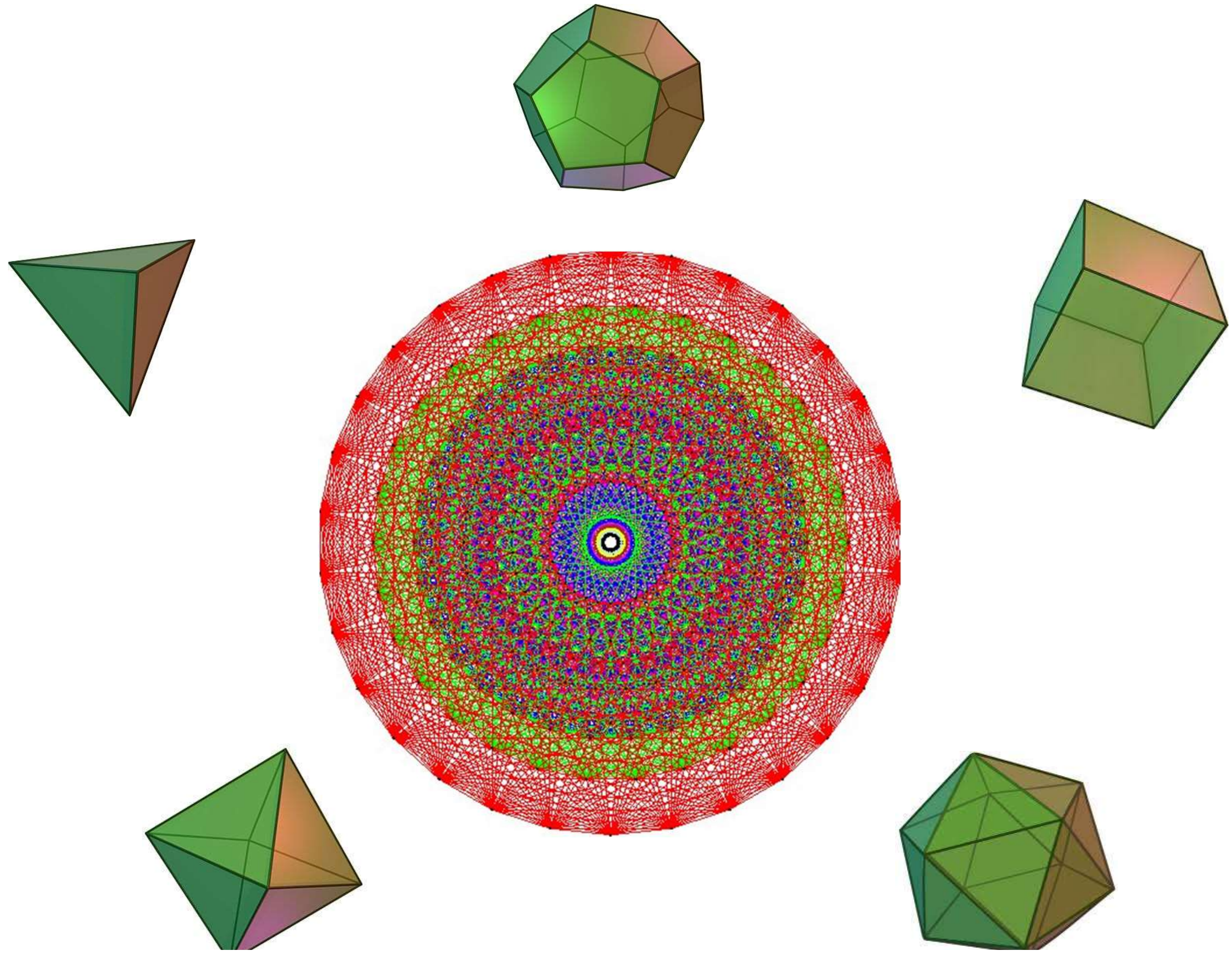
Philosophy – 1/2

We can divide physicists into two classes:

Our world is a random choice drawn from a huge ensemble:

Philosophy – 2/2

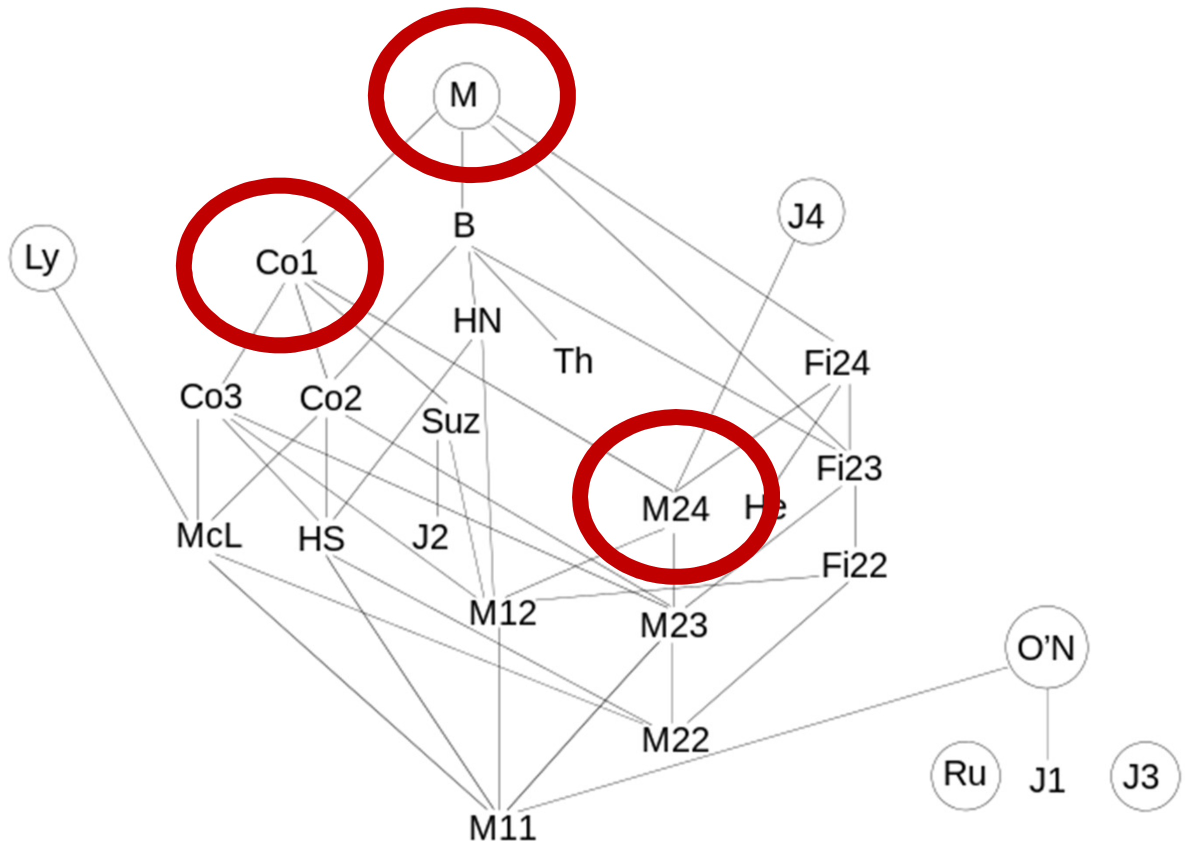
The fundamental laws of nature are based on some beautiful exceptional mathematical structure:



