Day 4: Chern, Pontrjagin, and Euler classes

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Today’s plan

- Chern classes and a few of their basic properties and applications
- Pontrjagin classes and...
- The Euler class and...
- The splitting principle for principal $G$-bundles
This week’s plan

- **Monday**: four perspectives on characteristic classes
- **Tuesday**: Stiefel-Whitney classes (real vector bundles, mod 2 cohomology)
- **Yesterday**: Steenrod squares and Wu classes (more mod 2 cohomology)
- **Today**: Chern, Pontrjagin, and Euler classes (real and complex vector bundles, $\mathbb{Z}$ cohomology)
- **Tomorrow**: Chern-Weil theory (de Rham cohomology)
Chern classes: it’s déjà vu all over again

- Chern classes $c_i(E) \in H^{2i}(X; \mathbb{Z})$ are characteristic classes of complex vector bundles $E \to X$
- $c_i(E) = 0$ if $i > \text{rank}_\mathbb{C}(E)$
- Naturality, stability, Whitney sum formula as before
- $c_i(\overline{E}) = (-1)^i c_k(E)$ (note: $\overline{E} \cong E^*$)
- If $M$ is a stably almost complex manifold, $\langle c_{\text{top}}(M), [M] \rangle = \chi(M)$
- Reduction mod 2: $c_k \mapsto w_{2k}$, and if $E$ is complex, $w_{2k+1}(E) = 0$
- $c_1$ classifies complex line bundles, and every class in $H^2(X; \mathbb{Z})$ is $c_1$ of a line bundle
Defining a bordism theory of complex manifolds (or even almost complex manifolds) as before doesn’t work: an a.c. manifold can’t be the boundary of an a.c. manifold

So instead we use stably almost complex manifolds (put a complex structure on $TM \oplus \mathbb{R}^n$)

Milnor-Novikov computed $\Omega^U_* \cong \mathbb{Z}[x_2, x_4, \ldots]$, where $|x_{2i}| = 2i$

Two stably almost complex manifolds are bordant iff their Chern numbers agree

After tensoring with $\mathbb{Q}$, $\Omega^U_*$ is generated by complex projective spaces; this is not true integrally
Digression: the chromatic program

- There is an identification of $\Omega^U_*$ with something called Lazard’s ring of formal group laws.
- Roughly implies that if $E$ is a multiplicative generalized cohomology theory with a map $\Omega^U_* \to E$, the algebraic geometry of formal group laws can be used to study $E$.
- This turns out to be a significant organizing principle in stable homotopy theory, and a useful one.
- The field which studies this is called *chromatic homotopy theory*.
There isn’t a single characteristic class obstructing complex or almost complex structures

These questions can be difficult in general

But, stably almost complex $\iff$ has Chern classes

So we know $w_{2k+1}(E) = 0$ and $w_{2k}(E)$ is the reduction of an integral class

Looking at $k = 0$, we need orientability and spin$^c$ (in fact, a stably almost complex structure determines an orientation and a spin$^c$ structure)
Recall that the *complexification* of a real vector bundle $E \to X$ is $E_\mathbb{C} := E \otimes \mathbb{R} \mathbb{C}$, a complex vector bundle.

Define the $i$\textsuperscript{th} Pontrjagin class $p_i(E) := (-1)^i c_{2i}(E_\mathbb{C}) \in H^{4i}(X; \mathbb{Z})$ (note: index multiplied by 4!)

- Note: not everyone uses the same sign convention; be careful!

$p_i(TM)$ and the total Pontrjagin class defined as usual

$H^*(BO_n; \mathbb{Z}) \cong \mathbb{Z}[p_1, \ldots, p_n]$
Pontrjagin classes don’t satisfy the Whitney sum formula!

- See title
- However, the difference $p(E)p(E) - p(E \oplus F)$ is 2-torsion, so if you work over $\mathbb{Z}[1/2]$ (or over $\mathbb{Q}$, or $\mathbb{R}$), the Whitney sum formula holds
- Likewise if $H^*(X; \mathbb{Z})$ lacks 2-torsion, e.g. for $\mathbb{C}\mathbb{P}^n$
Pontrjagin numbers are oriented bordism invariants

