A biased survey of Low-brow Topological Gauge Theory

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I will review some progress in understanding gauge theory input into equivariant topology. This leads to a 'retro' flavored lecture, but we will touch upon some recent work on *Coulomb branches* and gauged Gromov-Witten theory. Some topics:

1. The Verlinde ring of a compact group, a 2D TQFT, and the twisted equivariant K-theory ${}^{\tau}K_G(G)$, $\tau \in H^4(BG;\mathbb{Z})$
2. Deformations: Higgs bundles, gauged linear σ-model
3. Interpretation via Coulomb branches of 3D, 4D gauge theory
4. Gauge theory as representation theory on categories,
5. Flag varieties as irreducibles, Whittaker picture and ΓW-classes
6. (Will not discuss 4D and Langlands duality)

Some names: Atiyah-Bott, Segal, Hitchin, Moore, Seiberg, Witten, Jeffrey-Kirwan, Givental, Kontsevich, Costello, Hopkins, Lurie, Bezrukavnikov, Braverman, Finkelberg, Kapustin, Nakajima, Neitzke, Nadler, Ben-Zvi, ...
and of course the Birthday Boy.

Prehistory

$\{$Compact group G, $\tau \in H^4(BG;\mathbb{Z})\} \rightsquigarrow$ 2D *conformal field theory*.
Hilbert space of states: $\bigoplus H \otimes H^*$, over τ'-projective positive energy irr. representations H of the free loop group LG.

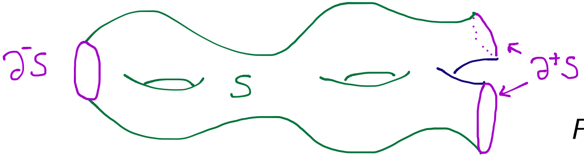
On a conformal surface, the CFT is assembled from its chiral/anti-chiral components using the *Chern-Simons* 3D topological field theory. Reduction over a circle gives the $2D$ *Verlinde theory* V, a K-theory version of $2D$ topological Yang-Mills theory.

$V : S^1 \mapsto$ the Verlinde ring $R(G,\tau)$ of PERs with their *fusion product*, (merging a pair of circles to a single one via a conformal pair of pants). When $\pi_0 G = \pi_1 G = 1$, $R(G,\tau) =$ quotient of the representation ring $R(G)$ by an explicit ideal.

A theorem in topology (Freed, Hopkins, -) identifies $R(G,\tau)$ with the Pontryagin ring ${}^{\tau}K_G^{top}(G)$. Viewing $G \ltimes G$ as the flat connections on S^1, the Pontryagin product is induced by the pair of pants, matching the fusion operation.

TQFT structure from topology

A general surface induces operations on ${}^{\tau}K_G^{top}(G)$, leading to a TQFT structure on V over the K-theory spectrum (enhanced from $\mathbb{C}$).



$$\begin{array}{ccc} & Flat_G(S) & \\ \rho_- \swarrow & & \searrow \rho_+ \\ Flat_G(\partial^- S) & & Flat_G(\partial^+ S) \end{array}$$

When $\partial^+ S \neq \emptyset$, the correspondence diagram $\{\rho_\pm\}$ defines a map

$$(\rho_+)_! \circ \rho_-^* : {}^{\tau}K_G(G) \to {}^{\tau}K_G(G) \otimes {}^{\tau}K_G(G)$$

which (together with a certain trace) assemble to a K-linear TQFT.

[**However**, this enhanced theory is not *fully extended* (to points).]

Verlinde deformations

V : (closed surface S) $\mapsto$ (dim of) the space of Θ-functions for G at level τ.

This can be enhanced, for instance to Θ-sections over the moduli of *flat* $G_{\mathbb{C}}$-bundles. Keep this finite by using graded dims, for the filtration by degree along the (affine) fibers of the projection from flat to all bundles. This gives a formal deformation in a parameter t (returning V at $t = 0$).

Closely related is the *gauged linear σ-model* for a unitary representation E. Here, $S \mapsto$ the index of the Θ-line bundle over the holomorphic mapping space of S to the quotient stack $G_{\mathbb{C}} \ltimes E$, instead of just over $G_{\mathbb{C}}$-bundles.

The mapping space consists of pairs: one (algebraic) $G_{\mathbb{C}}$-bundle and one section of the associated E-bundle on S, plus an obstruction bundle H^1:

$$H^* \left(Bun_{\mathbb{C}} G(S); \Theta^{\tau} \otimes \mathrm{Sym}\ \mathrm{Index}_S^{\vee}(E) \right).$$

Tracking symmetric degrees by t leads to a deformation of $V, R(G, \tau(t))$.