Background: Finite-Simple Groups

Jordan-Holder Theorem: Finite simple groups are the atoms of finite group theory.

$\mathbb{Z}_p$ p = prime $\quad A_n \; n \geq 5 \quad SL_n(\mathbb{F}_p)$ etc.



Background: McKay & Conway-Norton 1978-1979

$$J = \sum_n J_n q^n = q^{-1} + 196884\, q + 21493760\, q^2 + 864299970\, q^3 + \cdots$$

Now list the dimensions of irreps of $\mathbb{M}$

R_n = 1, 196883, 21296876, 842609326, 18538750076, 19360062527, 293553734298,…. , $3^{12} \cdot 5^7 \cdot 13^3 \cdot 17 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \sim 2.6 \times 10^{26}$

$$J_{-1} = R_1 \qquad J_1 = R_1 + R_2$$

$$J_2 = R_1 + R_2 + R_3 \qquad J_3 = 2R_1 + 2R_2 + R_3 + R_4$$

A way of writing J_n as a positive linear combination of the R_j for all n is a ``***solution of the Sum-Dimension Game***.''

There are infinitely many such solutions!!

Background: Monstrous Moonshine

Which, if any, of these solutions is interesting?

Every solution defines an infinite-dimensional $\mathbb{Z}$-graded representation of $\mathbb{M}$

$$V = q^{-1} R_1 \oplus q(R_1 \oplus R_2) \oplus q^2(R_1 \oplus R_2 \oplus R_3) \oplus \cdots$$

Now for every $g \in \mathbb{M}$ we can compute the character:

$$\chi(q; g) \coloneqq Tr_V \, g \, q^N$$

A solution of the Sum-Dimension game is *modular* if the $\chi(q; g)$ is a modular function in $\Gamma_0(m)$ where $g^m = 1$.

There is a unique modular solution of the Sum-Dimension game! Moreover the $\chi(q; g)$ have very special properties.

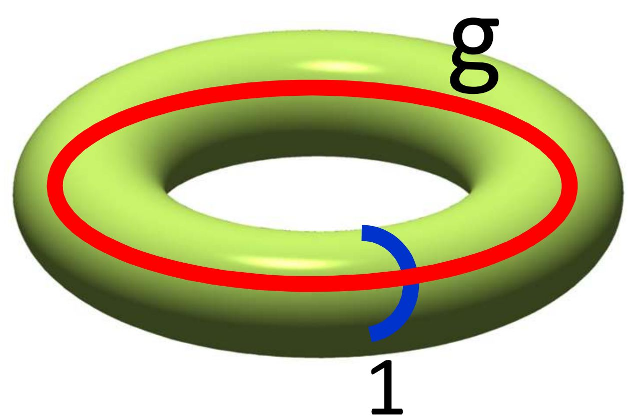
FLM Construction -1/3

String theory compactification of chiral bosonic string on a torus:

$$\partial_z x^j = -i \sum_n \alpha_n^j e^{inz} \qquad j = 1, \dots, 24$$
$$z = \sigma + \tau$$

$$[\alpha_n^i, \alpha_m^j] = n\delta^{ij}\delta_{n+m,0} \qquad \alpha_0^j = p^j \in \Lambda$$

OPE of conformal fields form a VOA: $\wp(\partial^* x) e^{i\, p\cdot x}$

g  1 $:= Tr_{\mathcal{H}}\, g\, q^{L_0 - \frac{c}{24}} =$ 

⇒ Modularity

Leech & Golay

FLM use torus associated to Leech lattice Λ:

Definition:[Cohn,Kumar,Miller,Radchenko,Viazovska]
$\Lambda \subset \mathbb{R}^{24}$ is the best sphere packing in d=24

$$Aut(\Lambda) = Co_0 \subset SO(24)$$

Λ can be constructed using the Golay code $\mathcal{G} \subset \mathbb{F}_2^{24}$

$\mathcal{G}$ is a special 12-dimensional subspace with nice error-correcting properties. Discovered @ Bell Labs in 1949 and used by Voyager 1&2 to send color photos

Definition: $M24 \subset S_{24}$ is the subgroup of permutations preserving the set $\mathcal{G}$

FLM Construction – 2/3

Now ``orbifold'' by $\vec{x} \to -\vec{x}$ for $\vec{x} \in \mathbb{R}^{24}/\Lambda$

''Orbifold by a symmetry G of a CFT'':
Gauge the symmetry

Different G-bundles over the circle

$\mathcal{H}_{untwisted}$ $\Phi(\sigma + 2\pi) = \Phi(\sigma)$

$\mathcal{H}_{[g]-twisted}$ $\Phi(\sigma + 2\pi) = \mathrm{g} \cdot \Phi(\sigma)$

$\mathbb{M}$ As An Automorphism Group

FLM & Borcherds: Automorphisms of the OPE algebra of the quotient theory = $\mathbb{M}$

Heisenberg group of translations on 24 fixed points + Co_1 + a ``quantum symmetry'' generate the Monster.

