Day 4: Chern, Pontrjagin, and Euler classes

Arun Debray

July 9, 2020

Today's plan

- Chern classes and a few of their basic properties and applications
- Pontrjagin classes and…
- The Euler class and…
- The splitting principle for principal G-bundles

This week's plan

- Monday: four perspectives on characteristic classes
- Tuesday: Stiefel-Whitney classes (real vector bundles, mod 2 cohomology)
- Yesterday: Steenrod squares and Wu classes (more mod 2 cohomology)
- Today: Chern, Pontrjagin, and Euler classes (real and complex vector bundles, $\mathbb{Z}$ cohomology)
- Tomorrow: Chern-Weil theory (de Rham cohomology)

Chern classes: it's déjà vu all over again

- Chern classes $c_i(E) \in H^{2i}(X;\mathbb{Z})$ are characteristic classes of complex vector bundles $E \to X$
- $c_i(E) = 0$ if $i > \operatorname{rank}_{\mathbb{C}}(E)$
- Naturality, stability, Whitney sum formula as before
- $c_i(\overline{E}) = (-1)^i c_k(E)$ (note: $\overline{E} \cong E^*$)
- If M is a stably almost complex manifold, $\langle c_{\mathrm{top}}(M), [M]\rangle = \chi(M)$
- Reduction mod 2: $c_k \mapsto w_{2k}$, and if E is complex, $w_{2k+1}(E) = 0$
- c_1 classifies complex line bundles, and every class in $H^2(X;\mathbb{Z})$ is c_1 of a line bundle

Chern classes and bordism

- Defining a bordism theory of complex manifolds (or even almost complex manifolds) as before doesn't work: an a.c. manifold can't be the boundary of an a.c. manifold
- So instead we use stably almost complex manifolds (put a complex structure on $TM \oplus \underline{\mathbb{R}}^n$)
- Milnor-Novikov computed $\Omega_*^{\mathrm{U}} \cong \mathbb{Z}[x_2, x_4, \dots]$, where $|x_{2i}| = 2i$
- Two stably almost complex manifolds are bordant iff their Chern numbers agree
- After tensoring with $\mathbb{Q}$, Ω_*^{U} is generated by complex projective spaces; this is not true integrally

Digression: the chromatic program

- There is an identification of Ω_*^{U} with something called Lazard's ring of formal group laws.
- Roughly implies that if E is a multiplicative generalized cohomology theory with a map $\Omega_*^{\mathrm{U}} \to E$, the algebraic geometry of formal group laws can be used to study E
- This turns out to be a significant organizing principle in stable homotopy theory, and a useful one
- The field which studies this is called *chromatic homotopy theory*

Obstructing (stably) almost complex structures

- There isn't a single characteristic class obstructing complex or almost complex structures
- These questions can be difficult in general
- But, stably almost complex $\Leftrightarrow$ has Chern classes
- So we know $w_{2k+1}(E) = 0$ and $w_{2k}(E)$ is the reduction of an integral class
- Looking at $k = 0$, we need orientability and spin^c (in fact, a stably almost complex structure *determines* an orientation and a spin^c structure)

Pontrjagin classes

- Recall that the *complexification* of a real vector bundle $E \to X$ is $E_{\mathbb{C}} := E \otimes_{\mathbb{R}} \underline{\mathbb{C}}$, a complex vector bundle
- Define the i^{th} *Pontrjagin class* $p_i(E) := (-1)^i c_{2i}(E_{\mathbb{C}}) \in H^{4i}(X;\mathbb{Z})$ (note: index multiplied by 4!)
    - Note: not everyone uses the same sign convention; be careful!
- $p_i(TM)$ and the total Pontrjagin class defined as usual
- $H^*(BO_n;\mathbb{Z}) \cong \mathbb{Z}[p_1,\ldots,p_n]$

Pontrjagin classes don't satisfy the Whitney sum formula!

- See title
- However, the difference $p(E)p(E) - p(E \oplus F)$ is 2-torsion, so if you work over $\mathbb{Z}[1/2]$ (or over $\mathbb{Q}$, or $\mathbb{R}$), the Whitney sum formula holds
- Likewise if $H^*(X;\mathbb{Z})$ lacks 2-torsion, e.g. for $\mathbb{CP}^n$

Pontrjagin numbers are oriented bordism invariants

- The answer is not as clean as for unoriented or complex bordism: Ω_*^{SO} contains 2-torsion, and its description is somewhat complicated (see Thom, Wall)
- $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q} \cong \mathbb{Q}[x_4, x_8, \dots]$, and $x_{4i} = [\mathbb{CP}^{2i}]$
- Two oriented n-manifolds are oriented bordant iff their Pontrjagin and Stiefel-Whitney numbers agree
- The lowest-degree torsion is $\Omega_5^{\mathrm{SO}} \cong \mathbb{Z}/2$, generated by the *Wu manifold* $\mathrm{SU}_3/\mathrm{SO}_3$

