The finite path integral

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Outline

1. Quick review of what we need from Tuesday's lecture
2. The finite path integral as modeling things in physics
3. Defining the finite path integral
4. Examples (Dijkgraaf-Witten, Yetter, Quinn's finite homotopy TFT, ...)

Goals today

- Construct interesting and nontrivial examples whose partition functions and state spaces are not too hard to calculate, and which (unlike yesterday) doesn't require much homotopy theory to digest

Review

- If $\varphi : \Omega_n^H \to \mathbb{C}^\times$ is a bordism invariant, it defines an invertible TFT $Z_\varphi : \mathcal{B}ord_n^H \to s\mathcal{V}ect_{\mathbb{C}}$ with partition function φ
- Good examples of bordism invariants: integrate characteristic classes or natural cohomology classes for manifolds with H-structure

Path integral quantization

- Consider a gauge theory in physics
- This means: one of the fields is a principal G-bundle and connection Θ
- The classical theory is defined using a *Lagrangian action* $S\colon F \to \mathbb{R}$ (F the space of fields)
- The system evolves along extremal trajectories under S

Path integral quantization

- One computes the partition function of the quantum theory by exponentiating the action and integrating over the space of fields:
$$Z = \int_F e^{-S} \, \mathrm{d}\varphi$$
- Problem: F is typically infinite-dimensional, hence such a measure cannot exist (e.g. if the fields include connections on principal G-bundles, for G positive-dimensional)
- Today is about a setting in which this *does* exist, and can be used to give more examples of topological field theories

Making this rigorous: take G to be a finite group!

- When G is a finite group, there is a procedure "summing over the space of principal G-bundles which takes a TFT $Z^{c\ell}\colon \mathcal{B}ord_n^{\xi\times G} \to \mathcal{V}ect_{\mathbb{C}}$ and produces a new TFT $Z\colon \mathcal{B}ord_n^{\xi} \to \mathcal{V}ect_{\mathbb{C}}$
- Due to Freed-Quinn, Freed-Hopkins-Lurie-Teleman, Morton, Trova, Schweigert-Woike
- So examples of $Z^{c\ell}$ (e.g. invertible TFTs, e.g. by bordism invariants) give new examples of TFTs
- Notable examples: Dijkgraaf-Witten theories, indexed by $\alpha \in H^n(BG;\mathbb{R}/\mathbb{Z})$ (bordism invariant: exponentiate the "classical action" $\int \alpha(P)$)

Somewhat more general example

- Let X be a space with *finite total homotopy*: $\pi_i(X)$ is finite, and is zero for all but finitely many i
- Then, there is a finite path integral summing over maps to X: go from TFTs of ξ-manifolds with a map to X, to TFTs of ξ-manifolds
- Recovers TFTs such as *Quinn's finite homotopy TFT* (again, $Z^{c\ell}$ defined by integrating cohomology classes of X) and the *Yetter model* (X has only two nonzero homotopy groups)
- Can be interpreted as summing over principal bundles for a finite higher group…
- Can also sum over things like spin structures

Sketch of the construction

- We will give the construction in detail for summing over principal G-bundles, then discuss the general case more quickly and sketchily
- Use $Z^{c\ell}$ to build a functor $\mathcal{B}ord_n^{\xi} \to \mathcal{C}orr$, a category of spans of groupoids equipped with vector bundles
  - Idea: send M to the space (groupoid) of fields on M; bordisms induce correspondences of this data
  - The vector bundle comes from applying $Z^{c\ell}$ at each point of this space

Sketch of the construction

- Then, quantize: define a functor $\mathcal{C}orr \to \mathcal{V}ect_{\mathbb{C}}$ by taking sections of the vector bundle
- For correspondences, we need a pushforward map, which is "sum over the fibers"
- This crucially uses that G is finite, so that this sum is finite

