Day 1: four perspectives on characteristic classes

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Today’s plan

- Brief review/introduction to vector bundles and principal $G$-bundles
- What are characteristic classes? And why?
- Brief introduction different perspectives on Chern classes (axiomatic definition; linear dependency of sections; Chern-Weil theory; classifying spaces)
This week’s plan

- Today: four perspectives on characteristic classes
- Tomorrow: Stiefel-Whitney classes (real vector bundles, mod 2 cohomology)
- Wednesday: Steenrod squares and Wu classes (more mod 2 cohomology)
- Thursday: Chern, Pontrjagin, and Euler classes (real and complex vector bundles, \( \mathbb{Z} \) cohomology)
- Friday: Chern-Weil theory (de Rham cohomology)
Vector bundles

- Idea: a continuously varying, locally trivial family of vector spaces over a base space $X$

- Formally, a map $\pi: V \to X$ such that there exists an open cover $\mathcal{U}$ of $X$ and homeomorphisms $\varphi_U: \pi^{-1}(U) \to U \times \mathbb{R}^n$ which commute with to the projections down to $X$

- and, for all pairs $U, V \in \mathcal{U}$, the transition function $g_{UV} := \varphi_V \varphi_U^{-1}$ is $\text{GL}_n(\mathbb{R})$-equivariant

- Finally, the cocycle condition on triple intersections $U \cap V \cap W$: $g_{WU}g_{VW}g_{UV} = \text{id}$

- This defines a real vector bundle, and its rank is $n$; using $\mathbb{C}^n$ gives a complex vector bundle
  - Rank 1 vector bundles are called line bundles
Examples of vector bundles

- On any smooth manifold $M$, the tangent bundle $TM \to M$ and the cotangent bundle $T^*M \to M$
- Direct sums, tensor products, duals, etc. of vector bundles
  - but: not an abelian category
- The *pullback* of a vector bundle $V \to X$ by a continuous map $f: Y \to X$ is a vector bundle $f^*V \to Y$ whose fiber at $y \in Y$ is $V_{f(y)}$
- The *tautological bundle* $S \to \mathbb{R}P^n$ or $S \to \mathbb{C}P^n$, a real, resp. complex line bundle
  - A point $x$ of $\mathbb{R}P^n$ is a line in $\mathbb{R}^{n+1}$; the fiber of $S$ at $x$ is that line
  - Generalizes to Grassmannians and tautological vector bundles
Vector bundles: when are we gonna have to use this?

- Much of differential geometry is stated and proven in terms of vector bundles (and things called connections on them): $TM$ and $T^*M$, but also spinor bundles and the like.

- In general, vector bundles interpolate between geometry and homotopy theory.
  - They feel more like geometric objects (especially if you choose a connection).
  - …but their classification depends only on the homotopy type of $X$.

- Upshot: allows information from differential geometry to be used in homotopy theory and vice versa!
For the next few slides, $G$ is any topological group

A $G$-torsor is a space $X$ with a free transitive right $G$-action

Choosing a basepoint on $X$ provides an identification $X \cong G$ — but the point is, we (usually) have no canonical choice
$G$-torsors: examples

- Circles are $\text{SO}_2$-torsors, lines are $\mathbb{R}$-torsors
- Affine $n$-space $\mathbb{A}^n$ is an $\mathbb{R}^n$-torsor
- The set of bases of a vector space $V$ is a $\text{GL}(V)$-torsor
- The set of orientations on an orientable manifold $M$ is an $H^0(M; \mathbb{Z}/2)$-torsor
A principal $G$-bundle $P \to X$ is a continuously varying family of $G$-torsors over $X$

(so, local trivializations, continuous transition maps...)