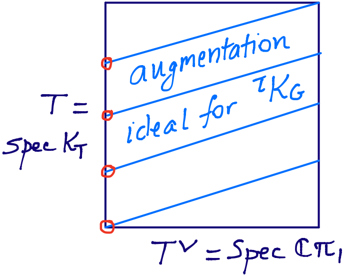
A computation of ${}^\tau K_G(G)$

Theorem (C. Douglas)

When $\pi_0 = \pi_1 = \{1\}$, ${}^\tau K_G(G) = K_G \otimes_{K_*^G(\Omega G)} {}^\tau K_G$.

Here, ΩG carries the Pontryagin product and G acts by conjugation.

The augmentations are:
the push-forward $\Omega G \to \mathrm{pt}$, and
the same after a line bundle $\mathcal{O}(\tau)$ twist.
Geometrically, this is the intersection of two subvarieties in $\mathrm{Spec} K_0^G(\Omega G)$.
They turn out to be *Lagrangian* for a symplectic form (obvious for $G = T$).



This extends to the deformed Verlinde rings: t-deform ${}^\tau K_G$ by twisting by the total symmetric power of the $\mathbb{P}^1$-*index bundle* of E over ΩG.

Fact of note: the augmentation fiber remains Lagrangian.

Coulomb branches for 3/4D pure gauge theory

The following spaces are now called the *Coulomb branches* of pure 3/4D pure gauge theory:

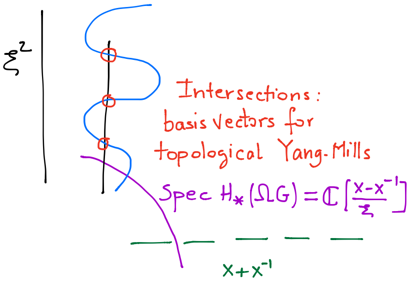
$$C_3(G) := \mathrm{Spec} H_*^G(\Omega G; \mathbb{C}); \quad C_4(G) := \mathrm{Spec} K_*^G(\Omega G; \mathbb{C})$$

Theorem (Bezrukavnikov, Finkelberg, Mirkovic)

1. *Spec* $H_*^G(\Omega G)$ *is an affine resolution of singularities of* $(T^* T_{\mathbb{C}}^\vee)/W$.
2. *Spec* $H_*^G(\Omega G)$ *is algebraic symplectic, and Spec* $H_*(\Omega G)$ *Lagrangian.*
3. *Spec* $K_*^G(\Omega G)$ *is an affine resolution of singularities of* $(T_{\mathbb{C}} \times T_{\mathbb{C}}^\vee)/W$.
4. *Spec* $K_*^G(\Omega G)$ *is algebraic symplectic, and Spec* $K_*(\Omega G)$ *Lagrangian.*
5. $C_3(G), C_4(G)$ *are the phase spaces of the Toda completely integrable systems for* G*, under projection to* $\mathfrak{t}_{\mathbb{C}}/W$*, resp.* $T_{\mathbb{C}}/W$.

The Verlinde theory calculation in $C_4(G)$ is not accidental, it is part of a "character calculus" for gauged 2D σ-models (H for C_3, K for C_4).

Example: $G = \mathrm{SU}(2)$



C_3 with Neumann, Dirichlet and Yang-Mills/Verlinde Lagrangians. x, ξ : coordinates on $T^\vee, \mathfrak{t}$.

The Toda projection to the axis ξ^2 of adj. orbits gives an integrable system, which is also an abelian group scheme over the base $\mathfrak{t}/W$. The unit section is black.

C_4 is vertically periodicized; the picture is good near $1 \in T^\vee$ at large level.

Rigidifying ${}^{\tau(t)}K_G^{top}(G)$ by Matrix Factorization

The complexified ring of ${}^{\tau}K_G^{top}(G)$ is the Jacobian ring of the function

$$g \mapsto \Psi(g) := \frac{1}{2}\log^2 g + \mathrm{Tr}_E\left(\mathrm{Li}_2(tg^{-1})\right)$$

on regular conjugacy classes g, with the proviso that "critical point of Ψ" means "$d\Psi$ is a weight". We square in the quadratic form defined by τ. This formula for deformed Verlinde theories (Woodward, -) generalizes Witten's for topological Yang-Mills theory (which appears as $\tau \to \infty$).