- The answer is not as clean as for unoriented or complex bordism: $\Omega_*^{SO}$ contains 2-torsion, and its description is somewhat complicated (see Thom, Wall).
- $\Omega_*^{SO} \otimes \mathbb{Q} \cong \mathbb{Q}[x_4, x_8, \ldots]$, and $x_{4i} = [\mathbb{C}P^{2i}]$.
- Two oriented $n$-manifolds are oriented bordant iff their Pontrjagin and Stiefel-Whitney numbers agree.
- The lowest-degree torsion is $\Omega_5^{SO} \cong \mathbb{Z}/2$, generated by the Wu manifold $SU_3/SO_3$. 
Ultimately because $\text{Spin}_n \to \text{SO}_n$ is a double cover, the forgetful map $\Omega^\text{Spin}_* \to \Omega^\text{SO}_*$ is an isomorphism after tensoring with $\mathbb{Z}[1/2]$ (or $\mathbb{Q}$). In particular, $\Omega^\text{Spin}_* \otimes \mathbb{Q} \cong \mathbb{Z}[\tilde{x}_4, \tilde{x}_8, \ldots]$, but we no longer can take $\mathbb{CP}^{2i}$ as representatives. e.g. $\mathbb{CP}^2$ isn’t spin! $\Omega^\text{Spin}_4 \cong \mathbb{Z}$ is generated by the K3 surface. In fact, $\Omega^\text{Spin}_4 \cong \mathbb{Z} \to \Omega^\text{SO}_4 \cong \mathbb{Z}$ can be identified with $(\cdot 16) : \mathbb{Z} \to \mathbb{Z}$. 2-torsion also differs: $\Omega^\text{Spin}_1 \cong \mathbb{Z}/2$, for example. Anderson-Brown-Peterson show that characteristic classes don’t suffice for spin bordism unless one uses characteristic classes in a generalized cohomology theory called real $K$-theory.
The Euler class

- The Euler class $e(V) \in H^k(X : \mathbb{Z})$ is an unstable class for oriented real vector bundles (here, $k := \text{rank } V$).
- Arises because $H^*(BO_n) \to H^*(BSO_n)$ is not surjective.
- Or, can define it as the Poincaré dual to the homology class of the zero set of a generic section of $E$.
  - “generic” means “transverse to the zero section,” which exists by standard difftop arguments.
Properties of the Euler class

- Natural but *not stable*: \( e(V \oplus \mathbb{R}) = 0 \)
- Whitney sum formula: \( e(V_1 \oplus V_2) = e(V_1)e(V_2) \)
- If \( V \) has a nonvanishing section, \( e(V) = 0 \)
- Reversing orientation: \( e(-V) = -e(V) \)
- If \( M \) is a closed, oriented manifold, \( \langle e(TM), [M] \rangle = \chi(M) \) (hence the name)
- If \( k \) is odd, \( E \) is 2-torsion
Relations to other characteristic classes

- Reduction mod 2: $e(V) \mod 2 = w_k(V)$
- If $V$ is complex, $c_{\text{top}}(V) = e(V)$
- $e(V)^2 = e(V_\mathbb{C})$, so if $k$ is even, $e(E)^2 = p_{k/2}(E)$
The Gysin map

- Let \( \pi : E \to M \) be a fiber bundle, where \( M \) is an \( n \)-dimensional manifold, the fiber is \( k \)-dimensional, and \( M \) and \( E \) are oriented (or we use \( \mathbb{Z}/2 \) coefficients)

- For any \( j \), we have maps

\[
\cdots \to H^{k+j}(E) \xrightarrow{\text{PD}} H_{n-j}(E) \xrightarrow{\pi^*} H_{n-j}(M) \xrightarrow{\text{PD}} H^{j}(M)
\]

(where PD denotes Poincaré duality)

- The composition \( H^{k+j}(E) \to H^{j}(M) \) is called the Gysin map and denoted \( \pi! \)
  - many other fun names: surprise map, wrong-way map, Umkehr map, shriek map, pushforward map — indeed, a covariant map in cohomology??

- The Gysin map is not functorial (because Poincaré duality isn’t)

- In de Rham cohomology, this corresponds to “integration along the fiber”
Now specialize to $E \to M$ the unit sphere bundle of a rank-$k$ vector bundle $V \to M$ (need a metric on $V$ to define that, but not a problem).

With orientation or coefficients hypotheses as above, there is a long exact sequence called the Gysin sequence:

$$\cdots \to H^m(E) \xrightarrow{\pi_!} H^{m-k+1}(M) \xrightarrow{e(V)} H^{m+1}(M) \xrightarrow{\pi^*} H^{m+1}(E) \to \cdots$$

Behind the curtain: this is a special case of the Serre spectral sequence.
The splitting principle

- Idea: a way to describe characteristic classes of vector bundles in terms of characteristic classes of line bundles
- We’ll discuss a general approach in principal $G$-bundles due to Borel-Hirzebruch
- At the end, we’ll specialize to Chern, Pontrjagin, and Stiefel-Whitney classes
Maximal tori

- Recall that a compact, connected, abelian Lie group is isomorphic to $\mathbb{T}^n$ for some $n$
- A torus in a Lie group $G$ is a compact, connected, abelian Lie subgroup $T \subset G$
- A maximal torus is a torus not contained in a larger torus
- Theorem: maximal tori exist, and any two choices are conjugate
The generalized splitting principle: setup

- Fix our Lie group $G$ and a maximal torus $T$ of rank $n$ (i.e. $T \cong \mathbb{T}^n$)
- Via the inclusion $i: T \hookrightarrow G$, we have models of $BG$ as $EG/G$ and $BT$ as $EG/T$, so $Bi: BT \to BG$ is a fiber bundle with fiber $G/T$
  - (Note: $T$ is generally not normal in $G$)
- Let $P \to X$ be a principal $G$-bundle classified by a map $f_P: X \to BG$, and $q: Y \to X$ be the pullback of $Bi$:

\[
\begin{array}{c}
Y \xrightarrow{g} BT \\
\downarrow q \downarrow Bi \\
X \xrightarrow{f_P} BG.
\end{array}
\]
The generalized splitting principle

Here’s the diagram again:

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & BT \\
\downarrow{q} & & \downarrow{Bi} \\
X & \xrightarrow{f_P} & BG.
\end{array}
\]

Theorem, part 1: there is a canonical reduction of structure group of \( q^*P \to Y \) to \( T \)

Theorem, part 2: \( q^* : H^*(X; \mathbb{Q}) \to H^*(Y; \mathbb{Q}) \) is injective
So?