This is the gold standard for the conceptual explanation of Moonshine-modularity

(But a truly satisfying conceptual explanation of genus zero properties remains elusive.)

New Moonshine: Mathieu Moonshine

Eguchi, Ooguri, Tachikawa 2010

CFT $\mathcal{C}$: 2d sigma model with target space $\mathcal{X}$ a K3 surface with hyperkahler metric and flat B-field.

Elliptic genus: A character-valued index on the loop space $L\mathcal{X}$

Hyperkahler structure $\Rightarrow$ model has (4,4) superconformal symmetry. One can compute a character-valued index wrt $SO(2) \times SU(2) \times G$ for any $G \subset Aut(\mathcal{C})$

(Super-) Conformal Symmetry:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \qquad n, m \in \mathbb{Z}$$

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \qquad T(z)T(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots$$

Superconformal symmetry $\Rightarrow$ $\mathbb{Z}_2$ $-$graded extension

$$T_F(z) = \sum_r G_r z^{-r-\frac{3}{2}} \qquad T_F(z)T_F(w) \sim \frac{\frac{\hat{c}}{4}}{(z-w)^3} + \frac{\frac{1}{2}T(w)}{z-w} + \cdots$$

(p, q) superconformal symmetry $\Rightarrow$

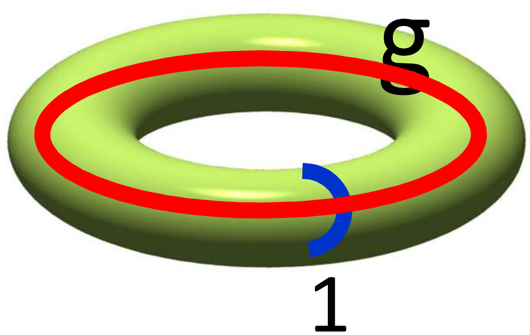
p holomorphic $T_F^i(z)$ and q anti-holomorphic $T_F^a(\bar{z})$

Elliptic Genus Of K3

Hyperkahler structure ⇒ model has (4,4) superconformal symmetry. One can compute a character-valued index wrt $SO(2) \times SU(2) \times G$ for any $G \subset Aut(\mathcal{C})$

$$\mathcal{E}_{\mathcal{C}}^{g}(z,\tau) = Tr_{\mathcal{H}_{RR}} (-1)^F \cdot \, g \, \cdot \, e^{2\pi i \tau \left(L_0 - \frac{c}{24}\right) + 2\pi i z J_0^3 - 2\pi i \bar{\tau}\left(\tilde{L}_0 - \frac{c}{24}\right)}$$

=



⇒ Modularity

Example:

$$\mathcal{E}_{\mathcal{C}}^{g=1}(z,\tau) = \quad 8\left[\left(\frac{\vartheta_2(z)}{\vartheta_2(0)}\right)^2 + \left(\frac{\vartheta_3(z)}{\vartheta_3(0)}\right)^2 + \left(\frac{\vartheta_4(z)}{\vartheta_4(0)}\right)^2\right]$$

New Moonshine: Mathieu Moonshine

Model has (4,4) susy so consider isotypical decomposition:

$$\mathcal{H}_{RR} = \bigoplus_{h,\ell;\tilde{h},\tilde{\ell}} D_{h,\ell;\tilde{h},\tilde{\ell}} \; R_{h,\ell} \otimes \tilde{R}_{\tilde{h},\tilde{\ell}}$$

$R_{h,\ell}$: Untry highest weight irrep of N=4 with $L_0 v = h\, v$ and $J_0^3 v = \ell\, v$

$$ch_{h,\ell}(z,\tau) := Tr_{R_{h,\ell}}\, e^{2\pi i\, \tau\left(L_0 - \frac{c}{24}\right) + 2\pi\, i\, z\, J_0^3}$$

$$\mathcal{E}_{\mathcal{C}}^{g}(z,\tau) =$$

$$\sum_{n\geq 0,\ell} \left(Tr_{D_{n+\frac{1}{4},\ell;\frac{1}{4},0}}(g) \; - 2\, Tr_{D_{n+\frac{1}{4},\ell;\frac{1}{4},\frac{1}{2}}}(g) \right) ch_{n+\frac{1}{4},\ell}(z,\tau)$$

Statement Of Mathieu Moonshine

[EOT 2010, M. Cheng 2011, Gaberdiel, Hohenegger, Volpato 2011; Gannon 2012]

There exist an infinite set of representations of the group $M24$

$H_{0,0}$, $H_{0,\frac{1}{2}}$, H_n , $n \geq 1$ For ALL $g \in M24$

$$\mathcal{E}^g(z,\tau) := Tr_{H_{0,0}}(g)\, ch_{\frac{1}{4},0} + Tr_{H_{0,\frac{1}{2}}}(g)\, ch_{\frac{1}{4},\frac{1}{2}} + \sum_{n=1}^{\infty} Tr_{H_n}(g)\, ch_{n+\frac{1}{4},\frac{1}{2}}$$

Has the expected modularity properties ***AS IF*** $g \in Aut(\mathcal{C})$

Umbral Moonshine: A nontrivial generalization of this statement:
There is one example for each of the 23 Niemeier lattices based on root systems.
[Cheng, Duncan, Harvey, 2012]

But there are no known physical/conceptual explanations of Mathieu or Umbral Moonshine.