Spin bordism

- Ultimately because $\mathrm{Spin}_n \to \mathrm{SO}_n$ is a double cover, the forgetful map $\Omega_*^{\mathrm{Spin}} \to \Omega_*^{\mathrm{SO}}$ is an isomorphism after tensoring with $\mathbb{Z}[1/2]$ (or $\mathbb{Q}$)
- In particular, $\Omega_*^{\mathrm{Spin}} \otimes \mathbb{Q} \cong \mathbb{Z}[\widetilde{x}_4, \widetilde{x}_8, \dots]$, but we no longer can take $\mathbb{CP}^{2i}$ as representatives
  - e.g. $\mathbb{CP}^2$ isn't spin! $\Omega_4^{\mathrm{Spin}} \cong \mathbb{Z}$ is generated by the K3 surface
  - In fact, $\Omega_4^{\mathrm{Spin}} \cong \mathbb{Z} \to \Omega_4^{\mathrm{SO}} \cong \mathbb{Z}$ can be identified with$(\cdot 16)\colon \mathbb{Z} \to \mathbb{Z}$
- 2-torsion also differs: $\Omega_1^{\mathrm{Spin}} \cong \mathbb{Z}/2$, for example
- Anderson-Brown-Peterson show that characteristic classes don't suffice for spin bordism unless one uses characteristic classes in a generalized cohomology theory called *real K-theory*

The Euler class

- The Euler class $e(V) \in H^k(X : \mathbb{Z})$ is an unstable class for oriented real vector bundles (here, $k := \operatorname{rank} V$)
- Arises because $H^*(BO_n) \to H^*(BSO_n)$ is not surjective
- Or, can define it as the Poincaré dual to the homology class of the zero set of a generic section of E
  - "generic" means "transverse to the zero section," which exists by standard difftop arguments

Properties of the Euler class

- Natural but *not stable*: $e(V \oplus \mathbb{R}) = 0$
- Whitney sum formula: $e(V_1 \oplus V_2) = e(V_1)e(V_2)$
- If V has a nonvanishing section, $e(V) = 0$
- Reversing orientation: $e(-V) = -e(V)$
- If M is a closed, oriented manifold, $\langle e(TM), [M] \rangle = \chi(M)$ (hence the name)
- If k is odd, E is 2-torsion

Relations to other characteristic classes

- Reduction mod 2: $e(V) \bmod 2 = w_k(V)$
- If V is complex, $c_{\mathrm{top}}(V) = e(V)$
- $e(V)^2 = e(V_{\mathbb{C}})$, so if k is even, $e(E)^2 = p_{k/2}(E)$

The Gysin map

- Let $\pi\colon E \to M$ be a fiber bundle, where M is an n-dimensional manifold, the fiber is k-dimensional, and M and E are oriented (or we use $\mathbb{Z}/2$ coefficients)
- For any j, we have maps

$$\cdots \longrightarrow H^{k+j}(E) \xrightarrow{\text{PD}} H_{n-j}(E) \xrightarrow{\pi_*} H_{n-j}(M) \xrightarrow{\text{PD}} H^j(M)$$

  (where PD denotes Poincaré duality)
- The composition $H^{k+j}(E) \to H^j(M)$ is called the *Gysin map* and denoted $\pi_!$
  - many other fun names: surprise map, wrong-way map, Umkehr map, shriek map, pushforward map — indeed, a covariant map in cohomology??
- The Gysin map is not functorial (because Poincaré duality isn't)
- In de Rham cohomology, this corresponds to "integration along the fiber"

Euler classes and the Gysin sequence

- Now specialize to $E \to M$ the unit sphere bundle of a rank-k vector bundle $V \to M$ (need a metric on V to define that, but not a problem)
- With orientation or coefficients hypotheses as above, there is a long exact sequence called the *Gysin sequence*

$$\cdots \longrightarrow H^m(E) \xrightarrow{\pi_!} H^{m-k+1}(M) \xrightarrow{\cdot e(V)} H^{m+1}(M) \xrightarrow{\pi^*} H^{m+1}(E) \longrightarrow$$

- Behind the curtain: this is a special case of the Serre spectral sequence

The splitting principle

- Idea: a way to describe characteristic classes of vector bundles in terms of characteristic classes of line bundles
- We'll discuss a general approach in principal G-bundles due to Borel-Hirzebruch
- At the end, we'll specialize to Chern, Pontrjagin, and Stiefel-Whitney classes

Maximal tori

- Recall that a compact, connected, abelian Lie group is isomorphic to $\mathbb{T}^n$ for some n
- A *torus* in a Lie group G is a compact, connected, abelian Lie subgroup $T \subset G$
- A *maximal torus* is a torus not contained in a larger torus
- Theorem: maximal tori exist, and any two choices are conjugate

The generalized splitting principle: setup

- Fix our Lie group G and a maximal torus T of rank n (i.e. $T \cong \mathbb{T}^n$)
- Via the inclusion $i\colon T \hookrightarrow G$, we have models of BG as EG/G and BT as EG/T, so $Bi\colon BT \to BG$ is a fiber bundle with fiber G/T
  - (Note: T is generally not normal in G)
- Let $P \to X$ be a principal G-bundle classified by a map $f_P\colon X \to BG$, and $q\colon Y \to X$ be the pullback of Bi:

$$
\begin{array}{ccc}
Y & \xrightarrow{g} & BT \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle Bi} \\
X & \xrightarrow{f_P} & BG.
\end{array}
$$

