CHARACTERISTIC CLASSES: EXERCISES

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These exercises are not in order. (3) is the most important one. Do the ones that look the most interesting to you.

(1) If you have not seen the associated bundle construction before, verify that it produces a vector bundle.

(2) Think about why the associated bundle to the frame bundle and the standard representation of $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{R}^n$ is the tangent bundle, and the associated bundle for the dual representation on $(\mathbb{R}^n)^*$ is the cotangent bundle.

(3) In this exercise, we'll compute $c(\mathbb{CP}^n) = (1+x)^{n+1}$, where $x \in H^2(\mathbb{CP}^n) \cong \mathbb{Z}$ is a generator, Poincaré dual to $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$.

  (a) Let $Q = \underline{\mathbb{C}}^{n+1}/S$, the universal quotient bundle: its fiber over an $\ell \in \mathbb{CP}^n$ is $\mathbb{C}^{n+1}/\ell$. Show that $\mathrm{Hom}(S, Q) \cong T\mathbb{CP}^n$. (Hint: let ℓ be a complex line in $\mathbb{C}^{n+1}$ and $\ell^\perp$ be a complimentary subspace, i.e. $\ell \oplus \ell^\perp \cong \mathbb{C}^{n+1}$. Then, $\mathrm{Hom}(\ell, \ell^\perp)$ can be identified with the neighborhood of $\ell \in \mathbb{CP}^n$ of lines which are graphs of functions $\ell \to \ell^\perp$.)

  (b) Using this, show that $T\mathbb{CP}^n \oplus \mathrm{Hom}(S, S) \cong (S^*)^{\oplus(n+1)}$.

  (c) If E is any line bundle, show that $\mathrm{Hom}(E, E)$ is trivial.

  (d) If $E \to \mathbb{CP}^n$ is a line bundle, show that $c_1(E^*) = -c_1(E)$. (Hint: use the fact that $E^* \cong \overline{E}$ and naturality of Chern classes.)

  (e) Applying (3c) and (3d) to (3b), conclude $c(\mathbb{CP}^n) = (1+x)^{n+1}$.

(4) Let's construct some classifying spaces.

  (a) Suppose G is a discrete group. Show that any $K(G, 1)$ (i.e. a connected space with $\pi_1 = G$ and all higher homotopy groups vanishing) is a model for BG. Thus S^1 is a $B\mathbb{Z}$ and $\mathbb{RP}^\infty$ is a $B\mathbb{Z}/2$.

  (b) In the rest of exercise, we construct $B\mathrm{O}_n$ as an infinite-dimensional manifold. Fix a separable Hilbert space, such as ℓ^2. The Stiefel manifold $\mathrm{St}_n(\ell^2)$ is the set of linear isometric embeddings $\mathbb{R}^n \hookrightarrow \ell^2$ (i.e. injective linear maps preserving the inner product), topologized as a subspace of $\mathrm{Hom}(\mathbb{R}^n, \ell^2)$. O_n acts on $\mathrm{St}_k(\ell^2)$ by precomposition.
  The infinite-dimensional Grassmannian $\mathrm{Gr}_n(\ell^2)$ is the space of n-dimensional subspaces of ℓ^2, topologized in a similar way to finite-dimensional Grassmannians. There's a projection $\pi\colon \mathrm{St}_n(\ell^2) \twoheadrightarrow \mathrm{Gr}_n(\ell^2)$ sending a map $b\colon \mathbb{R}^n \to \ell^2$ to its image.

    (i) Why is the unit sphere in an infinite-dimensional Hilbert space contractible?

    (ii) Show that $\mathrm{St}_n(\ell^2)$ is contractible. (Hint: if e_i denotes the sequence with a 1 in position i and 0 everywhere else, define two homotopies, one which pushes any embedding to one orthogonal to the standard embedding $s\colon \mathbb{R}^n \to \ell^2$ as the first n coordinates, and the other which contracts the subspace of embeddings orthogonal to s onto s).

    (iii) Show that the O_n-action on $\mathrm{St}_n(\ell^2)$ is free, so $\mathrm{St}_n(\ell^2)$ is an $E\mathrm{O}_n$.

    (iv) Show that $\pi\colon \mathrm{St}_n(\ell^2) \to \mathrm{Gr}_n(\ell^2)$ is the quotient by the O_n-action, so $\mathrm{Gr}_n(\ell^2)$ is a $B\mathrm{O}_n$.[1]

(5) If you have seen Čech cohomology, this exercise will construct c_1 using that perspective.

  (a) Analogously to the transition functions of a vector bundle, describe how the data of a principal G-bundle $P \to X$ can be specified by a trivializing open cover $\mathfrak{U}$ of P and transition functions $g_{UV}\colon U \cap V \to G$.

  (b) Let $h_{UV} \coloneqq \det \circ g_{UV}\colon U \cap V \to \mathrm{U}_1$. The h_{UV} collectively define a U_1-valued Čech 1-cochain. Show it is a cocycle.

[1] Using complex vector spaces, Grassmannians, and Stiefel manifolds everywhere gives $E\mathrm{U}_n \to B\mathrm{U}_n$. Now, if $G \subset \mathrm{U}_n$, $E\mathrm{U}_n/G$ is a BG. Since every compact Lie group admits a finite-dimensional unitary representation, this gives models for BG for any compact Lie group G.

(c) Consider the long exact sequence in cohomology associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong \mathrm{U}_1 \to 0$. We define the *first Chern class* of P to be the image of $[\{h_{UV}\}] \in \check{H}^1(X; \mathrm{U}_1)$ under the connecting morphism to $\check{H}^2(X; \mathbb{Z})$. Show that if P is nontrivial, $\{h_{UV}\}$ is not in the image of $\check{Z}^1(X; \mathbb{R}) \to \check{Z}^1(X; \mathrm{U}_1)$. (Hint: if this is the case, P admits a reduction of structure group to a principal $\mathbb{R}$-bundle, where $\mathbb{R}$ carries the usual topology. Why are all principal $\mathbb{R}$-bundles trivial?)

(d) If $n = 1$, why is $c_1(P) \neq 0$ iff P is nontrivial?

(e) Why does $c_1(P) = 0$ iff P admits a reduction of structure to SU_n?

(6) If $E \to M$ is a vector bundle, its determinant bundle $\mathrm{Det}E \to M$ is its top exterior power, which is a line bundle. Use the locus-of-dependency definition of Chern classes to show that $c_1(E) = c_1(\mathrm{Det}E)$.

(7) Use Chern-Weil theory to compute the first Chern class of $\mathbb{CP}^1$.

Less crucial but perhaps still interesting:

(8) There is a finite analogue of the correspondence between principal GL_n-bundles and vector bundles, relating n-fold covering spaces $X' \to X$ and principal S_n-bundles.[2]

(a) Consider a variant of the associated bundle construction for sets. Let $[n] := \{1, \ldots, n\}$, and given a principal S_n-bundle $P \to X$, let $P \times_{S_n} [n]$ denote the quotient of $P \times [n]$ by the equivalence relation $(p \cdot \sigma, i) \sim (p, \sigma(i))$ for all $\sigma \in S_n$. Show $P \times_{S_n} [n] \to X$ is an n-fold covering map. (Don't get too bogged down in the details.)

(b) Can you define a map going the other way, from n-fold covering maps to principal S_n-bundles? Hint: first work over a point. Given a set with n elements, how can one canonically build an S_n-torsor?

(c) Show that these constructions are mutually inverse.

[2] This is not just a party trick, and in fact is useful for counting Hurwitz numbers, but that's out of scope for now.