Why Is It Moonshine?

There is no obvious action of M24 on $\mathcal{C}$ nor on the highest weight states $D_{n+\frac{1}{4},\ell;\frac{1}{4},\tilde{\ell}}$

Why should these objects have good modular properties?

What do these representations have to do with (any) K3 sigma model?

… or with the space of loops on K3 ?

Can we define an action of M24 on $\mathcal{C}$ or on the highest weight states $D_{n+\frac{1}{4},\ell;\frac{1}{4},\tilde{\ell}}$???

There is no known analog of the FLM construction.

Despite 9 years of intense effort by a small, but devoted, community of physicists and mathematicians..

Strategies

1. Find a special point in the moduli space of K3 sigma models where the symmetry is manifest.

Ruled out by the Quantum Mukai Theorem (see below).

2. Find an algebraic structure of the highest weight "BPS states" - and ``algebra of BPS states'' and study the automorphisms of this structure. (Some success with this approach for the original monstrous moonshine example achieved by Natalie Paquette, Daniel Persson, and Roberto Volpato. But the discussion still very incomplete.)

3. ``Symmetry surfing'' [Anne Taormina & Katrin Wendland]. Try to ``combine'' symmetries of the CFT's at different points in moduli space. Problem: Requires a notion of connection on the bundle of CFT's over the moduli space. This raises many, many, very interesting, but unresolved issues.

4. Reduced symmetry (New proposal): Study the symmetries commuting with just the (4,1) superconformal algebra, since only right-moving N=1 is required to define the elliptic genus!

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Quantum Mukai Theorem

M. Gaberdiel, S. Hohenegger, R. Volpato 2011

There is a 1-1 correspondence between

(a.) Automorphisms of K3 sigma-models commuting with (4,4) supersymmetry.

(b.) Subgroups of Co_0 fixing sublattices of Λ of rank ≥ 4.

Proof: Uses Aspinwall-Morrison theorem identifying a K3 sigma model with a choice of positive definite 4-plane in $\mathbb{R}^{4,20}$ up to action of U-duality

$$O_{\mathbb{Z}}(II^{20;4})\backslash O_{\mathbb{R}}(20;4)/\left(O_{\mathbb{R}}(20)\times O_{\mathbb{R}}(4)\right)$$

(A simple conceptual explanation follows from ``heterotic/typeII string duality'' - Harvey-Moore 2016 .)

Fixed Sublattices Of The Leech Lattice

The culmination of a long line of work is the classification by Hohn and Mason of the 290 isomorphism classes of fixed-point sublattices of the Leech lattice:

99	4	245760	$2^8{:}M_{20}$	$2_{\mathrm{II}}^{-2}4_{\mathrm{II}}^{-2}$	0	1	1	1	1	-	Mon_a*
100	4	30720	$[2^9].A_5$	$2_{\mathrm{II}}^{-4}5^{-1}$	0	1	1	1	1	-	Mon_a*
101	4	29160	$3^4{:}A_6$	$3^{+2}9^{+1}$	1	1	1	1	1	-	S*
102	4	20160	$L_3(4)$	$2_{\mathrm{II}}^{-2}3^{-1}7^{-1}$	2	1	1	1	2	1	M_{23}*
103	4	12288	$[2^{12}3]$	$2_{\mathrm{II}}^{+2}4_3^{+1}8_1^{+1}$	0	1	2	1	1	-	Mon_a
104	4	9216	$[2^{10}3^2]$	$2_{\mathrm{II}}^{+4}3^{+2}$	0	1	2	1	1	-	Mon_a*
105	4	6144	$[2^{11}3]$	$2_{\mathrm{II}}^{-2}4_6^{+2}3^{-1}$	0	1	4	1	1	-	Mon_a
106	4	5760	$2^4{:}A_6$	$4_5^{-1}8_1^{+1}3^{+1}$	2	1	1	1	2	1	M_{23}*
107	4	4096	$2^{1+8}{:}2^3$	4_4^{+4}	0	1	8	1	1	-	Mon_a
108	4	2520	A_7	$3^{+1}5^{+1}7^{+1}$	3	1	1	1	2	1	M_{23}*
109	4	1944	$3^{1+4}{:}2.2^2$	$2_2^{+2}3^{+3}$	1	1	1	1	1	-	S
110	4	1920	$2^4{:}S_5$	$4_3^{-1}8_1^{+1}5^{-1}$	2	1	2	1	3	1	M_{23}
111	4	1344	$2^3{:}L_2(7)$	$4_2^{+2}7^{+1}$	2	1	1	1	3	1	M_{23}*
112	4	1152	$Q(3^2{:}2)$	$8_6^{-2}3^{-1}$	2	1	2	1	2	1	M_{23}*

Remarks

1. Mukai Theorem: There is a 1-1 correspondence between

(a.) Holomorphic symplectic automorphisms of K3 surfaces

(b.) Subgroups of $M23$ with at least 5 orbits on the permutation representation.

2. String theory explanation of QMT suggests an interesting variant: (arising from a discussion with Dave Morrison):

There is a 1-1 correspondence between

(a.) Hyperkahler isometries of K3 surfaces

(b.) Subgroups of Co_0 fixing lattices of rank ≥ 5.

Remarks

3. From the viewpoint of explaining Mathieu Moonshine, the QMT is a huge disappointment:

(a.) Some GHV groups are NOT subgroups of M24

(b.) M24 is not a subgroup of any quotient of any GHV group.

4. However, only (4,1) susy is needed to define the elliptic genus. It turns out that Stab(4,1) is much bigger than Stab(4,4). Whether it is ``big enough'' is unknown. We will discuss what is known about Stab(4,1) at the end.

