Day 2: Stiefel-Whitney classes

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Today's plan

- Stiefel-Whitney classes: definition and a few basic properties
- An application: tangential structures (e.g. orientation, spin, ...)
- An application: unoriented bordism
- An application: nonembedding results

This week's plan

- Yesterday: four perspectives on characteristic classes
- Today: Stiefel-Whitney classes (real vector bundles, mod 2 cohomology)
- Tomorrow: Steenrod squares and Wu classes (more mod 2 cohomology)
- Thursday: Chern, Pontrjagin, and Euler classes (real and complex vector bundles, $\mathbb{Z}$ cohomology)
- Friday: Chern-Weil theory (de Rham cohomology)

A definition of Stiefel-Whitney classes

- Stiefel-Whitney classes are characteristic classes of real vector bundles living in mod 2 cohomology
- Theorem: $H^*(BO; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2, \dots]$ with $|w_i| = i$, so for every real vector bundle $V \to X$, we obtain its i^{th} *Stiefel-Whitney class* $w_i(V) \in H^i(X; \mathbb{Z}/2)$
- Define $w(V) := w_0(V) + w_1(V) + w_2(V) + \dots$ (here $w_0(V) := 1$)
- If M is a manifold, its Steifel-Whitney classes means the Stiefel-Whitney classes of $TM \to M$

Some basic properties of Stiefel-Whitney classes

- Naturality (of course)
- The *Whitney sum formula* $w(V \oplus W) = w(V)w(W)$, or equivalently,

$$w_n(V \oplus W) = \sum_{i+j=n} w_i(V) w_j(W) \tag{1}$$

- If $S \to \mathbb{RP}^n$ denotes the tautological line bundle, then $w_1(S)$ is the nontrivial element of $H^1(\mathbb{RP}^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$
- Stability: $w(V \oplus \underline{\mathbb{R}}) = w(V)$
- If $n > \operatorname{rank}(V)$, then $w_n(V) = 0$
  - More generally, if V has ℓ everywhere linearly independent sections, $w_n(V) = 0$ for all $n > \operatorname{rank} V - \ell$

Tangential structures

- Tangential structures provide a general framework for studying orientations, spin structures, etc.
- Works for “topological” structures, not “geometric” ones (e.g. metric, connection, complex structure, symplectic structure, contact structure)

Reduction of structure groups

- We need a homomorphism $f\colon G \to H$ of topological groups
- Given f, and a principal G-bundle $P \to X$, there is an associated principal H-bundle $P \times_G H \to X$ defined in the same way as associated vector bundles
- Now, given a principal H-bundle $Q \to X$, a *reduction of structure group to G* is data of
  1. a principal G-bundle $P \to X$, and
  2. an isomorphism $Q \xrightarrow{\cong} P \times_G H$ of principal H-bundles

Orientations and the bundle of frames

- The principal $\mathrm{GL}_n(\mathbb{R})$-bundle of frames admits a "canonical" reduction of structure group to the principal O_n-bundle of orthonormal frames: the reduction of structure group is asking, which bases are orthonormal?
  - given by the data of a Euclidean metric on the bundle, but this is a contractible choice, hence "canonical"
- An orientation is the data of declaring which half of the bases of T_xM are positively oriented, consistently over the entire space
  - The space of positively oriented orthonormal bases of an oriented inner product space V is an $\mathrm{SO}(V)$-torsor
  - So an orientation tells us compatible SO_n-torsors inside $\mathscr{B}_{\mathrm{O}}(V) \to X$
- That is: *an orientation of a vector bundle is precisely a reduction of structure group from* O_n *to* SO_n

Tangential structures

- Tangential structures take that idea and run with it
- Given a group G_n and a homomorphism $\rho_n\colon G_n \to \mathrm{O}_n$, a *G-structure* on a vector bundle is a reduction of structure group across ρ_n
  - This is data, and need not exist or be unique (like for orientations)
- A G-structure on a manifold M is a G-structure on TM
- Examples: orientation, *spin structure* $\rho_n\colon \mathrm{Spin}_n \overset{2:1}{\twoheadrightarrow} \mathrm{SO}_n \hookrightarrow \mathrm{O}_n$, spin^c structure, …
- Note: for $\mathrm{U}_{n/2}$ or $\mathrm{Sp}_{n/4}$ this only gives almost complex, resp. almost symplectic structures

Alternative ways to think about tangential structures

- Let $\mathfrak{U}$ be an open cover of X with respect to which V is trivial, so we obtain transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{R})$ for $U_\alpha, U_\beta \in \mathfrak{U}$
- A G-structure is a (homotopy class of) lifts of $g_{\alpha\beta}$ across $G_n \to \mathrm{GL}_n(\mathbb{R})$:

$$\begin{array}{ccc} & & G_n \\ & \overset{\widetilde{g}_{\alpha\beta}}{\nearrow} & \big\downarrow \rho_n \\ V & \xrightarrow[g_{\alpha\beta}]{} & \mathrm{GL}_n(\mathbb{R}) \end{array} \tag{2}$$

The lifts still must satisfy the cocycle condition $\widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma}\widetilde{g}_{\gamma\alpha} = 1$

Alternative ways to think about tangential structures

- The map $\rho_n : G_n \to \mathrm{GL}_n(\mathbb{R})$ lifts to a map $B\rho_n : BG \to B\mathrm{GL}_n(\mathbb{R})$, and we could ask for a (homotopy class of a) lift of the classifying map $X \to B\mathrm{GL}_n(\mathbb{R})$ across $B\rho_n$
- All three of these perspectives are really the same thing
- One can stabilize and ask about a lift of the map $X \to B\mathrm{GL}_n(\mathbb{R})$ across a map $B\rho : BG \to B\mathrm{GL}(\mathbb{R})$, giving *stable tangential structures*
  - For many things (orientation, spin structure) this isn't different
  - For some (e.g. framing), though, it is!