Jacobian rings are Hochschild cohomologies of *Matrix factorization* categories associated to a superpotential, of brany fame in 2D mirror symmetry, where they define mirror models of Gromov-Witten theory. (Matrix factorizations: 2-periodic curved complexes $E^0 \rightleftarrows E^1$, with differential squaring to Ψ.)
So this is a "mirror" of 2D gauge theory.

MFs also appeared in a partial rigidification of the FHT theorem.

An invertible function (like e^{Ψ}) is a $B\mathbb{Z}$-action on the category of vector bundles on a space: this is a rigid form of a circle action.
The quotient stack $G \ltimes G$, a rigid version of the loop space LBG, has a loop rotation $B\mathbb{Z}$-action — even after τ-twisting.

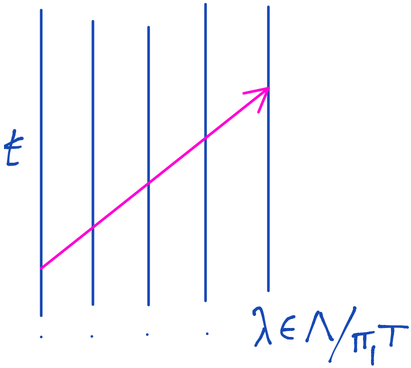
Theorem (Freed,-)

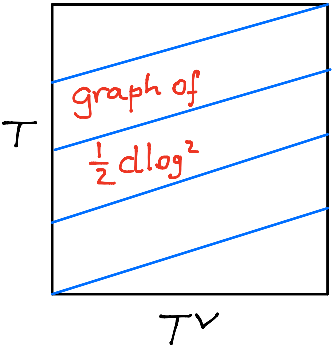
The G-equivariant, τ-twisted MF category over G with respect to the $B\mathbb{Z}$ loop-rotation action is equivalent to the category of τ'-PERs of LG.

(Reason why: interpret $G \ltimes G$ as flat connections, $LG \ltimes \Omega^1(S^1; \mathfrak{g})$.
A boson-fermion correspondence converts this into the crossed product algebra $LG \ltimes \text{Cliff}(L\mathfrak{g})$, with Dirac operator as (curved) differential.
A basis of objects in the category consists of $H \otimes S^{\pm}$, where H runs over appropriate LG-representations and $S^{\pm}$ are the $L\mathfrak{g}$-spinors.)

Fact: these two matrix factorization descriptions are closely related.

Reconciliation





$\mathrm{Spec}(T \ltimes_\tau \mathbb{C}[T]) \cong (\mathfrak{t} \times \Lambda)/\pi_1$, differs from $\mathrm{Spec}(T \ltimes \mathbb{C}[T]) = T \times \Lambda$ by coupling the π_1 action on $\mathfrak{t}$ to Λ. $\Psi = \frac{1}{2}\xi^2 + \lambda(\xi)$, with one critical point on each sheet.

$C_4(T) = T_{\mathbb{C}} \times T_{\mathbb{C}}^\vee$ with the graph of the Verlinde Lagrangian $\frac{1}{2} d\log^2$. Its intercepts with $T \times \{1\}$ are in bijection with the Verlinde points $\Lambda/\pi_1 T$.

Coulomb branches as classifying spaces for gauge theory I

The spaces $C_{3,4}$ carry a *character theory* for 2D ("A-model") gauge theories, serving as *classifying spaces* for the latter. In examples coming from gauged Gromov-Witten theory, the character calculus carries much TQFT information (Seidel operators, J-function...).

Precisely: 2D TQFTS are generated by categories; gauge symmetry is a G-action on the category.
For B-models, the meaning is straightforward, but for A-model theories (GW) the action must factor through the topology of the group.
C_3 carries the characters of gauged A-models.

C_4 should do that for K-linear categories (gauge TQFTs over K), or for LG-actions on linear categories. (I only understand this from examples.)

One precise statement is a (partially verified) equivalence between two 3D TQFTs: pure G-gauge theory PG versus Rozansky-Witten theory of C_3. An analogue for C_4 is less clear (LG-gauge theory?).

Coulomb branches as classifying spaces for gauge theory II

Precise consequences of this may be stated (and proved in examples). Namely, gauged $2D$ theories give rise to a sheaf of categories with Lagrangian support on C_3 (C_4, in the K-linear case).

A construction has been proposed [KRS] of a 2-category of sheaves of categories on hyper-Kaehler manifolds, related to Rozansky-Witten TQFT. A precise claim would be that this KRS 2-category is equivalent to the 2-category of $\mathbb{C}$-linear categories with G-action.

