Cluster duality & mirror symmetry for Grassmannians & Schubert varieties

Lauren Williams

Joint work w/ Konstanze Rietsch

Austin, January 2019

Plan

0. Introduction

1. Cluster structures on the Grassmannian

2. Newton-Okounkov bodies for Grassmannians

   || Thm (RW)

3. Superpotential polytopes for Grassmannians

4. Conjectural LG-models for Schubert varieties

0. Introduction: mirror symmetry for smooth projective Fano varieties $/\mathbb{C}$

$\mathbb{X}$ - d-dim'l Fano variety $\longleftrightarrow$ Landau-Ginzburg model $(\check{\mathbb{X}}^\circ, W)$ where

$\check{\mathbb{X}}^\circ$ a d-dim'l Kahler mfld and $W: \check{\mathbb{X}}^\circ \to \mathbb{C}$ a holomorphic function s.t. $\langle \cdots \rangle$

Construction of LG models: Hori-Vafa, Batyrev, Givental, Rietsch, many others...

Correspondence between invariants on both sides, e.g.

quantum cohom of $\mathbb{X}$ $\longleftrightarrow$ Jacobian ring assoc to W $\quad \mathbb{C}[\check{\mathbb{X}}^\circ][q,q^{-1}]/\partial W_q$

- Toric case: Batyrev '93
- G/P: Rietsch

Introduction: mirror symmetry for smooth projective Fano varieties $/\mathbb{C}$

$\mathbb{X}$ – d-dim'l Fano variety $\longleftrightarrow$ Landau-Ginzburg model $(\check{\mathbb{X}}^\circ, W)$ where

$\check{\mathbb{X}}^\circ$ a d-dim'l Kahler mfld and $W: \check{\mathbb{X}}^\circ \to \mathbb{C}$ a holomorphic function s.t. $\langle \cdots \rangle$

Today's talk: New polytopal correspondence between two sides, i.e.

Newton-Okounkov body Δ of $(\mathbb{X}, D)$ $\longleftrightarrow$ Superpotential polytope of W

ample divisor

defined as convex hull

lattice pts of $r\Delta \leftrightarrow$ basis of space of sections $H^0(\mathbb{X}, \mathcal{O}(rD))$

$$\left(\frac{\text{NO-body}}{\text{arbitrary var}} = \frac{\text{moment polytope}}{\text{toric variety}} \right)$$

defined by inequalities

related to geom. crystals (BK)

Similar objects come up in Goncharov-Shen, Gross-Hacking-Keel-Kontsevich

Notation for Grassmannians

Def: The Grassmannian $Gr_k(\mathbb{C}^n) = \{V \subset \mathbb{C}^n \mid \dim V = k\}$

Represent elements by (full rank) $k \times n$ matrices M

Let $[n] = \{1, 2, \ldots, n\}$

For $I \in \binom{[n]}{k}$, $p_I(M) =$ det of $k \times k$ minor of M located in columns I

Plucker coordinate

LG model for Grassmannian

$\mathbb{X} = Gr_{n-k}(\mathbb{C}^n)$

Fix ample divisor
$D = \{ P_{12\ldots(n-k)} = 0 \} \subset \mathbb{X}$

$(\check{\mathbb{X}}^{\circ}, W)$ - LG model

where

$\check{\mathbb{X}} = Gr_k(\mathbb{C}^n)$

$\check{\mathbb{X}}^{\circ} =$ complement of anti canonical divisor in $Gr_k(\mathbb{C}^n)$:

remove locus where
$p_{12\ldots k} = 0$ or $p_{23\ldots(k+1)} = 0$ or...

$W : \check{\mathbb{X}}^{\circ} \to \mathbb{C}(q)$ the Marsh-Rietsch superpotential

$\mathbb{X} = Gr_{n-k}(\mathbb{C}^n)$

$(\check{\mathbb{X}}^\circ, W)$ - LG model

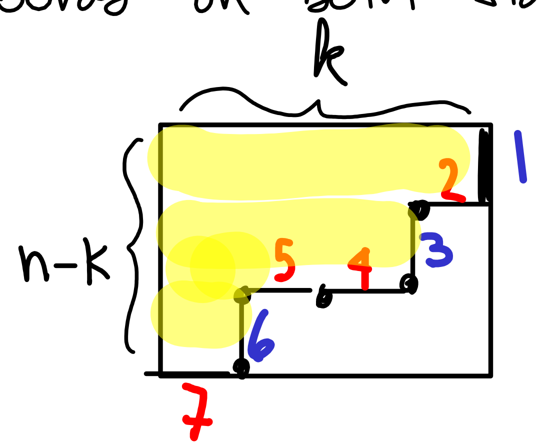
$\check{\mathbb{X}}^\circ =$ complement of anti canonical divisor in $Gr_k(\mathbb{C}^n)$

$$\dim \mathbb{X} = \dim \check{\mathbb{X}} = k(n-k) =: N$$

Plucker coords:

P_J for $J \in \binom{[n]}{n-k}$ $\qquad$ p_I for $I \in \binom{[n]}{k}$

Index Plucker coords on both sides by partitions $\lambda \subseteq (n-k)\times k$



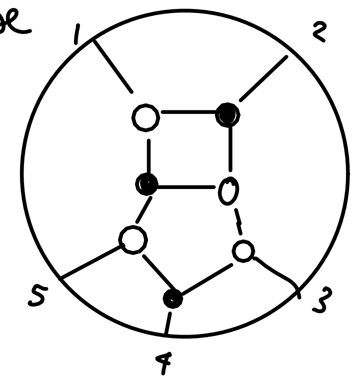
vert steps $\{1,3,6\}$

horiz. steps $\{2,4,5,7\}$

1. Cluster structures on the Grassmannian

— Cluster ($\mathcal{X}$ or $\mathcal{A}$) variety (Fomin-Zelevinsky, Fock-Goncharov) is variety covered by tori, glued together along specific birational maps.

— Cluster $\mathcal{X}$ and $\mathcal{A}$ tori for the Grassmannian can be understood via Postnikov's <u>planar bicolored</u> <u>(plabic)</u> graphs G.



G

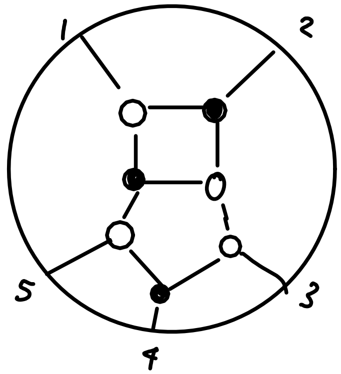
Postnikov

Scott

"network" chart $\Phi_G^{\mathcal{X}} : (\mathbb{C}^*)^N \to \mathbb{X}$

cluster chart $\Phi_G^{\mathcal{A}} : (\mathbb{C}^*)^N \to \check{\mathbb{X}}$

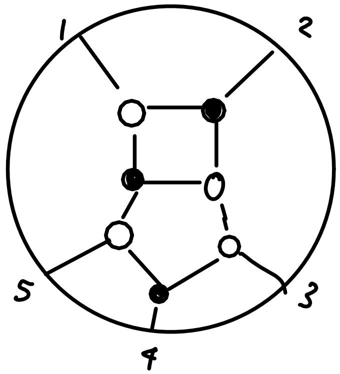
<u>Def:</u> A <u>plabic graph</u> is planar graph in disk with n boundary vertices labeled 1… n. Each boundary vertex incident to unique edge. Internal vertices ● or ○.



(For <u>reduced</u> G)

<u>Def / Lem:</u> "**Rules of road**": Turn right at ●, left at ○. Given G, the <u>trip</u> T_i starts at i & follows rules to end at another bdy vertex $\pi_G(i)$. Defines permutation $\pi_G \in S_n$.

<u>Ex:</u> $$\pi_G = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 4 & 5 & 1 & 2 \end{matrix}$$



$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 4 & 5 & 1 & 2 \end{array}$$

$k=3,\ n=5$

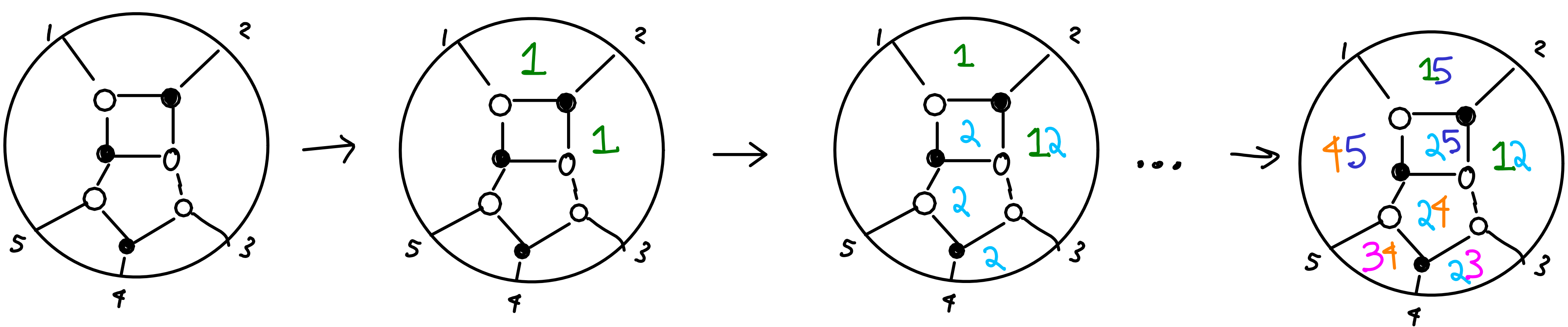
Def: A plabic graph has <u>type (k,n)</u> if it has $k(n-k)+1$ regions and

