EXERCISES

ARUN DEBRAY
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These problems aren't essential for this week's material, but could be helpful or at least interesting to think about. Do not feel the need to go in order or to do all of the problems, and pick the questions that look most interesting to you.

(1) (a) Read the full definitions of symmetric monoidal category and symmetric monoidal functor; it will be clear why most introductions to TFT do not give the whole definition!
 (b) In this problem, we define symmetric monoidal category $s\mathcal{V}ect_{\mathbb{C}}$ of "super-vector spaces." As a category, this is the same as the category of $\mathbb{Z}/2$-graded vector spaces, where morphisms must preserve the grading; the symmetric monoidal structure implements the Koszul sign rule
$$a \otimes b = (-1)^{\deg(a)\deg(b)} b \otimes a$$
where a and b are homogeneous elements. Define this symmetric monoidal category. (Hint: everything except the symmetry is the same as for vector spaces. Don't spend way too much time verifying all the details of the coherence conditions.)

(2) (a) If X is a space, functions $X \to \mathbb{C}$ have a "pointwise multiplication" $(fg)(x) \coloneqq f(x)g(x)$. In the same way, define a "pointwise tensor product" structure on TFTs (where the dimension and tangential structure are fixed).
 (b) This does not work for direct sums, because not all closed manifolds are connected. However, we can define a direct sum operation on TFTs (tangential structure and dimension are fixed), where $(Z_1 \oplus Z_2)(X) = Z_1(X) \oplus Z_2(X)$ when X is a connected object or morphism; on disconnected manifolds, use the symmetric monoidal requirement to define $Z_1 \oplus Z_2$, and check that it defines a TFT.

(3) (a) Compute Ω_0^{O}, Ω_1^{O}, Ω_0^{SO}, Ω_1^{SO}, and Ω_2^{SO}. (Ω_k^{O}, resp. Ω_k^{SO}, denotes the group of bordism classes of unoriented, resp. oriented, k-manifolds.)
 (b) Show that $\Omega_2^{\mathrm{O}} \cong \mathbb{Z}/2$, generated by $\mathbb{RP}^2$.
 (c) If you have seen characteristic classes before, show that $\int_M p_1(M)$ defines a bordism invariant $\Omega_4^{\mathrm{SO}} \to \mathbb{Z}$. Use this to show that $\mathbb{CP}^2$ doesn't bound, and moreover that $\mathbb{CP}^2$ admits no orientation-reversing diffeomorphism!

(4) Classify 1-dimensional oriented TFTs (1-dimensional means the bordisms are 1-dimensional.)