GTVW Model

Largest group $Stab(4,4) \cong 2^8 : M20$ associated with a distinguished K3 sigma model investigated by Gaberdiel, Taormina, Volpato, Wendland.

2d susy sigma model with target:

$$X = \frac{T(Spin8)}{\mathbb{Z}_2}$$

Special B-field

$$B(v,w) = g(v,w) \; mod \; 2 \quad v, w \in \pi_1(T)$$

Equivalence To A WZW Model

Amazing result of GTVW:
This model is isomorphic to the product of 6 copies of the ***bosonic*** k=1 SU(2) WZW model !

WZW with $G = SU(2)^6$ and
$k = (1,1,1,1,1,1) \in H^4(BG;\mathbb{Z}) \cong \mathbb{Z}^6$

$LSU(2)^{k=1}$ has 2 unitary hw irreps: V_0 and V_1

$$\mathcal{H} \cong \left(V_0 \otimes \tilde{V}_0 \oplus V_1 \otimes \tilde{V}_1 \right)^{\otimes 6} \cong \oplus_{a \in \mathbb{F}_2^6} V_a \otimes \tilde{V}_a$$

Frenkel-Kac-Segal

Gaussian model: $S = \frac{R^2}{4\pi} \int \partial x \tilde{\partial} x \quad x \sim x + 2\pi$

$$e^{\frac{i}{\sqrt{2}}\left(\frac{n}{R}+w\,R\right)x}(z) \otimes e^{\frac{i}{\sqrt{2}}\left(\frac{n}{R}-w\,R\right)\tilde{x}}(\tilde{z})$$

At R=1 we have a theory equivalent to the $SU(2)_1$ WZW model

("Witten's nonabelian bosonization" or "FKS construction")

$$J^3(z) = \frac{1}{\sqrt{2}}\partial x(z), J^{\pm}(z) = e^{\pm i\sqrt{2}x}(z)$$

$$\tilde{J}^3(\tilde{z}) = \frac{1}{\sqrt{2}}\partial \tilde{x}(\tilde{z}), \tilde{J}^{\pm}(\tilde{z}) = e^{\pm i\sqrt{2}\tilde{x}}(\tilde{z})$$

Gives an $su(2)_L \oplus su(2)_R$ current algebra.

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Beauty and the Beast: Superconformal Symmetry in a Monster Module

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Abstract. Frenkel, Lepowsky, and Meurman have constructed a representation of the largest sporadic simple finite group, the Fischer–Griess monster, as the automorphism group of the operator product algebra of a conformal field theory with central charge $c=24$. In string terminology, their construction corresponds to compactification on a $\mathbf{Z}_2$ asymmetric orbifold constructed from the torus $\mathbf{R}^{24}/\Lambda$, where Λ is the Leech lattice. In this note we point out that their construction naturally embodies as well a larger algebraic structure, namely a super-Virasoro algebra with central charge $\hat{c}=16$, with the supersymmetry generator constructed in terms of bosonic twist fields.

HOLOMORPHIC SCFTS WITH SMALL INDEX

DAVIDE GAIOTTO AND THEO JOHNSON-FREYD

ABSTRACT. We observe that every self-dual ternary code determines a holomorphic $\mathcal{N}=1$ superconformal field theory. This provides ternary constructions of some well-known holomorphic $\mathcal{N}=1$ SCFTs, including Duncan's "supermoonshine" model and the fermionic "beauty and the beast" model of Dixon, Ginsparg, and Harvey. Along the way, we clarify some issues related to orbifolds of fermionic holomorphic CFTs. We give a simple coding-theoretic description of the supersymmetric index and conjecture that for every self-dual ternary code this index is divisible by 24; we are able to prove this conjecture except in the case when the code has length 12 mod 24. Lastly, we discuss a conjecture of Stolz and Teichner relating $\mathcal{N}=1$ SCFTs with Topological Modular Forms. This conjecture implies constraints on the supersymmetric indexes of arbitrary holomorphic SCFTs, and suggests (but does not require) that there should be, for each k, a holomorphic $\mathcal{N}=1$ SCFT of central charge $12k$ and index $24/\gcd(k,24)$. We give ternary code constructions of SCFTs realizing this suggestion for $k \leq 5$.

Chiral Fields Of Dimension 3/2

At the price of extending VOA's to super-VOA's we can introduce holomorphic fields of half-integral dimension:

$$V_{\epsilon_1,\epsilon_2,\dots,\epsilon_6} := \exp\left(\frac{i\sqrt{2}}{2}(\epsilon_1 X_1 + \epsilon_2 X_2 + \cdots + \epsilon_6 X_6)\right) \quad \epsilon_i \in \{\pm 1\}$$

$\Rightarrow$ 2^6 vertex operators of conformal dimension = $\left(\frac{1}{4}\right) \times 6 = \frac{3}{2}$

View $|\epsilon_1, \dots, \epsilon_6\rangle \in (\mathbb{C}^2)^{\otimes 6}$ as a state in a system of 6 q-bits

Extend by linearity to define a 2^6 −dimensional space of (3/2,0) vertex operators V_s for $s \in (\mathbb{C}^2)^{\otimes 6}$

The N=4 supercurrents are V_s for very special quantum states $s \in (\mathbb{C}^2)^{\otimes 6}$

N=4 Generators As States In A System Of 6 q-bits

$$[1] \coloneqq | - , + , + , + , + , + \rangle$$

$$[135] \coloneqq | - , + , - , + , - , + \rangle$$

$$Q^{+} = \frac{i-1}{2} \left([\emptyset] + [3456] + [245] + [236] - i[246] - i[235] + i[56] + i\,[34] \right)$$

$$Q^{-} = \frac{i-1}{2} \left(-[1] - [13456] + i[1246] + i[1235] - i([156] + [134]) - [1245] - [1236] \right)$$

$$\bar{Q}^{+} = \frac{i-1}{2} \left(i\,[23456] + i[2] + [35] + [46] + [234] + [256] - i([36] + [45]) \right)$$

$$\bar{Q}^{-} = \frac{i-1}{2} \left(i\,[123456] + i[12] + [135] + [146] + [1234] + [1256] - i([136] + [145]) \right)$$