2D TQFTs with gauge symmetry are boundary conditions for PG. Equivalence with RW would give a correspondence of boundary conditions.

While the KRS program is not fully developed, enough can be established for $C_{3,4}$ using their integrable system structure ($\rightsquigarrow$ preferred co-ordinates).

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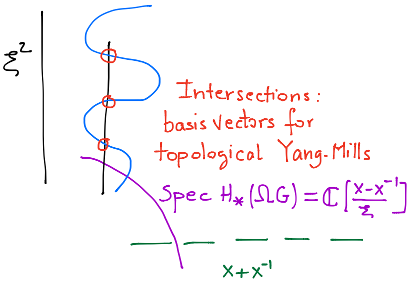

C_3 with Neumann, Dirichlet and Verlinde ($\exp d\Psi$) Lagrangians. x, ξ are the coordinates on $T^\vee, \mathfrak{t}$.

The Toda projection to the axis ξ^2 of adj. orbits gives an integrable system, which is also an abelian group scheme over the base $\mathfrak{t}/W$. The unit section is in black.

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General 2D gauge TQFT are generated by categories with G-action and have a "character" in $C_{3,4}$.
The group structure along the Toda fibers corresponds to the tensor product of TQFTs. Hom categories are computed by matrix factorizations.

Whittaker construction

The space C_3 has two important alternative constructions – Adjoint and Whittaker – as an algebraic symplectic quotient:

$$C_3 \cong T^*_{reg} G^\vee_{\mathbb{C}} /\!/_{Ad}\, G^\vee_{\mathbb{C}} \cong N_\chi \backslash\!\backslash T^* G^\vee_{\mathbb{C}} /\!/_\chi N$$

with reduction by the Adjoint and natural actions, but moment value a regular nilpotent character χ of $\mathfrak{n} \subset \mathfrak{g}^\vee_{\mathbb{C}}$. The second, *Toda isomorphism* is compatible to the projection to $\mathfrak{h}/W = (\mathfrak{g}^\vee)^* /\!/ G^\vee_{\mathbb{C}}$, the base of the Toda integrable system, and captures the diagonalization of the latter.

The fibers of the Whittaker construction correspond to the GW flag varieties of G (their Fukaya categories as G-representations).

The symplectic structures of $C_{3,4}$ lead to deformation quantizations $NC_{3,4}$, which are realized as loop-rotation-equivariant versions of $H, K^*_G(\Omega G)$.

Remark

I do not know a Whittaker construction for C_4.

Whittaker interpretation in the Coulomb branch

Braverman established a striking connection between the (small) quantum J-function of (full) flag varieties and Whittaker matrix coefficients.

Recall the *universal Whittaker module* $Wh = U(\mathfrak{g}) \otimes_{\mathfrak{n}} \mathbb{C}_\chi$.

Theorem (A. Braverman 04, w/ some Coulomb branch improvements)

1. *$QH^*_G(G/T)$ deforms naturally to a module over NC_3 (as HC^{per}).*
2. *This can be additively identified with $U(\mathfrak{h}^\vee) \cong \mathbb{C}_{q(\chi)} \otimes_{\overline{\mathfrak{n}}} Wh^\vee$. In particular, the $Z(\mathfrak{g}^\vee)$- and $H^*(BG)$-module structure match.*
3. *Thereunder, 1 corresponds to the equivariant J-function: the (lowest-weight-regularized) matrix coefficient $\langle w|1|w\rangle$ is J.*
4. *The solution function to the NC_3-module over the Toda base is the equivariant Gromov-Witten Gamma-function $\Gamma W_G(G/T)$.*

Remark

1. The ΓW-function is the quantum S-matrix (2-point) image of the characteristic Γ-class, which is associated to the series $x \mapsto \Gamma_h(h+x)$.
2. ΓW can be interpreted as the inverse Euler class of X within the space of centered holomorphic disks within X.
3. This interpretation explains heuristically the "solution" property of ΓW_G: it reduces the near-triviality of its G-gauged version to the fact that holomorphic $G_{\mathbb{C}}$-bundles on the disk are trivial.
4. The solution property is known in the toric GW world
5. The class $\Gamma W_G(E)$ of a representation can be used to construct, from $C_{3,4}$ and $NC_{3,4}$, the Coulomb branch of the gauged linear σ-model.
6. Examples suggest a relation akin to the Fourier transform between $\Gamma W_G(X)$ and the gauged $\Gamma W(X)$, in the torus case. Specifically, the respective modules in NC_3 seem to be mutual Laplace transforms (?).