$$\pi_G = \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & k+2 & \cdots & n \\ \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ n-k+1 & n-k+2 & & n & 1 & 2 & & n-k \end{pmatrix}$$

[They exist... & give toric charts on Grassmannian]

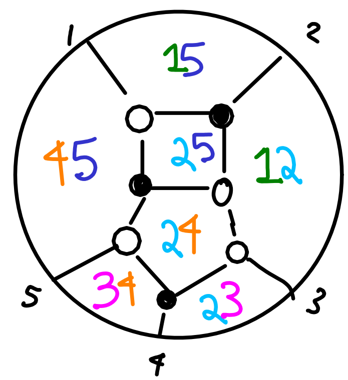
Given G, use <u>trips</u> (right at •, left at ∘) to label faces of G by partitions $\subseteq (n-k)\times k$

T_i divides disk into 2 parts (L & R): put i in each face to left.



Given G, use trips to label faces of G by partitions $\subseteq (n-k)\times k$

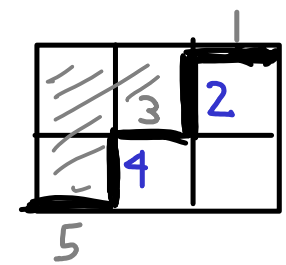
T_i divides disk into 2 parts (L & R): put i in each face to left.

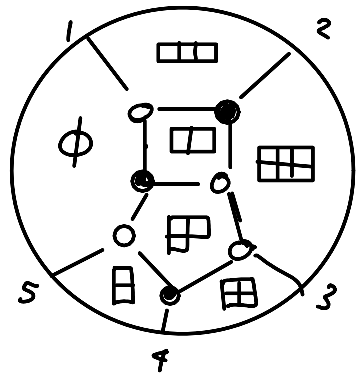


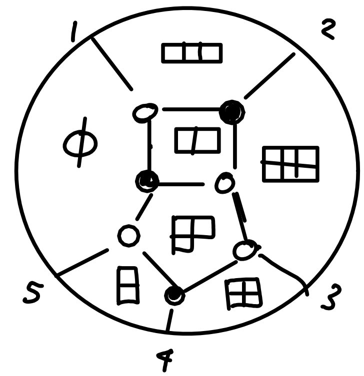
Has type (k,n) for $k=3, n=5$

Map $(n-k)$-element subsets of $[n]$ to Young diagrams $\subseteq (n-k)\times k$ rect

i.e. $\{2,4\} \mapsto$



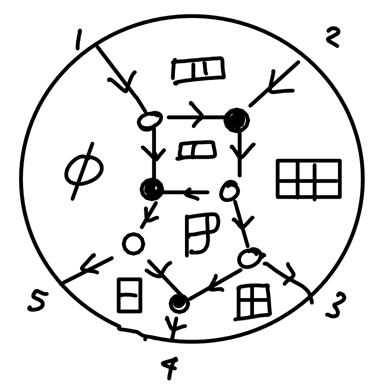


Labeled graph G  of type (k,n)

$\rightsquigarrow$ "network" chart $\Phi_G^{\mathcal{X}} : (\mathbb{C}^*)^N \to \mathcal{X} = Gr_{n-k}(\mathbb{C}^n)$.