N=1 Generator

Up to $SU(2)_R$ symmetry there is a unique N=1 generator:

$$G = Q^{+} + \bar{Q}^{-} \qquad\qquad G = \frac{i-1}{2}\,\Psi$$

$$\Psi = [\emptyset] + i\,[123456] + ([1234] + [3456] + 1256]) + i([12] + [34] + [56]) \\ + ([135] + [245] + [236] + [146]) - i([246] + 235] + [136] + [145])$$

Is there a <u>*code*</u> governing this quantum state?

Yes!! It is the ``hexacode’’

$\mathbb{F}_4$ And The Hexacode

Finite field of $4 = 2^2$ elements: $\mathbb{F}_4 = \{\, 0, 1, \omega, \overline{\omega} \,\}$

Addition: $1 + \omega = \overline{\omega}$ $\quad 1 + \overline{\omega} = \omega$ $\quad \omega + \overline{\omega} = 1$ $\quad \mathbb{F}_4^{+} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

Multiplication: $\omega \cdot \omega = \overline{\omega}$ $\quad \omega \cdot \overline{\omega} = 1$

Hexacode: $\mathcal{H}_6 \subset \mathbb{F}_4^6$

$$w = \left(a, b, c, \Phi_{abc}(1), \Phi_{abc}(\omega), \Phi_{abc}(\overline{\omega})\right)$$

$$\Phi_{abc}(x) := a\, x^2 + b\, x + c$$

Relation To Quaternion Group

$$Q \cong \{ \pm 1, \pm i\,\sigma^1, \pm i\,\sigma^2, \pm i\,\sigma^3 \} \subset SU(2)$$

Group of special unitary bit-flip and phase-flip errors in theory of QEC.

$$1 \to \{\pm 1\} \to Q \underset{\overset{h}{\longleftarrow}}{\to} \mathbb{F}_4^+ \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 0$$

$$h(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad h(1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$h(\omega) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad h(\bar{\omega}) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$h(x)h(y) = c_{x,y}\, h(x+y)$$

$c_{x,y}$ is a nontrivial cocycle with some nice properties.

N=1 Generator And The Hexacode

For $w = (x_1, x_2, \ldots, x_6) \in \mathbb{F}_4^6$ define

$$h(w) := h(x_1) \otimes h(x_2) \otimes \cdots \otimes h(x_6) \in SU(2)_6 \subset End(\mathbb{C}^2)^{\otimes 6}$$

$$SU(2)_6 := SU(2)^6 / \mathbb{Z}_2^5$$

$$h(w_1)h(w_2) = \chi(w_1, w_2) h(w_1 + w_2)$$

For general $w_1, w_2 \in \mathbb{F}_4^6$ this is a nontrivial cocycle.

Nontrivial fact: The cocycle is trivial when restricted to $\mathcal{H}_6$!

$$P = 2^{-6} \sum_{w \in \mathcal{H}_6} h(w)$$

One dimensional projection operator

$$\Psi \in Im(P)$$

Consequences: 1/2

V_Ψ generates an N=1 superconformal symmetry:

$$V_s(z_1)V_s(z_2) \sim \frac{\bar{s}s}{z_{12}^3} + \frac{\bar{s}s}{z_{12}}T(z_2) + \frac{\bar{s}\Sigma^A s}{z_{12}^2}J^A(z_2) + \frac{\bar{s}\Sigma^{AB}s}{z_{12}}J^A J^B(z_2) + \cdots$$

J^A : generators of $SU(2)^6$ affine Lie algebra, $A = 1, \dots, 3 \cdot 6 = 18$

Σ^A, Σ^{AB} generate 1- and 2- qubit errors

$$\bar{\Psi}\Sigma^A\Psi = 0 \quad \& \quad \bar{\Psi}\Sigma^{AB}\Psi = 0$$

Because Ψ is in a QEC. $\Rightarrow T_F = V_\Psi$

Consequences: 2/2

$Stab_{SU(2)_6}(\Psi)$ is a finite group

Again follows from the error-correcting properties of the hexacode because the generators of $SU(2)_6$ are the Σ^A

Restricting to the error group $Q^6 \subset SU(2)^6$

$$0 \to \mathbb{Z}_2^5 \to Stab_{Q^6}(\Psi) \cong \{ (\epsilon_1 h(x_1), \dots, \epsilon_6 h(x_6)) | (x_1, \dots, x_6) \in \mathcal{H}_6 \quad \prod_i \epsilon_i = 1\}$$

$$\to \mathcal{H}_6 \to 0$$

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RR Sector And The Golay Code

The crucial RR groundstates in the WZW description form a rep of $(SU(2)_L \times SU(2)_R)^6$

$$V_{RR}^{gnd} \cong \oplus_{\alpha=1}^{6} \left(\frac{1}{2}\right)_L^{(\alpha)} \otimes \left(\frac{1}{2}\right)_R^{(\alpha)}$$