Wait… weren't we talking about characteristic classes?

- Often, a given kind of tangential structure exists on a vector bundle $V \to X$ iff some characteristic class of V vanishes
- Example: V is orientable iff $w_1(V) = 0$. If V is orientable, its orientations are an $H^0(X;\mathbb{Z}/2)$-torsor
- V admits a spin structure iff $w_1(V) = 0$ and $w_2(V) = 0$. If V is spinnable, its spin structures are an $H^1(X;\mathbb{Z}/2)$-torsor

Bordism

- Two closed n-manifolds M and N are *bordant* if there is a compact $(n+1)$-dimensional manifold X with an identification $\partial X \cong M \oplus N$
- E.g. the pair of pants is a bordism from $S^1 \amalg S^1$ to S^1
- Disjoint union turns the set of equivalence classes of n-manifolds under bordism into an abelian group, denoted Ω_n^{O} and called the n^{th} *bordism group*. The empty set is the identity
- Cartesian product turns $\Omega_*^{\mathrm{O}} := \bigoplus_{n\geq 0} \Omega_n^{\mathrm{O}}$ into a graded commutative ring
- Variants Ω_*^{SO} (oriented bordism), spin bordism, ...

Bordism groups

- A *bordism invariant* generally refers to a group homomorphism $\Omega_n^{\mathrm{O}} \to A$ or a ring homomorphism $\Omega_*^{\mathrm{O}} \to A$
- Example: one can prove that if $[M] = 0$ in Ω_n^{O} (so, M bounds an $(n+1)$-manifold), then $\chi(M)$ is even, so $\chi(M) \bmod 2 \colon \Omega_n^{\mathrm{O}} \to \mathbb{Z}/2$ is a bordism invariant

Stiefel-Whitney numbers

- Why we care: Stiefel-Whitney classes can be used to make bordism invariants
- Let $i_1 + \cdots + i_k = n$ be a partition of n, and define the *Stiefel-Whitney number* of a closed n-manifold M as

$$w_{i_1 \cdots i_k}(M) := \langle w_{i_1}(M) \cdots w_{i_k}(M), [M] \rangle, \tag{3}$$

  where $[M] \in H_n(M; \mathbb{Z}/2)$ is the fundamental class and $\langle -, - \rangle$ is the cap product pairing
- These are all bordism invariants

Determination of the bordism ring

- Thom completely determined Ω_*^{O}:

$$\Omega_*^{\mathrm{O}} \cong \mathbb{Z}/2[x_i \mid i \neq 2^j - 1] = \mathbb{Z}/2[x_2, x_4, x_5, x_6, \dots]$$

- Some but not all x_i can be realized by $\mathbb{RP}^i$
- Two closed n-manifolds are bordant iff all of their Stiefel-Whitney numbers agree
- In general, Stiefel-Whitney numbers are "too much" information: some coincide. We'll see some tomorrow

Stiefel-Whitney classes obstruct immersions

- If $N \hookrightarrow M$ is an immersion, there is a short exact sequence $0 \to TN \to TM|_N \to \nu \to 0$, where ν is the normal bundle, and this noncanonically splits
- This, together with information about the Stiefel-Whitney classes of M, imposes constraints on the Stiefel-Whitney classes of N

Example: $\mathbb{RP}^9$ does not immerse into $\mathbb{R}^{14}$

- In the exercises, you'll show $w(\mathbb{RP}^n) = (1+x)^{n+1}$, where x generates $H^1(\mathbb{RP}^n; \mathbb{Z}/2)$. For $n = 9$, this is $1 + x^2 + x^8$
- $w(\mathbb{R}^n) = 1$, because $T\mathbb{R}^n$ is a trivial bundle
- Therefore if there is such an immersion, $(1 + x^2 + x^8)w(\nu) = 1$ — solving, $w(\nu) = 1 + x^2 + x^4 + x^6$
- But $\operatorname{rank}(\nu) = 5$, so w_6 must vanish
- This is a simple test for nonimmersions, but usually provides non-sharp bounds

A few more useful Stiefel-Whitney facts

- For all vector bundles $V \to X$, $w_1(V) = w_1(\operatorname{Det} V)$ (the determinant is defined to be the top exterior power)
- Line bundles are classified up to isomorphism by w_1, and $w_1(L_1 \otimes L_2) = w_1(L_1) + w_1(L_2)$
- The top Stiefel-Whitney number of M, $\langle w_n(M), [M] \rangle$, equals its mod 2 Euler characteristic