The generalized splitting principle

- Here's the diagram again:

$$\begin{CD} Y @>g>> BT \\ @VqVV @VVBiV \\ X @>f_P>> BG. \end{CD}$$

- Theorem, part 1: there is a canonical reduction of structure group of $q^*P \to Y$ to T
- Theorem, part 2: $q^* : H^*(X;\mathbb{Q}) \to H^*(Y;\mathbb{Q})$ is injective

So?

- Suppose c is a characteristic class for principal G-bundles
- Via Bi, it also defines a characteristic class for principal T-bundles
- Since q^* is injective, that characteristic class for the principal T-bundle we obtained over Y determines $c(P)$
- Since $T \cong \mathbb{T}^n$, that principal T-bundle decomposes (in a sense) as a product of n principal $\mathbb{T}$-bundles
- Therefore the characteristic class also factors as a product $\prod_{i=1}^{n}(1+x_i)$, where the x_i are the c_1s of the principal $\mathbb{T}$-bundle summands. The x_i are called the *roots* of P

Very quick proof sketch

- For part 1 (reduction of structure group), this applies universally to reduce $(Bi)^*EG \to BT$ to $ET \to BT$; then pull back and use commutativity of the diagram
- For part 2 (injectivity on cohomology): uses Borel's computation of cohomology of G/T and BG and the Serre spectral sequence
  - Key fact: $H^*(BG; \mathbb{Q})$ is a polynomial ring with even-degree generators, so the spectral sequence collapses
  - In fact, you can replace $\mathbb{Q}$ with any ring A such that if $H_*(G; \mathbb{Z})$ has p-torsion, then p is a unit in A

Example: Chern classes

- The diagonal matrices are a maximal torus in U_n of rank n
- Using associated bundles to pass between principal U_n-bundles and complex vector bundles, this tells us that a complex vector bundle $V \to X$ splits as a sum of line bundles $L_1, \ldots, L_n$ when pulled back to Y
- $c_1(L_i)$ is called the i^{th} *Chern root*, and $c_k(V)$ is the k^{th} symmetric polynomial in the Chern roots
- $H^*(\mathrm{U}_n; \mathbb{Z})$ is free, so we can work over $\mathbb{Z}$

Chern classes and the flag manifold

- Y has a more concrete description in this case
- Namely, the *flag manifold* for $V \to X$
- A *flag* of an inner product space W is a decomposition of W as a sum of one-dimensional, orthogonal subspaces
- The flag manifold $Y \to X$ is a fiber bundle whose fiber at $x \in X$ is the space of flags of V_x
  - (Ok, you need a Hermitian metric to define this, but the isomorphism type of the fiber bundle does not depend on this choice)

Example: Pontrjagin classes, part 1

- $G = \mathrm{SO}_{2n}$: one maximal torus is the diagonal matrices in $\mathrm{U}_n \subset \mathrm{SO}_n$, which has rank n
- Upshot: if V is an oriented rank-$2n$ vector bundle, q^*V splits as a sum of complex line bundles $L_1, \ldots, L_n$, but the symmetric polynomial gets squared:

$$p_i(q^*V) = \sigma_i(c_1(L_1), \ldots, c_1(L_n))^2.$$

- This is because the Pontrjagin classes of V are the Chern classes of $V_{\mathbb{C}}$, and the Chern roots of $V_{\mathbb{C}}$ come in pairs $\pm x_1, \ldots, \pm x_n$
- The Euler class also splits:

$$e(q^*V) = \sigma_n(c_1(L_1), \ldots, c_n(L_n))$$

Example: Pontrjagin classes, part 2

- $G = \mathrm{SO}_{2n+1}$ has a similar story: one maximal torus is the diagonal matrices in $\mathrm{U}_n \subset \mathrm{SO}_{2n+1}$ (so the last diagonal entry is always 1)
- So an oriented rank-$(2n+1)$ real vector bundle V, pulled back to Y, splits as a direct sum of n complex line bundles and a trivial real line bundle
- The Pontrjagin and Euler classes of V admit the same description as in the case SO_{2n}

Example (sort of): Stiefel-Whitney classes

- O_n isn't connected, so we can't use the theorem
- Nonetheless, enough of the structure persists with $\mathbb{Z}/2$ coefficients and the subgroup $O_1^n \subset O_n$ to prove something via similar methods
- One can prove q^* is an injection on mod 2 cohomology and q^*P admits a canonical reduction of structure group to a principal O_1^n-bundle
- Upshot: a rank-n real vector bundle V, after pullback to Y, splits as a direct sum of n real line bundles $L_1, \ldots, L_n$, and

$$w_k(V) = \sigma_i(w_1(L_1), \ldots, w_1(L_n))$$