Day 3: Stable cohomology operations and Wu classes

Arun Debray

July 8, 2020

Today's plan

- Steenrod squares are stable cohomology operations (also, what are stable cohomology operations?)
- Using Steenrod squares to define Wu classes
- Using Wu classes to do cool stuff

This week's plan

- Monday: four perspectives on characteristic classes
- Yesterday: Stiefel-Whitney classes (real vector bundles, mod 2 cohomology)
- Today: Steenrod squares and Wu classes (more mod 2 cohomology)
- Tomorrow: Chern, Pontrjagin, and Euler classes (real and complex vector bundles, $\mathbb{Z}$ cohomology)
- Friday: Chern-Weil theory (de Rham cohomology)

Cohomology operations

- Goal: use additional structure on $H^*(-;\mathbb{Z}/2)$ to deduce properties of characteristic classes
- These come in the form of *cohomology operations*, namely natural transformations $H^p(-;A) \to H^q(-;B)$
- Example: $x \mapsto x^2$
- Example: the *Pontrjagin square* $\mathscr{P} : H^2(X;\mathbb{Z}/2) \to H^4(X;\mathbb{Z}/4)$ (idea: if you know x mod 2, you know x^2 mod 4)

Stable cohomology operations

- Apparently the algebraic structure of all unstable cohomology operations is complicated and messy
- Simplifies considerably by stabilizing
- A cohomology operation is *stable* if it commutes with the suspension isomorphism $H^k(X) \to H^{k+1}(\Sigma X)$
- All of the examples on the previous slide were unstable!

Examples of stable cohomology operations

- Given a short exact sequence of abelian groups $0 \to A \to B \to C \to 0$, there is a long exact sequence in cohomology, and all maps in the long exact sequence are stable cohomology operations
- The connecting map $\beta : H^k(X; C) \to H^{k+1}(X; A)$, called the *Bockstein map*, is most interesting
- Other examples tend to be harder to define. Some general results:
  - Over $\mathbb{Q}$, stable cohomology operations are trivial (i.e. only scalar multiplication)
  - All stable cohomology operations from $\mathbb{Z}$ to $\mathbb{Z}$ are torsion: reduction mod n, some operation in $\mathbb{Z}/n$ cohomology, then Bockstein $H^k(X; \mathbb{Z}/n) \to H^{k+1}(X; \mathbb{Z})$
  - With $\mathbb{Z}/n$ coefficients, the story is more interesting

The (mod 2) Steenrod algebra

- The set of stable cohomology operations $H^*(X;\mathbb{Z}/2) \to H^{*+k}(X;\mathbb{Z}/2)$ forms a graded $\mathbb{Z}/2$-algebra under composition, denoted $\mathscr{A}$ and called the *Steenrod algebra*
- Generated by *Steenrod squares* $\mathrm{Sq}^n : H^*(X;\mathbb{Z}/2) \to H^{*+n}(X;\mathbb{Z}/2)$, $n \geq 0$, satisfying some axioms and relations
  - For $n > 1$, definition is a bit technical

Axiomatic definition of Steenrod squares

- (implicit: group homomorphism, naturality, stability)
- $\mathrm{Sq}^0 = \mathrm{id}$ and Sq^1 is the Bockstein for $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$
- If $|x| = n$, $\mathrm{Sq}^n(x) = x^2$
- If $|x| < n$, $\mathrm{Sq}^n(x) = 0$
- The *Cartan formula*:

$$\mathrm{Sq}^n(xy) = \sum_{i+j=n} \mathrm{Sq}^i(x)\mathrm{Sq}^j(y),$$

  or, if $\mathrm{Sq}(x) := \mathrm{Sq}^0(x) + \mathrm{Sq}^1(x) + \cdots$, then $\mathrm{Sq}(xy) = \mathrm{Sq}(x)\mathrm{Sq}(y)$
- Theorem: these properties uniquely characterize the Steenrod squares and their action on mod 2 cohomology of spaces

Ádem relations

- $\mathscr{A}$ is not free (in fact generated by Sq^{2^n} for all n):

$$\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{k=0}^{\lfloor i/2 \rfloor} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k$$

- Summary: for any space X, $H^*(X;\mathbb{Z}/2)$ is an $\mathscr{A}$-module (*not* an $\mathscr{A}$-algebra), and pullback is always an $\mathscr{A}$-module homomorphism

Examples of Steenrod squares acting on cohomology

- $H^*(\mathbb{RP}^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^{n+1})$, with $|x| = 1$
- The axioms imply that for any degree-1 class, $\mathrm{Sq}(x) = x + x^2$
- The Cartan formula then tells us the rest: $\mathrm{Sq}(x^k) = (x + x^2)^k$
- Similar reasoning applies for $\mathbb{CP}^n$, $\mathbb{HP}^n$, and $n = \infty$

Steenrod squares + Poincaré duality = Wu classes

- One consequence of Poincaré duality is that $H^k(X;\mathbb{Z}/2)$ and $H^{n-k}(X;\mathbb{Z}/2)$ are canonically dual $\mathbb{Z}/2$-vector spaces for any closed n-manifold X
- That is, a functional $H^{n-k}(X;\mathbb{Z}/2) \to \mathbb{Z}/2$ is identified with an element of $H^k(X;\mathbb{Z}/2)$
- Let's apply that to $x \mapsto \langle \mathrm{Sq}^k(x), [M] \rangle$
  - Sq^k takes us to $H^n(X;\mathbb{Z}/2)$
  - Then evaluate on the fundamental class
- This determines an element $v_k \in H^k(X;\mathbb{Z}/2)$ called the k^{th} *Wu class* of X