$\implies$ the map $P \to X$ is the quotient map for the $G$-action
Examples of principal bundles

- On an n-manifold $M$, the frame bundle $\mathcal{B}_{\text{GL}}(M) \to M$ is the principal $\text{GL}_n(\mathbb{R})$-bundle whose fiber at $x$ is the $\text{GL}_n(\mathbb{R})$-torsor of bases of $T_xM$.
- The orientation bundle over a manifold $M$ has fiber at $x$ equal to the set of orientations of a small neighborhood of $x$.
  - A principal $\mathbb{Z}/2$-bundle
  - A trivialization is an orientation of $M$
- Unlike for vector bundles, principal $G$-bundles are nonlinear, so no duals, direct sums, etc.
- Like vector bundles, principal bundles pull back: given $P \to X$ and $f: Y \to X$, define $f^*P \to Y$ to have fiber at $y$ equal to the fiber of $P$ at $f(y)$.
The associated bundle construction

- Input data: $P \to X$ a principal $G$-bundle and $V$ a $G$-representation
- Output data: a vector bundle $P \times_G V \to X$, defined to be the quotient of $P \times V$ by the equivalence relation $(p \cdot g, v) \sim (p, g \cdot v)$
- Intuition: “using up” the $G$-actions on $P$ and $V$, or maybe using the ways in which $P$ is twisted to build a vector bundle twisted in the same ways
- Useful in geometry, where geometric aspects of a vector bundle you care about are secretly controlled by a principal $G$-bundle via this construction
Taking the associated bundle of a principal $O_n$-bundle and the standard representation of $O_n$ on $\mathbb{R}$ defines a bijection between isomorphism classes of principal $O_n$-bundles over a space and rank-$n$ real vector bundles over a space.

Likewise with $U_n$ and complex vector bundles; can instead use $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$ if you want.

More directly, the Gram-Schmidt algorithm defines a bijection between isomorphism classes of $GL_n(\mathbb{R})$-bundles and $O_n$-bundles (resp. $GL_n(\mathbb{C})$ and $U_n$).
Principal bundles: when are we gonna have to use this?

- Vector bundles of interest in differential geometry are all associated bundles for the bundle of frames: $TM$, $T^*M$, exterior powers; spinor bundles; and more
  - Often, one gets info on the bundle of frames, then uses the associated bundle construction to propagate that information to several vector bundles at once
- Useful for slickly defining orientations, spin structures, spin$^c$ structures, …
- Gauge theory is all about connections on principal bundles, both in math and in physics
Characteristic classes

- Fix some kind of vector or principal bundle (e.g. complex vector bundle; principal SU$_2$-bundle; etc.), a $d \in \mathbb{N}$, and a commutative ring $A$

- A characteristic class for these bundles is a procedure for associating to each bundle $E \to X$ a cohomology class $c(P) \in H^d(X;A)$ which is natural under pullback

- Naturality: given $f : Y \to X$, need $c(f^*E) = f^*c(E)$ in $H^d(Y;A)$
More algebraic invariants of geometric or topological information

Often detect or obstruct useful topological or geometric properties (orientability, flatness, null-bordism, …)

Sweet spot in “conservation of effort:” the best things in algebraic topology are both informative and computable
Approach 1: the axiomatic definition of Chern classes

- Chern classes are characteristic classes $c_i(V) \in H^{2i}(X; \mathbb{Z})$ for complex vector bundles $V \to X$, $i \geq 0$.
- Define them to satisfy a short list of axioms; it is a theorem of Grothendieck this uniquely characterizes them.
Approach 1: the axioms

- (implicit: naturality)
- $c_0(E) = 1$
- the Whitney sum formula $c(E \oplus F) = c(E)c(F)$
  - Here $c(E)$ is the total Chern class $c_0(E) + c_1(E) + c_2(E) + \ldots$
- Nontriviality: if $x$ is the generator of $H^2(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}$, then $c(S \to \mathbb{C}P^n) = 1 - x$.
  - Here $S \to \mathbb{C}P^n$ is the tautological line bundle
  - Use the orientation of $\mathbb{C}P^n$ to pick a specific isomorphism $H^2(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}$
Approach 1: advantages and disadvantages

▶ Succinct, but no intuition for what Chern classes are
▶ Can make some computations: if $E \oplus \mathbb{C}^k$ is trivial, $c(E) = 1$
  ▶ Also some calculations on projective complex manifolds
  ▶ … but pretty inflexible for computations
Approach 2: linear dependency of generic sections