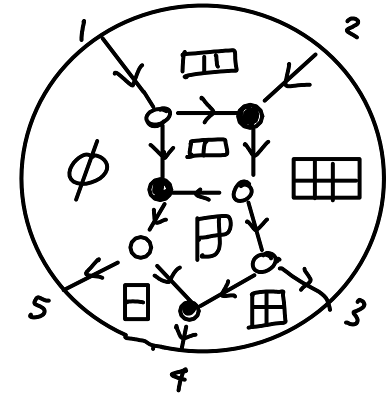
$\{x_\mu\} \mapsto \Phi_G^{\mathcal{X}}(\{x_\mu\})$

$\Phi_G^{\mathcal{X}}$ defined by:

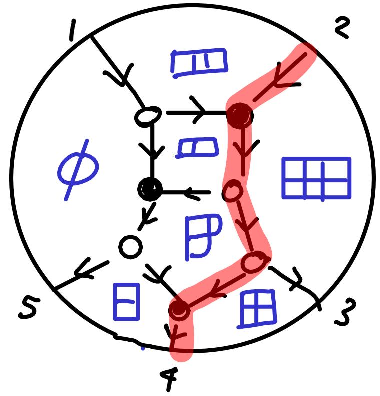


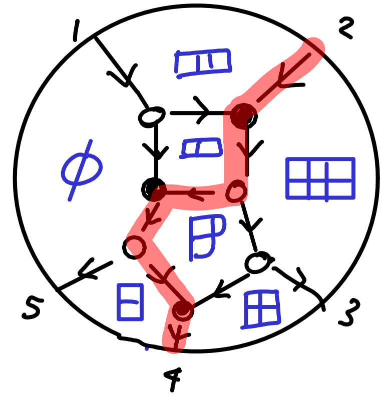
- putting variable x_μ in face labeled μ of G
- choosing canonical "perfect orientation" of G w/ sources at $I = \{1, 2, \dots, n-k\}$
- computing Plucker coord's P_J of image as gen. functions for collections of non-intersecting paths (flows) $I \to J$

$\underline{\text{Ex:}}$ To compute P_{14} of image $\{x_\mu\} \longmapsto \Phi_G^{x}(\{x_\mu\})$,
look at flows $\{1,2\} \to \{1,4\}$, ie.
path collections where $1 \to 1$, $2 \to 4$.



Two paths from $2 \to 4$ w/ weight $x_{\ydiagram{2,2}}\, x_{\ydiagram{3,3}}$ and $x_{\ydiagram{2,2}}\, x_{\ydiagram{3,3}}\, x_{\ydiagram{2,1}}$.

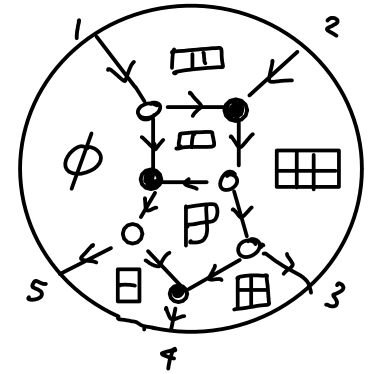




So $P_{14}\left(\Phi_G^{x}(\{x_\mu\})\right) \quad = \quad x_{\ydiagram{2,2}}\, x_{\ydiagram{3,3}}\,(1 + x_{\ydiagram{2,1}})$.

"network" chart $\Phi_G^{\mathcal{X}} : (\mathbb{C}^*)^N \to \mathcal{X} = Gr_{n-k}(\mathbb{C}^n)$

$$\{x_\mu\} \mapsto \Phi_G^{\mathcal{X}}(\{x_\mu\})$$

$G =$ 

$Gr_2(\mathbb{C}^5)$

Plucker coordinates of $\Phi_G^{\mathcal{X}}(\{x_\mu\})$

$P_{12} = 1$

$P_{13} = x_{(3,3)}$

$P_{14} = x_{(2,2)}\, x_{(3,3)} (1 + x_{(2,1)})$

$P_{15} = x_{(2,2)}\, x_{(3,3)}\, x_{(1,1)}\, x_{(2,1)}$

$P_{23} = x_{(3,3)}\, x_{(3)}$

$P_{24} = x_{(2,2)}\, x_{(3,3)}\, x_{(3)} (1 + x_{(2,1)} + x_{(2)}\, x_{(2,1)})$

$P_{25} = x_{(2,2)}\, x_{(3,3)}\, x_{(1,1)}\, x_{(3)}\, x_{(2,1)} (1 + x_{(2)})$

$P_{34} = x_{(3,3)}^2\, x_{(2,2)}\, x_{(2)}\, x_{(3)}\, x_{(2,1)}$

$P_{35} = x_{(3,3)}^2\, x_{(2,2)}\, x_{(2)}\, x_{(3)}\, x_{(2,1)}\, x_{(1,1)}$

$P_{45} = x_{(3,3)}^2\, x_{(2,2)}^2\, x_{(2)}\, x_{(3)}\, x_{(2,1)}\, x_{(1,1)}$

2. Newton-Okounkov bodies of Grassmannians

Recall divisor $D := \{P_I = 0\} \subset \mathbb{X} = Gr_{n-k}(\mathbb{C}^n)$ where $I = \{1, 2, \ldots, n-k\}$.