- Suppose $c$ is a characteristic class for principal $G$-bundles.
- Via $Bi$, it also defines a characteristic class for principal $T$-bundles.
- Since $q^*$ is injective, that characteristic class for the principal $T$-bundle we obtained over $Y$ determines $c(P)$.
- Since $T \cong \mathbb{T}^n$, that principal $T$-bundle decomposes (in a sense) as a product of $n$ principal $\mathbb{T}$-bundles.
- Therefore the characteristic class also factors as a product $\prod_{i=1}^{n} (1 + x_i)$, where the $x_i$ are the $c_1$s of the principal $\mathbb{T}$-bundle summands. The $x_i$ are called the roots of $P$. 
Very quick proof sketch

- For part 1 (reduction of structure group), this applies universally to reduce $(Bi)^*EG \to BT$ to $ET \to BT$; then pull back and use commutativity of the diagram.

- For part 2 (injectivity on cohomology): uses Borel’s computation of cohomology of $G/T$ and $BG$ and the Serre spectral sequence.

  - Key fact: $H^*(BG; \mathbb{Q})$ is a polynomial ring with even-degree generators, so the spectral sequence collapses.

  - In fact, you can replace $\mathbb{Q}$ with any ring $A$ such that if $H^*(G; \mathbb{Z})$ has $p$-torsion, then $p$ is a unit in $A$. 
Example: Chern classes

- The diagonal matrices are a maximal torus in $U_n$ of rank $n$
- Using associated bundles to pass between principal $U_n$-bundles and complex vector bundles, this tells us that a complex vector bundle $V \to X$ splits as a sum of line bundles $L_1, \ldots, L_n$ when pulled back to $Y$
- $c_1(L_i)$ is called the $i^{th}$ Chern root, and $c_k(V)$ is the $k^{th}$ symmetric polynomial in the Chern roots
- $H^*(U_n; \mathbb{Z})$ is free, so we can work over $\mathbb{Z}$
Y has a more concrete description in this case

Namely, the flag manifold for $V \to X$

A flag of an inner product space $W$ is a decomposition of $W$ as a sum of one-dimensional, orthogonal subspaces.

The flag manifold $Y \to X$ is a fiber bundle whose fiber at $x \in X$ is the space of flags of $V_x$

(Ok, you need a Hermitian metric to define this, but the isomorphism type of the fiber bundle does not depend on this choice)
Example: Pontrjagin classes, part 1

- $G = \text{SO}_{2n}$: one maximal torus is the diagonal matrices in $U_n \subset \text{SO}_n$, which has rank $n$

- Upshot: if $V$ is an oriented rank-2n vector bundle, $q^*V$ splits as a sum of complex line bundles $L_1, \ldots, L_n$, but the symmetric polynomial gets squared:

$$p_i(q^*V) = \sigma_i(c_1(L_1), \ldots, c_1(L_n))^2.$$

- This is because the Pontrjagin classes of $V$ are the Chern classes of $V_\mathbb{C}$, and the Chern roots of $V_\mathbb{C}$ come in pairs $\pm x_1, \ldots, \pm x_n$

- The Euler class also splits:

$$e(q^*V) = \sigma_n(c_1(L_1), \ldots, c_n(L_n))$$
Example: Pontrjagin classes, part 2

- $G = \text{SO}_{2n+1}$ has a similar story: one maximal torus is the diagonal matrices in $U_n \subset \text{SO}_{2n+1}$ (so the last diagonal entry is always 1)

- So an oriented rank-$(2n + 1)$ real vector bundle $V$, pulled back to $Y$, splits as a direct sum of $n$ complex line bundles and a trivial real line bundle

- The Pontrjagin and Euler classes of $V$ admit the same description as in the case $\text{SO}_{2n}$
Example (sort of): Stiefel-Whitney classes

- $O_n$ isn’t connected, so we can’t use the theorem
- Nonetheless, enough of the structure persists with $\mathbb{Z}/2$ coefficients and the subgroup $O_1^n \subset O_n$ to prove something via similar methods
- One can prove $q^*$ is an injection on mod 2 cohomology and $q^*P$ admits a canonical reduction of structure group to a principal $O_1^n$-bundle
- Upshot: a rank-$n$ real vector bundle $V$, after pullback to $Y$, splits as a direct sum of $n$ real line bundles $L_1, \ldots, L_n$, and

$$w_k(V) = \sigma_i(w_1(L_1), \ldots, w_1(L_n))$$