- Idea: at least on a manifold, produce submanifolds from the vector bundle data, giving classes in homology
- Then use Poincaré duality to turn these into cohomology classes
- It is a theorem that every complex vector bundle on a finite-dimensional CW complex pulls back from a vector bundle on an oriented manifold, so this suffices
Approach 2: Poincaré duality is best theorem

- An orientation of a closed $n$-manifold $M$ determines a fundamental class, an element $[M] \in H_n(M; \mathbb{Z})$ (well, with any coefficients)
- Cap product with this class defines an isomorphism called Poincaré duality $H^k(M; \mathbb{Z}) \to H_{n-k}(M; \mathbb{Z})$
- For $k = n$ this corresponds to integration in de Rham theory
- With field coefficients, the universal coefficient theorem reformulates this as a duality pairing $H^k(M; k) \otimes_k H^{n-k}(M; k) \to k$
- If you only need $\mathbb{Z}/2$ coefficients, no orientation is necessary
Defining Chern classes using Poincaré duality

- Given a closed, oriented $n$-manifold $M$ and a complex vector bundle $V \to M$, choose $k$ generic sections $s_1, \ldots, s_k$ and let $N \subset M$ be the subset on which $s_1, \ldots, s_k$ are linearly dependent
  - e.g., if $k = 1$, $N$ is the zero set of $s_1$
- For a generic choice of $s_1, \ldots, s_k$, $N$ is a closed, oriented submanifold of dimension $n - 2k$. Push its fundamental class forward to define $[N] \in H_{n-2k}(M)$
  - “Generic” means suitable transversality hypotheses, etc., and is satisfied on a subset of full measure (as usual with such constructions in differential topology)
  - The homology class $[N]$ does not depend on any of the choices we made
- Now, the $k^{th}$ Chern class of $V$ is the Poincaré dual of $[N]$, which is a degree $n - (n - 2k) = 2k$ cohomology class
Approach 2: advantages and disadvantages

▶ **Yay:** makes clearer what Chern classes are measuring: an obstruction to linearly independent sections
▶ Non-functoriality of Poincaré duality means proving naturality is a headache
▶ That characteristic classes pull back from manifolds is not an obvious theorem
Approach 3: Chern-Weil theory

- Define Chern classes in de Rham theory, using concepts from differential geometry
- This is how characteristic classes tend to appear in quantum field theory
Approach 3: connections

Let $M$ be a smooth manifold and $V \to M$ be a real vector bundle. A connection on $V$ is an $\mathbb{R}$-linear map $
abla : \Gamma(TM) \otimes \mathbb{R} \Gamma(V) \to \Gamma(V)$ which is $C^\infty(M)$-linear in the first argument and satisfies the *Leibniz rule*

$$\nabla_v(f\psi) = (v \cdot f)\psi + f\nabla_v\psi$$

where $\nabla_v(\psi) := \nabla(v, \psi)$

Idea: this is a way to differentiate sections of $V$

Locally, $\nabla = d + A$, where $A$ is a “matrix-valued one-form,” namely an element of $\Omega^1_M(\text{End} V) := \Gamma(T^*M \otimes \text{End} V)$

Theorem: the space of connections on $V$ is an infinite-dimensional affine space, and in particular nonempty
The curvature of a connection is a matrix-valued 2-form $F_\nabla \in \Omega^2_M(\text{End } V) := \Gamma(\Lambda^2 T^* M \otimes \text{End } V)$ defined by the formula

$$\nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]}$$

Vector bundles are locally trivial, but connections are not, and the curvature measures this.
Approach 3: Chern classes in de Rham cohomology

- The trace map induces a map $\text{tr}: \Omega^k_M(\text{End} V) \to \Omega^k_M$
- Given a complex vector bundle $V \to M$, choose a connection $\nabla$ on $V$
- Wedge its curvature form together $k$ times to get $(F_\nabla)^k \in \Omega^k_M(\text{End} V)$
- Take the trace, land in $\Omega^{2k}_M$
- Theorem: this is a closed form, and its de Rham cohomology class does not depend on the choice of $\nabla$!
- Define the $k^{\text{th}}$ Chern class of $V$ to be $(1/2\pi i)^k[\text{tr}((F_\nabla)^k)] \in H^{2k}_{dR}(M)$
Approach 3: advantages and disadvantages

