CHARACTERISTIC CLASSES: EXERCISES

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These exercises are not in order. Do the ones that look the most interesting to you. Some are a lot easier than others.

(1) Analogously to yesterday's calculation of $c(\mathbb{CP}^n)$, show that $w(\mathbb{RP}^n) = (1+x)^{n+1}$, where x is the nonzero element of $H^1(\mathbb{RP}^n;\mathbb{Z}/2) \cong \mathbb{Z}/2$.

(2) For which n is $\mathbb{RP}^n$ orientable? Spin?

(3) In this exercise, we show Stiefel-Whitney numbers are unoriented bordism invariants.[1]
   (a) Let W be a compact $(n+1)$-manifold with boundary M. Show that $TW|_M \cong TM \oplus \underline{\mathbb{R}}$, so $w(TW|_M) = w(TM)$.
   (b) Associated to the pair $W \hookrightarrow M$, we have long exact sequences in homology and cohomology. Let $\partial_\downarrow\colon H_{n+1}(W,M;\mathbb{Z}/2) \to H_n(M;\mathbb{Z}/2)$ and $\partial^\uparrow\colon H^n(M;\mathbb{Z}/2) \to H^{n+1}(W,M;\mathbb{Z}/2)$ be the boundary maps. Show that if $i_1 + \cdots + i_k = n$, then $\partial^\uparrow(w_{i_1}(M)\cdots w_{i_k}(M)) = 0$.
   (c) The cap product pairing is compatible with the long exact sequence of a pair, in that $\langle \partial^\uparrow x, y\rangle = \langle x, \partial_\downarrow y\rangle$ (this is true in general and not just here). Conclude $\langle w_{i_1}(M)\cdots w_{i_k}(M), [M]\rangle = 0$.
   (d) Which $\mathbb{RP}^n$, for $n \le 5$, are not null-bordant?

(4) In this exercise, we study orientations and spin structures from a Čech cohomology point of view.
   (a) Use the determinant $\det\colon \mathrm{O}_n \to \mathrm{O}_1 \cong \mathbb{Z}/2$ to mimic yesterday's exercise and build a characteristic class in $H^1(-;\mathbb{Z}/2)$ that is precisely the obstruction for a principal O_n-bundle P to admit a reduction of structure group to a principal SO_n-bundle. This class is $w_1(P)$, of course.
   (b) Recall that $\mathrm{Spin}_n \to \mathrm{SO}_n$ is the unique connected $(n>0)$ double cover of SO_n, with kernel denoted $\{\pm 1\}$. Assume that our open cover $\mathfrak{U}$ of X is *good* (all $U \in \mathfrak{U}$, as well as pairwise, triple, etc. intersections of elements of $\mathfrak{U}$, are contractible; such a cover exists on any manifold). Let $P \to X$ be a principal SO_n-bundle, and for each pair $U, V \in \mathfrak{U}$, choose a lift of the transition function $g_{UV}\colon U\cap V \to \mathrm{SO}_n$ to $\widetilde{g}_{UV}\colon U\cap V \to \mathrm{Spin}_n$. Comparing these lifts on triple intersections defines a class $\{\widetilde{g}_{UV}\widetilde{g}_{VW}\widetilde{g}_{UW}\} \in \check{C}^2(X,\mathfrak{U},\{\pm 1\})$. Why is it a cocycle?
   (c) Suppose we chose different lifts to Spin_n. Show that the resulting cocycle differs by a coboundary. Thus we have defined a characteristic class obstructing a lift from an SO_n-bundle to a Spin_n-bundle; this is $w_2(P)$, of course.
   (d) The set of spin structures on a vector bundle admitting a spin structure is a torsor for $H^1(X;\mathbb{Z}/2)$. Describe the $H^1(X;\mathbb{Z}/2)$-action from this perspective.

(5) Show that the top Stiefel-Whitney class of an odd-dimensional closed manifold vanishes. (Hint: what is the mod 2 Euler characteristic of an odd-dimensional closed manifold?)

(6) If $M \subset \mathbb{R}^n$ is a closed, embedded $(n-1)$-dimensional manifold, show that M is orientable. (Hint: $\mathbb{RP}^2$ immerses in $\mathbb{R}^3$, so this uses the fact that it's an embedding.)

(7) Show that if $n > 1$, $\mathbb{RP}^n$ does not embed in $\mathbb{R}^{n+1}$, and if in addition $n \ne 2^k - 1$, $\mathbb{RP}^n$ does not immerse in $\mathbb{R}^{n+1}$.

(8) Bordism classes of real projective spaces generate a subring of Ω_*^{O}. What is the lowest-degree element that is not contained in this subring? Recall that $\Omega_*^{\mathrm{O}} \cong \mathbb{Z}/2[x_i \mid i \ne 2^j - 1]$.

(9) Show there is no vector bundle $V \to \mathbb{RP}^\infty$ such that $V \oplus S$ is trivial. (A vector bundle over a finite-dimensional CW complex always has such a "complement.")

[1] The basic argument is quite general, and will apply to the other characteristic numbers we meet this week.