CHARACTERISTIC CLASSES: EXERCISES

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These exercises are not in order. Do the ones that look the most interesting to you. Some are a lot easier than others.

(1) For which n is $\mathbb{CP}^n$ spin?
(2) Which Stiefel-Whitney numbers of closed 4-manifolds coincide, analogously to $w_2 = w_1^2$ for 2-manifolds? (Hint: $\Omega_4^{\mathrm{O}} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, so you should obtain two linearly independent Stiefel-Whitney numbers.)
(3) The *Wu manifold* $W \coloneqq \mathrm{SU}_3/\mathrm{SO}_3$ has cohomology ring $H^*(W;\mathbb{Z}/2) \cong \mathbb{Z}/2[z_2, z_3]/(z_2^2, z_3^2)$, with $|z_i| = i$, and $\mathcal{A}$-action $\mathrm{Sq}(z_2) = z_2 + z_3$ and $\mathrm{Sq}(z_3) = z_3 + z_5$. Show W is not null-bordant (in fact, it generates $\Omega_5^{\mathrm{O}} \cong \mathbb{Z}/2$).
(4) Let $4 \mid n$. Then $H^*(B\mathbb{Z}/n;\mathbb{Z}/2) \cong \mathbb{Z}/2[x,y]/(x^2)$ with $|x| = 1$ and $|y| = 2$. The axiomatics of Steenrod squares almost completely determine the $\mathcal{A}$-action on $H^*(B\mathbb{Z}/n;\mathbb{Z}/2)$. Show $\mathrm{Sq}^1(y) = 0$. (Hint: show that y is the mod 2 reduction of a class $\widetilde{y} \in H^2(B\mathbb{Z}/n;\mathbb{Z})$, which is therefore c_1 of a nontrivial line bundle...)
(5) Show that $\Omega_2^{\mathrm{Spin}^c}$ and $\Omega_4^{\mathrm{Spin}^c}$ are both infinite by finding an analogue to Stiefel-Whitney numbers (we can use $\mathbb{Z}$ cohomology because a spin^c structure induces an orientation) which maps to $\mathbb{Z}$, and producing examples where it doesn't vanish. (Hint: find a map $\mathrm{Spin}_n^c \to \mathrm{U}_1$, allowing one to define an associated principal U_1-bundle given a principal Spin_n^c-bundle.)
(6) Let M denote the mapping torus for complex conjugation on $\mathbb{CP}^2$ (i.e. the space $\mathbb{CP}^2 \times [0,1]$, modulo the relation $(0,x) \sim (1,\overline{x})$). Is this orientable? Can you determine its mod 2 cohomology and Stiefel-Whitney classes?
(7) There is a Čech-cohomological perspective on spin^c structures, where one chooses a good cover $\mathfrak{U}$ and transition functions $g_{UV}\colon U \cap V \to \mathrm{SO}_n$ as before, and lifts them to $\widetilde{g}_{UV}\colon U \cap V \to \mathrm{Spin}_n$ as before, possibly incoherently. From this perspective, a spin^c structure is "a principal U_1-bundle that doesn't quite work, but in the same ways as the spin structure doesn't work," i.e. transition functions $h_{UV}\colon U \cap V \to \mathrm{U}_1$ whose cocycle condition might not be satisfied, but such that for all triple intersections $U \cap V \cap W$,
$$\widetilde{g}_{UV}\widetilde{g}_{VW}\widetilde{g}_{WU} = h_{UV}h_{VW}h_{WU}.$$
Show this is the same data as a spin^c structure. Can you also show how this provides a preimage of w_2 under $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z}/2)$?
(8) Show that if w_k is the first nonzero Stiefel-Whitney class of a vector bundle, then k is a power of 2.