Let $\mathcal{L}_r :=$ space of sections $H^0(\mathbb{X}, \mathcal{O}(rD))$

$$= \left\langle \frac{\text{deg } r \text{ homog. poly's in Plucker coords}}{P_I^r} \right\rangle$$

$\equiv 1$ in network coord's

Define valuation $\mathrm{val}_G : \mathcal{L}_r \to \mathbb{R}^N$ using network chart $\Phi_G^{\mathcal{X}}$ + choosing leading term (lowest deg) after writing Pluckers in terms of $\{x_\mu\text{'s}\}$

Plucker coordinates $\xrightarrow{\ val_G\ }$ Integer lattice point in $\mathbb{R}^N$ obtained by choosing leading (lowest degree) term in $\frac{P_J}{P_{12}} = P_J$.

$$P_{12} = \underline{1}$$

$$P_{13} = x_{\ydiagram{3,3}}$$

$$P_{14} = x_{\ydiagram{2,2}}\, x_{\ydiagram{3,3}} (1 + x_{\ydiagram{2,1}})$$

$$P_{15} = x_{\ydiagram{2,2}}\, x_{\ydiagram{3,3}}\, x_{\ydiagram{1,1}}\, x_{\ydiagram{2,1}}$$

$$P_{23} = x_{\ydiagram{3,3}}\, x_{\ydiagram{3}}$$

$$P_{24} = x_{\ydiagram{2,2}}\, x_{\ydiagram{3,3}}\, x_{\ydiagram{3}} (1 + x_{\ydiagram{2,1}} + x_{\ydiagram{2}}\, x_{\ydiagram{2,1}})$$

$$P_{25} = x_{\ydiagram{2,2}}\, x_{\ydiagram{3,3}}\, x_{\ydiagram{1,1}}\, x_{\ydiagram{3}}\, x_{\ydiagram{2,1}} (1 + x_{\ydiagram{2}})$$

$$P_{34} = x_{\ydiagram{3,3}}^2\, x_{\ydiagram{2,2}}\, x_{\ydiagram{2}}\, x_{\ydiagram{3}}\, x_{\ydiagram{2,1}}$$

$$P_{35} = x_{\ydiagram{3,3}}^2\, x_{\ydiagram{2,2}}\, x_{\ydiagram{2}}\, x_{\ydiagram{3}}\, x_{\ydiagram{2,1}}\, x_{\ydiagram{1,1}}$$

$$P_{45} = x_{\ydiagram{3,3}}^2\, x_{\ydiagram{2,2}}^2\, x_{\ydiagram{2}}\, x_{\ydiagram{3}}\, x_{\ydiagram{2,1}}\, x_{\ydiagram{1,1}}$$

always 0 so ignore

$\ydiagram{2,2}$	$\ydiagram{3,3}$	$\ydiagram{1,1}$	$\ydiagram{2}$	$\ydiagram{3}$	$\ydiagram{2,1}$	ϕ
0	0	0	0	0	0	0
0	1	0	0	0	0	0
1	1	0	0	0	0	0
1	1	1	0	0	1	0
0	1	0	0	1	0	0
1	1	0	0	1	0	0
1	1	1	0	1	1	0
1	2	0	1	1	1	0
1	2	1	1	1	1	0
2	2	1	1	1	1	0

Def: The <u>Newton-Okounkov body</u> is

$$\Delta_G := \overline{\text{ConvexHull}\left(\bigcup_{r=1}^{\infty} \frac{1}{r}\, val_G(\mathcal{L}_r \setminus \{0\}) \right)}$$

Rk: Not obviously a rat'l polytope or even polytope...