The Wu formula

- Wu Wenjun proved that $\mathrm{Sq}(v) = w$
  - Here $v := v_0 + v_1 + \dots$ as usual, and as usual, $v_0 = 1$
- There is also the *Wu formula*:

$$\mathrm{Sq}^i(w_k) = \sum_{j=0}^{i} \binom{k+j-i-1}{j} w_{i-j} w_{k+j}$$

- Can interpret as telling you $\mathscr{A}$-module structure on $H^*(BO_n; \mathbb{Z}/2)$, or as universal relations between Stiefel-Whitney classes
  - Brown-Peterson proved all universal relations between Stiefel-Whitney classes arise this way

The Wu formula: quick corollaries

- Stiefel-Whitney classes of manifolds are homotopy invariants
- Homotopy equivalent closed manifolds are unoriented bordant
  - "unoriented" is important: counterexamples in, e.g. pin^+ bordism
- Can define Stiefel-Whitney classes for closed topological manifolds

The Wu formula imposes constraints on Stiefel-Whitney classes

- Example: if M is a closed manifold of dimension at most 3, then $w_1^2 = w_2$
  - Hence M orientable $\Rightarrow$ M spin
  - In general $w_1^2 = w_2$ means M admits a pin$^-$ structure
- Proof: $w_1 = \mathrm{Sq}^0 v_1 + \mathrm{Sq}^1 v_0 = v_1$
- $w_2 = \mathrm{Sq}^2 v_0 + \mathrm{Sq}^1 v_1 + \mathrm{Sq}^0 v_2 = v_1^2 + v_2$
- But $v_2 = 0$, because $\mathrm{Sq}^2 \colon H^{n-2}(X; \mathbb{Z}/2) \to H^n(X; \mathbb{Z}/2)$ vanishes when $n \leq 3$
- Hence $w_2 = w_1^2$

The Wu formula and the intersection pairing on 4-manifolds

- Theorem: if M is a simply connected closed 4-manifold, M is spin iff its intersection pairing $H^2(M) \otimes H^2(M) \to \mathbb{Z}$ is even
- Since M is simply connected, $H^1(M; \mathbb{Z}/2) = 0$, so the kernel of mod 2 reduction $H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{Z}/2)$ is precisely $2H^2(M; \mathbb{Z})$
  - Why? Use the long exact sequence in cohomology associated to $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$
  - Upshot: we can check in $H^*(M; \mathbb{Z}/2)$, because if $x \in H^2(M; \mathbb{Z})$ dies mod 2, it's twice something, so its intersection pairing with anything is twice something

The Wu formula and the intersection pairing on 4-manifolds

- Now we want to prove for all $a \in H^2(M;\mathbb{Z}/2)$, $\langle a^2, [M]\rangle = 0$
- Well, $a^2 = \mathrm{Sq}^2 a = v_2 a = w_2 a$
  - Here we use orientability: in general, $v_2 = w_2 + w_1^2$
- Since cup-product-then-evaluate is a perfect pairing $H^2(M;\mathbb{Z}/2) \otimes H^2(M;\mathbb{Z}/2) \to \mathbb{Z}/2$, $w_2 = 0$ iff $\langle w_2 a, [M]\rangle = 0$ for all $a \in H^2(M;\mathbb{Z}/2)$
- Note: some hypothesis on π_1 is necessary: otherwise the *Enriques surface*, with $\pi_1 \cong \mathbb{Z}/2$, is a counterexample
- Note: nothing about this needed smoothness! In particular, applies to the E_8-manifold, a topological spin 4-manifold with intersection form the E_8 lattice

Spin^c structures

- The Lie group $\mathrm{Spin}_n^c := \mathrm{Spin}_n \times_{\{\pm 1\}} \mathrm{U}_1$
- The map $\mathrm{Spin}_n \to \mathrm{SO}_n$ extends to Spin_n^c, so we can consider spin^c structures on oriented manifolds
- It turns out that M (assumed oriented) admits a spin^c structure iff $w_2(M)$ is the reduction of a class $c_1 \in H^2(M;\mathbb{Z})$
- e.g. $\mathbb{CP}^2$ is not spin, but it is spin^c (as are all almost complex manifolds)

Spin^c structures

- Let $\beta : H^k(X;\mathbb{Z}/2) \to H^{k+1}(X;\mathbb{Z})$ denote the Bockstein for $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$
- Then, M admits a spin^c structure iff $\beta w_2 = 0$
  - The classes $W_{n+1} := \beta w_n$ are sometimes called "integral Stiefel-Whitney classes"
- Note: compatibility of the above short exact sequence with $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$ means that β composed with reduction mod 2 is Sq^1
  - This is sometimes called the "Bock-to-Sq^1 lemma"

Foreshadowing the exercises

- Computing some Steenrod squares
- Using Steenrod squares to compute some Stiefel-Whitney classes
- Applying the Wu formula
- Čech cohomology and spin^c structures