There is a distinguished basis of RR groundstates:

$$\mathbb{H} \cong \left[\left(\frac{1}{2}\right)_L \otimes \left(\frac{1}{2}\right)_R \right]_{\mathbb{R}}$$

as $SU(2)_L \times SU(2)_R$ representations

The usual basis $1, \mathfrak{i}, \mathfrak{j}, \mathfrak{k}$ of quaternions corresponds to 4 distinguished spin states:

$$1 \leftrightarrow \frac{1}{\sqrt{2}}\left(|+,-\rangle - |-,+\rangle \right) := |1\rangle$$

$$\mathfrak{i} \leftrightarrow \frac{1}{\sqrt{2}}\left(|+,+\rangle + |-,-\rangle \right) := |2\rangle$$

$$\mathfrak{j} \leftrightarrow \frac{i}{\sqrt{2}}\left(|+,+\rangle - |-,-\rangle \right) := |3\rangle$$

$$\mathfrak{k} \leftrightarrow \frac{i}{\sqrt{2}}\left(|+,-\rangle + |-,+\rangle \right) := |4\rangle$$

In this basis the action of $\left(\epsilon_L\, h(x), \epsilon_R\, h(x)\right)$ $\epsilon_{L,R} \in \{\pm 1\}$ is diagonal, e.g. $\left(h(1), h(1)\right)$ takes:

$|1\rangle \rightarrow |1\rangle$, $|2\rangle \rightarrow |2\rangle$, $|3\rangle \rightarrow -|3\rangle$, $|4\rangle \rightarrow -|4\rangle$

Column Interpretations Of Hexacode Digits

$| 1 \rangle \to | 1 \rangle$, $|2\rangle \to | 2 \rangle$, $|3\rangle \to -| 3 \rangle$, $|4\rangle \to -| 4 \rangle$

$$\begin{bmatrix} + \\ + \\ - \\ - \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

	$h(0)$	$h(1)$	$h(\omega)$	$h(\overline{\omega})$
$\vert 1 \rangle$	0	0	0	0
$\vert 2 \rangle$	0	0	1	1
$\vert 3 \rangle$	0	1	0	1
$\vert 4 \rangle$	0	1	1	0

So when we consider a general element of $Stab_{Q^6}(\Psi)$ acting on the entire RR sector we get a 4×6 array of 0's and 1's

	$h(x_1)$	$h(x_2)$	$h(x_3)$	$h(x_4)$	$h(x_5)$	$h(x_6)$
$\vert 1\rangle$						
$\vert 2\rangle$						
$\vert 3\rangle$						
$\vert 4\rangle$						

Example:

	$h(1)$	$h(1)$	$h(\omega)$	$h(\omega)$	$h(\bar{\omega})$	$h(\bar{\omega})$
$\vert 1\rangle$	0	0	0	0	0	0
$\vert 2\rangle$	0	0	1	1	1	1
$\vert 3\rangle$	1	1	0	0	1	1
$\vert 4\rangle$	1	1	1	1	0	0

Golay Code & The MOG

Nontrivial statement: The length 24 codewords generated from $Stab_{Q^6}(\Psi)$ are Golay code words. This gives half the Golay code $\mathcal{G}^+$

To get the full Golay code include worldsheet parity (exchanging left- and right-moving dof). This acts as the parity operator in $O(4)$

$$\begin{bmatrix} - \\ + \\ + \\ + \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ ``odd interpretations of hexacode digits ''

Golay Code & The MOG

The action of the stabilizer of $\Psi_L - \Psi_R$ within $\langle P, Q^6 \rangle \subset \left(Pin(4)\right)^6$ in the canonical basis of RR states defines the full Golay code.

This presentation of the Golay code is the Miracle Octad Generator of Curtis and Conway.

So What?

The Golay code can be found in this action of symmetries commuting with (1,1) supersymmetry.

By definition, the automorphism group of the Golay code is $M24$

Conjecture 1: Suppose $Stab_{Q^6}(\Psi) \subset SU(2)^6$ is a normal subgroup of $Stab(4,1) \cap SU(2)^6 \subset SU(2)^6$

Then $Stab(4,1)$ has a quotient that contains a subgroup of the automorphisms of the (even) Golay code.

Conjecture 2: This quotient is a maximal subgroup of $Aut(\mathcal{G}^+)$

What We (Don't) Know About $Stab(4,1)$

1. $Stab(4,1) \cap SU(2)^6$ is strictly larger than $Stab_{Q^6}(\Psi)$

2. $Stab(4,1) \subset \left(Pin(4)\right)^6$: S_6 is strictly larger than $Stab(4,4)$.

$$|Stab(4,4)| = |2^8{:}M20| = 2^{14} \cdot 3 \cdot 5$$

$Stab(4,1)$ contains a subgroup of order $2^{17} \cdot 3^2 \cdot 5$ and (thanks to JEFF and GAP) we understand the structure of this group.

The full group might be much larger.

3. For comparison $|M24| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$

We do not know if there are any elements of order 7,11,23 in $Stab(4,1)$.

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5. **Conway Group Moonshine**
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Conway Group Moonshine

[Frenkel, Lepowsky, Meurman; Duncan; Duncan-Mack-Crane]

Susy sigma model with target
$\mathfrak{X}$ = Cartan torus of E8, with special B-field.

Equivalent to 24 real fermions λ_i
(Holomorphic half of 24 Ising models.)

R groundstates = (a chiral) spinor S of Spin(24)

For $s \in S$ the vertex operator V_s has
conformal dimension $h = \frac{24}{16} = \frac{3}{2}$

Spin operators for the 24 fermions have OPE:

$$V_s(z_1)V_s(z_2) \sim \frac{\bar{s}s}{z_{12}^3} + \frac{\bar{s}s}{z_{12}}T(z_2) + \frac{\bar{s}\gamma^{ij}s}{z_{12}^2}\lambda_i\lambda_j + \frac{\bar{s}\gamma^{ijkl}s}{z_{12}}\lambda_i\lambda_j\lambda_k\lambda_l + \cdots$$

So, in general V_s does not generate an N=1 superconformal algebra …

Choose ON basis for $\mathbb{R}^{24}$ and Clifford generators $\{\gamma_i, \gamma_j\} = 2\,\delta_{ij}$ acting on irreducible module of $\mathbb{R}$ −dimension 2^{12}.