3. Superpotential polytopes for Grassmannians

Marsh-Rietsch superpotential is $W : \check{X}^\circ \longrightarrow \mathbb{C}(q)$ where W expressed in terms of Plucker coordinates;

$$\check{X}^\circ \subset Gr_k(\mathbb{C}^n)$$

Ex: When $\check{X} = Gr_3(\mathbb{C}^5)$, Plucker coords $\longleftrightarrow$ Young diagrams $\leq$ $(3,3)$, and

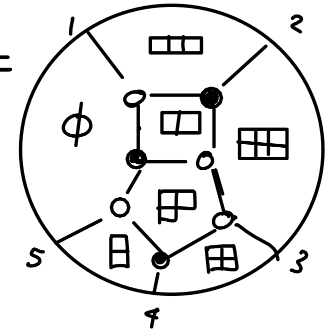
$$W = \frac{p_{\square}}{p_{\emptyset}} + q \underbrace{\frac{p_{(2)}}{p_{(3,3)}} + \frac{p_{(3,1)}}{p_{(3)}}}_{\text{widest rectangles}} + \underbrace{\frac{p_{(3,2)}}{p_{(2,2)}} + \frac{p_{(2,1)}}{p_{(1,1)}}}_{\text{tallest rectangles}}$$

Rk: Pieri rule

Rk: The Plucker coord's appearing in W above not alg. indep. To associate sensible polytope to W, need to rewrite using alg indep. variables...

Superpotential polytopes for Grassmannians

- We used cluster $\mathcal{X}$-torus from plabic G to define NO-bodies
- Now we use cluster $\mathcal{A}$-torus from G to define superpotential polytope
- Cluster $\mathcal{A}$-structure on Grassmannian (Scott) $\Longrightarrow$ for each plabic G of type (k,n), the Plucker coord's assoc to face labels are an $\mathcal{A}$-cluster (alg. indep, gen. $\mathbb{C}(\check{\mathbb{X}})$)

<u>Ex:</u> When $\check{\mathbb{X}} = Gr_3(\mathbb{C}^5)$ and $G =$ 

we can rewrite

$$W = \frac{p_{(1)}}{p_{\phi}} + q\,\frac{p_{(2)}}{p_{(3,3)}} + \frac{p_{(3,1)}}{p_{(3)}} + \frac{p_{(3,2)}}{p_{(2,2)}} + \frac{p_{(2,1)}}{p_{(1,1)}}$$

in terms of Pluckers of face variables

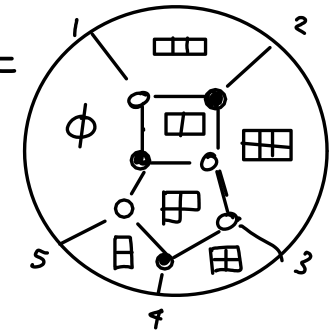
$$= \frac{p_{(2,2)}}{p_{(2,1)}} + \frac{p_{(1,1)}\,p_{(2)}}{p_{(2,1)}} + q\,\frac{p_{(2)}}{p_{(3,3)}} + \frac{p_{(2,1)}}{p_{(1,1)}} + \frac{p_{(3,2)}}{p_{(2)}\,p_{(2,1)}} + \frac{p_{(3,2)}\,p_{(1,1)}}{p_{(2,1)}\,p_{(2,2)}} + \frac{p_{(2,1)}}{p_{(2)}} + \frac{p_{(3,2)}}{p_{(1)}\,p_{(3)}} + \frac{p_{(3)}}{p_{(2)}}$$

(Also set $p_{\phi} = 1$)

Superpotential polytopes for Grassmannians

Now $W = \dfrac{p_{\yng(2,2)}}{p_{\yng(2,1)}} + \dfrac{p_{\yng(1,1)}\, p_{\yng(2)}}{p_{\yng(2,1)}} + q\, \dfrac{p_{\yng(2)}}{p_{\yng(3,3)}} + \dfrac{p_{\yng(2,1)}}{p_{\yng(1,1)}} + \dfrac{p_{\yng(3,3)}}{p_{\yng(2)}\, p_{\yng(2,1)}} + \dfrac{p_{\yng(3,3)}\, p_{\yng(1,1)}}{p_{\yng(2,1)}\, p_{\yng(2,2)}} + \dfrac{p_{\yng(2,1)}}{p_{\yng(2)}} + \dfrac{p_{\yng(3,3)}}{p_{\yng(2)}\, p_{\yng(3)}} + \dfrac{p_{\yng(3)}}{p_{\yng(2)}}$

written in terms of alg indep variables, coming from $G =$



We define "superpotential polytope" Q_G by inequalities coming from tropicalization of W:

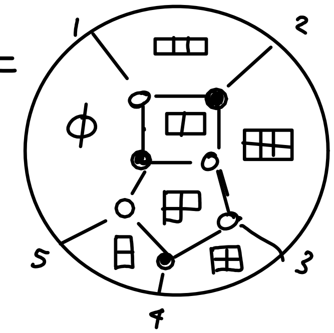
$b_{\yng(2,2)} - b_{\yng(2,1)} \geq 0$ $\quad$ $b_{\yng(1,1)} + b_{\yng(2)} - b_{\yng(2,1)} \geq 0$ $\quad$ $1 + b_{\yng(2)} - b_{\yng(3,3)} \geq 0$ $\quad$ $b_{\yng(2,1)} - b_{\yng(1,1)} \geq 0$

$b_{\yng(3,3)} - b_{\yng(2)} - b_{\yng(2,1)} \geq 0$ $\quad$ $b_{\yng(3,3)} + b_{\yng(1,1)} - b_{\yng(2,1)} - b_{\yng(2,2)} \geq 0$ $\quad$ $b_{\yng(2,1)} - b_{\yng(2)} \geq 0$ $\quad$ $b_{\yng(3,3)} - b_{\yng(2)} - b_{\yng(3)} \geq 0$ $\quad$ $b_{\yng(3)} - b_{\yng(2)} \geq 0$

Superpotential polytopes for Grassmannians

Now $W = \dfrac{p_{\ydiagram{2,2}}}{p_{\ydiagram{2,2,1}}} + \dfrac{p_{\ydiagram{1,1}}\, p_{\ydiagram{2}}}{p_{\ydiagram{2,2,1}}} + q\,\dfrac{p_{\ydiagram{2}}}{p_{\ydiagram{3,3}}} + \dfrac{p_{\ydiagram{2,1}}}{p_{\ydiagram{1,1}}} + \dfrac{p_{\ydiagram{3,3}}}{p_{\ydiagram{2}}\, p_{\ydiagram{2,2,1}}} + \dfrac{p_{\ydiagram{3,3}}\, p_{\ydiagram{1,1,1}}}{p_{\ydiagram{2,1}}\, p_{\ydiagram{2,2}}} + \dfrac{p_{\ydiagram{2,2,1}}}{p_{\ydiagram{2}}} + \dfrac{p_{\ydiagram{3,3}}}{p_{\ydiagram{2}}\, p_{\ydiagram{3}}} + \dfrac{p_{\ydiagram{3}}}{p_{\ydiagram{2}}}$

written in terms of alg indep variables, coming from $G =$



We define "superpotential polytope" Γ_G by inequalities coming from tropicalization of W:

$b_{\ydiagram{2,2}} - b_{\ydiagram{2,2,1}} \geq 0$ $\quad$ $b_{\ydiagram{1,1}} + b_{\ydiagram{2}} - b_{\ydiagram{2,2,1}} \geq 0$ $\quad$ $1 + b_{\ydiagram{2}} - b_{\ydiagram{3,3}} \geq 0$ $\quad$ $b_{\ydiagram{2,1}} - b_{\ydiagram{1,1}} \geq 0$

$b_{\ydiagram{3,3}} - b_{\ydiagram{2}} - b_{\ydiagram{2,2,1}} \geq 0$ $\quad$ $b_{\ydiagram{3,3}} + b_{\ydiagram{1,1}} - b_{\ydiagram{2,1}} - b_{\ydiagram{1,1,1}} \geq 0$ $\quad$ $b_{\ydiagram{2,2,1}} - b_{\ydiagram{2}} \geq 0$ $\quad$ $b_{\ydiagram{3,3}} - b_{\ydiagram{2}} - b_{\ydiagram{3}} \geq 0$ $\quad$ $b_{\ydiagram{3}} - b_{\ydiagram{2}} \geq 0$

Theorem (Rietsch-W):

$\Delta_G = \Gamma_G$ for any G of type (k,n)

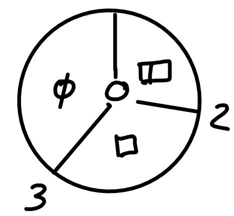
NObody defined from $\mathcal{X}$-torus of G, convex hull of points

Superpotential polytope from $\mathcal{A}$-torus of G inequalities

Pf tools: Canonical choice of $G \rightsquigarrow$ Gelfand Tsetlin polytope

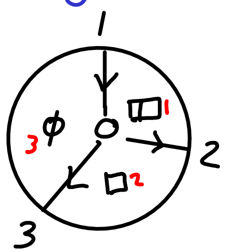
tropicalized mutation, Gross-Hacking-Keel-Kontsevich theta basis