- If you like geometry or quantum field theory, you’ll probably like this approach the best
- Again, tells you something about what Chern classes actually are
- But doesn’t work to get classes in integral cohomology, nor on non-manifolds, and it depends on choices
- Sometimes computable. Sometimes not.
Approach 4: the search for the universal bundle

- Idea: create a moduli space $BG$ of principal $G$-bundles, carrying the “universal” or “maximally twisted” principal $G$-bundle $EG \to BG$

- Key fact: every principal $G$-bundle $P \to X$ is isomorphic to $f^*EG \to X$ for some $f : X \to BG$, and moreover in a unique way ($f$ is unique up to homotopy)

- Then, cohomology classes on $BG$ give characteristic classes, and vice versa
Let $G$ be a topological group. A *classifying space* for $G$, denoted $BG$, is any space which can be realized as the quotient of a contractible space $EG$ by a free $G$-action.

Key facts about classifying spaces:
- Homotopy classes of maps $f : X \to BG$ are in natural bijection with isomorphism classes of principal $G$-bundles $P \to X$ via $f \mapsto (f^*EG \to X)$.
- $BG$ always exists but is not unique! But any two choices are homotopy equivalent.
- $BG$ is often infinite-dimensional.
Approach 4: examples of classifying spaces

- $S^1$ is a $B\mathbb{Z}$, because $\mathbb{R}/\mathbb{Z} \cong S^1$
- $\mathbb{RP}^\infty$ is a $B\mathbb{Z}/2$, because $S^\infty$ is contractible!
  - Can take either $S^\infty := \text{colim}_n S^n$, or the unit sphere in an infinite-dimensional Banach space; this produces homotopy-equivalent but non-homeomorphic models for $B\mathbb{Z}/2$
  - Similarly, $\mathbb{CP}^\infty$ is a $B\mathbb{U}_1$
- Classifying spaces of Lie groups tend to look like infinite-dimensional Grassmannians
Approach 4: stabilization

- $H^*(B\text{GL}_n(\mathbb{R}))$ gives characteristic classes of principal $\text{GL}_n(\mathbb{R})$-bundles, hence real rank-$n$ bundles via the associated bundle construction. So how do we obtain characteristic classes for all real bundles at once?

- The inclusion $\text{GL}_n(\mathbb{R}) \to \text{GL}_{n+1}(\mathbb{R})$ induces maps $B\text{GL}_n(\mathbb{R}) \to B\text{GL}_{n+1}(\mathbb{R})$. Let $B\text{GL}(\mathbb{R})$ denote the colimit of these maps
  - From the moduli POV, $B\text{GL}_n(\mathbb{R}) \to B\text{GL}_{n+1}(\mathbb{R})$ sends a vector bundle $V \to X$ to $V \oplus \mathbb{R} \to X$

- Thus $H^*(B\text{GL}(\mathbb{R}))$ gives characteristic classes for all real vector bundles at once

- All this works for complex vector bundles as well
Approach 4: stabilization

- We’ve been using $\text{GL}_n(\mathbb{R})$, but it’s more common to see $\text{O}_n$ in the literature.
- There’s essentially no difference (you do have to choose a metric to get a principal $\text{O}_n$-bundle of frames, but this is a contractible choice so don’t worry about it).
- Likewise with $\text{GL}_n(\mathbb{C})$ and $\text{U}_n$. 
Approach 4: defining Chern classes

- There is a theorem that $H^*(BGL(\mathbb{C});\mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, c_3, \ldots]$ with $|c_i| = 2i$

- So we define the $i^{th}$ Chern class of a complex vector bundle $V \to X$ to be $f^*c_i$, where $f: X \to BGL(\mathbb{C})$ is in the homotopy class of maps classifying the stabilization of $E$
Approach 4: advantages and disadvantages

- It’s nice to know that we got ’em all (for any $G$, or for real or complex vector bundles)
- $BG$ is often useful for other reasons
- But, $BG$ is mysterious, and often big
- We’re also black-boxing how $H^*(BG)$ is actually computed
Foreshadowing the problem session

- Hands-on examples of vector bundles and principal bundles
- Seeing what we can compute with these different approaches
- A fifth perspective??