For $w \in \mathbb{F}_2^{24}$ define $\gamma_w := \gamma_1^{w_1} \cdots \gamma_{24}^{w_{24}}$

γ_w generate a Heisenberg extension of $\mathbb{Z}_2^{24}$

Theorem: The cocycle on $\mathcal{G} \subset \mathbb{F}_2^{24}$ is trivializable: There is $b(w) \in \{\pm 1\}$ so that $\tilde{\gamma}_w := b(w)\gamma_w$ satisfies:

$$\tilde{\gamma}_{w_1}\tilde{\gamma}_{w_2} = \tilde{\gamma}_{w_1+w_2}$$

$$P := \frac{1}{2^{12}} \sum_{w \in \mathcal{G}} \tilde{\gamma}_w$$

Rank one projector

$$\Psi = P s_0$$

$$V_\Psi(z_1)V_\Psi(z_2) \sim \frac{\bar{\Psi}\Psi}{z_{12}^3} + \frac{\bar{\Psi}\Psi}{z_{12}} T(z_2) + \frac{\bar{\Psi}\gamma^{ij}\Psi}{z_{12}^2}\lambda_i\lambda_j + \frac{\bar{\Psi}\gamma^{ijkl}\Psi}{z_{12}}\lambda_i\lambda_j\lambda_k\lambda_l + \cdots$$

Vanishing of last two terms ***follows*** from error-correcting properties of Golay

$$T_F = V_\Psi$$

Stabilizer of the rank one image of P turns out to be the Conway group! [Duncan 2003, Duncan-Mack-Crane 2014]

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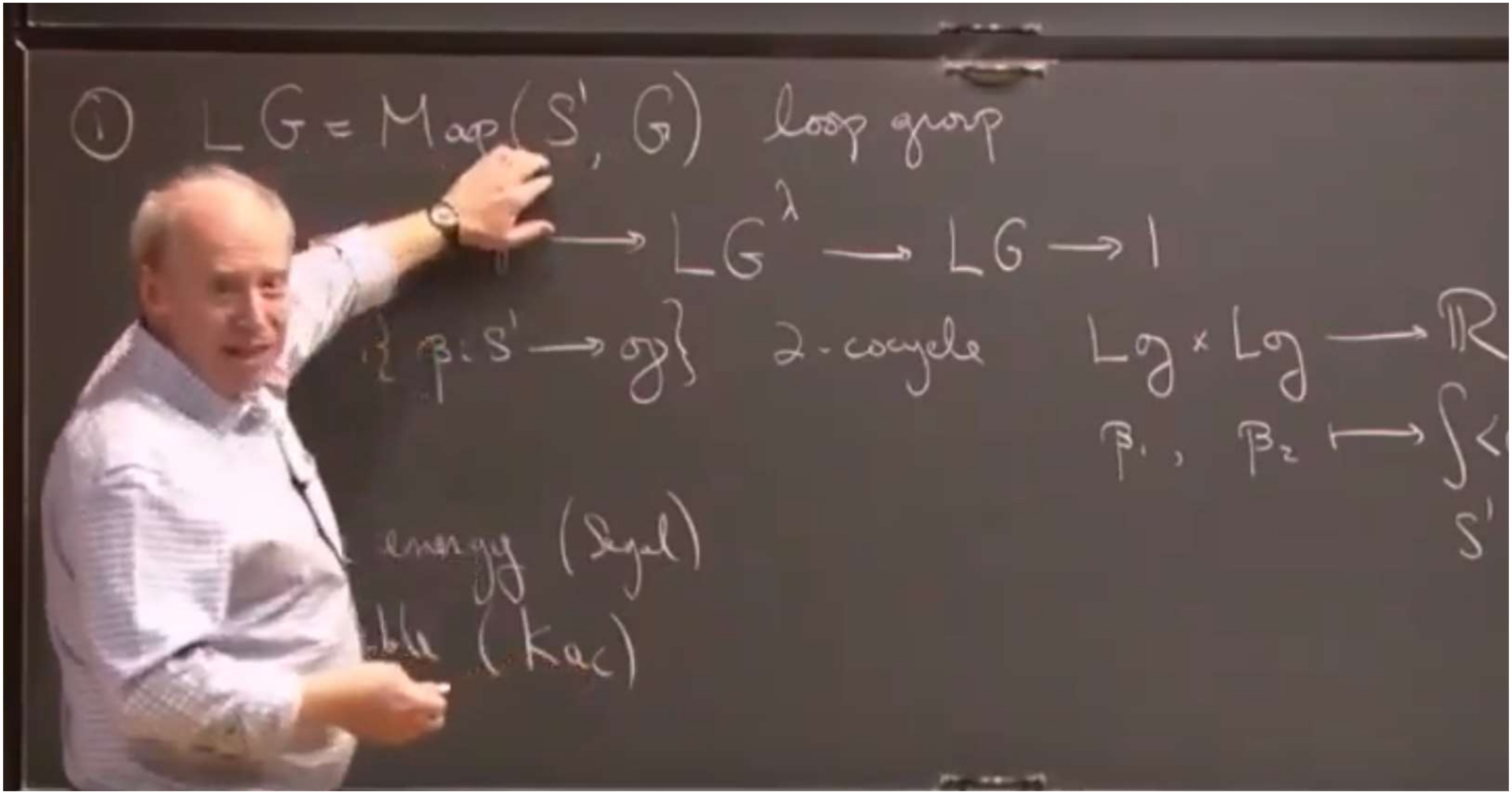
Conclusions

1. *Possible* explanation of Mathieu Moonshine based on Stab(4,1)

(But there is no evidence for/against elements of order 7,11,23.)

2. Interesting connections between QEC and 2d N=1 superconformal symmetry – raises many questions.



HAPPY BIRTHDAY DAN!!!