Example: $k=2,\ n=3$, $\mathbb{X} = Gr_1(\mathbb{C}^3) = \mathbb{P}^2_1$ $\check{\mathbb{X}} = Gr_2(\mathbb{C}^3)$

$G =$ 

● right
○ left

NO body from Network chart:



$\rightsquigarrow$ Pluckers

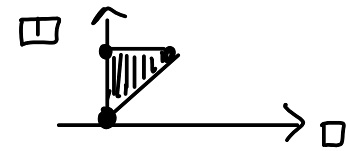
$P_1 = 1$

$P_2 = x_{\square\square}$

$P_3 = x_{\square} x_{\square\square}$

$\xrightarrow{val_G}$

$\square$	$\square\square$
0	0
0	1
1	1

$\rightsquigarrow$ $\Delta_G =$ 

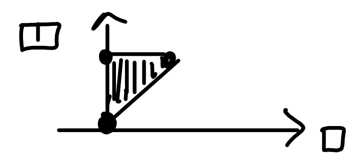
Superpotential polytope from Cluster chart:

Superpotential is $\dfrac{p_\square}{p_\phi} + q\dfrac{p_\phi}{p_{\square\square}} + \dfrac{p_{\square\square}}{p_\square}$ $\underset{\text{Set } p_\phi = 1}{=}$ $p_\square + \dfrac{q}{p_{\square\square}} + \dfrac{p_{\square\square}}{p_\square}$

$\Big\downarrow$ Tropicalize

$b_\square \geq 0$ $1 - b_{\square\square} \geq 0$ $b_{\square\square} - b_\square \geq 0$

$\Big\downarrow$

$\Gamma_G =$ 

Note: $\Delta_G = \Gamma_G$!

Work in progress

- The paradigm NObody $\overset{\star}{=}$ Superpotential polytope should hold more generally
- In situations where no superpotential is known, $\star$ suggests a way to construct it

- Schubert varieties X_λ in Grassmannian
  - Are singular in general ...
  - Admit cluster structures, again using plabic graphs (Leclerc, Serhiyenko-ShermanBennett - W.)

Work in progress

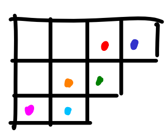
– Schubert varieties X_λ in Grassmannian

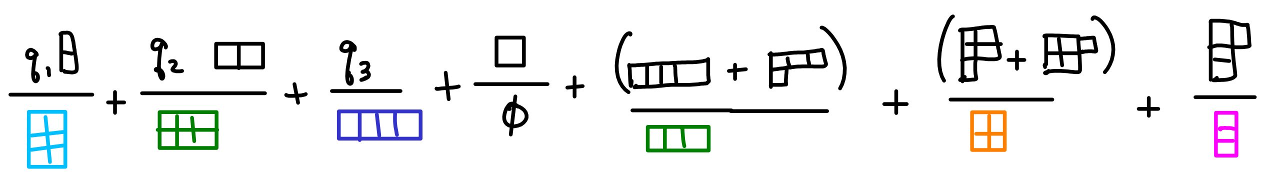
- Rietsch-W: We give conjectural generalized LG model for Schubert varieties:

$$X_\lambda \subseteq Gr_{n-k}(\mathbb{C}^n) \longleftrightarrow (\overset{\vee}{X}{}^{\circ}_\lambda, W_\lambda)$$

$$\overset{\vee}{X}{}^{\circ}_\lambda \subset Gr_k(\mathbb{C}^n)$$

defining W_λ s.t. for each G, NO body $\overset{\text{Thm (R.W.)}}{=}$ superpotential polytope

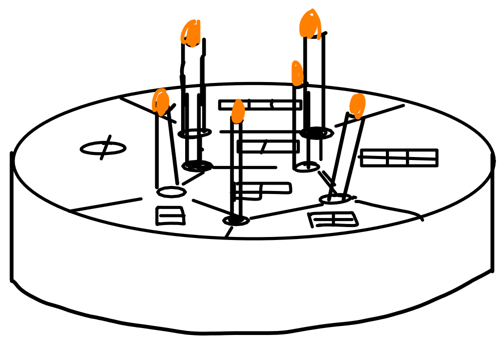
Ex: Let $\lambda =$ , so $X_\lambda \subseteq Gr_3(\mathbb{C}^7)$.

Then $W_\lambda =$ 

Thank you!!

References:

- Rietsch–W. "Cluster duality & mirror symmetry for Grassmannians," arXiv:1712.00447



Happy Birthday Dan!