Vector bundles over groupoids

- A *vector bundle over a groupoid* X is a functor $V : X \to \mathcal{V}ect_{\mathbb{C}}$. A line bundle is an invertible vector bundle (so it's valued in $\mathcal{V}ect_{\mathbb{C}}^{\times}$)
- That is: to every object x we assign a vector space V_x (the fiber) and to every morphism $x \to y$ we assign a linear map $V_x \to V_y$ (parallel transport)
- The *space of sections* of a vector bundle is its colimit
- For a line bundle, the space of sections is the free vector space on the subset of $\pi_0(X)$ such that the parallel transport maps for automorphisms act by the identity

Building the category of correspondences

- The objects of $\mathcal{C}orr$ are groupoids with vector bundles
- We require these groupoids are *finite*, meaning $\pi_0(X)$ is finite and all automorphism groups of objects of X are finite
- A morphism is a correspondence or span:

$$\begin{tikzcd} & Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X_1 & & X_2, \end{tikzcd}$$

  with vector bundles $V_i \to X_i$, together with data of an element of $\mathrm{Hom}(\Gamma(V_1), \Gamma(V_2))$
- (Well, we need to take isomorphism classes of such data, so that composition is associative on the nose)

Building the category of correspondences

- The identity is the correspondence $X \leftarrow X \rightarrow X$, with the vector bundle maps all equal to id
- Composition: given two correspondences, take the pullback

The first functor $\mathcal{B}ord_n^{\xi} \to \mathcal{C}orr$

- We start with $Z^{c\ell} : \mathcal{B}ord_n^{\xi \times G} \to \mathcal{V}ect_{\mathbb{C}}$
- So, given an $(n-1)$-manifold M, assign the groupoid $\mathcal{B}un_G(M)$ (finite because G is) with the vector bundle $P \mapsto Z^{c\ell}(M,P)$
- A bordism $X : M_0 \to M_1$ induces maps of manifolds $i_0 : M_0 \hookrightarrow X$ and $i_1 : X \hookleftarrow M_1$, hence pullback maps of principal bundles, giving a correspondence
- The element of $\mathrm{Hom}(\Gamma(V_1), \Gamma(V_2))$ that we choose is the one coming from applying $Z^{c\ell}$ to the bordism

The second functor: quantization $\mathcal{C}orr \to \mathcal{V}ect_{\mathbb{C}}$

- On objects, send $(X, V) \mapsto \Gamma(V)$
- Given a correspondence (morphism), act by the element of $\mathrm{Hom}(\Gamma(V_1), \Gamma(V_2))$

Partition functions for the quantum theory

$$Z(M) = \sum_{P \in \pi_0(\mathcal{B}un_G(M))} \frac{Z^{c\ell}(M,P)}{\#\,\mathrm{Aut}(P)}$$

i.e. integrate the function $P \mapsto Z^{c\ell}(M,P)$ in the “groupoid measure”

State spaces for the quantum theory

- The state space on N is the space of sections of a vector bundle on $\mathcal{B}un_G(N)$
- The fiber at $P \to M$ is $Z^{c\ell}(N, P)$, and the parallel transport for $\varphi \in \mathrm{Aut}(P)$ is $Z^{c\ell}$ of $M \times S^1$ with the mapping torus of P
- So the state space is free on the set of isomorphism classes of principal G-bundles for which these parallel transport maps are all trivial

What changes for the finite homotopy TFT?

- Groupoids are replaced with the space of maps $\mathrm{Map}(M,X)$
- Now we need to use the fact that $Z^{c\ell}(M,-)$ defines a vector bundle with connection over $\mathrm{Map}(M,X)$, where M is a closed $(n-1)$-manifold
  - Why? It suffices to know the parallel transports along paths in $\mathrm{Map}(M,X)$; a path gives a bordism of manifolds with a map to X, and $Z^{c\ell}$ turns that into a linear map, which is the parallel transport map
- Partition functions use the n-groupoid cardinality

$$\sum_{f\in\pi_0(\mathrm{Map}(M,X))}\prod_{k=1}^{n}\#\pi_k(\mathrm{Map}(M,X),f)\cdot Z^{c\ell}(M,f)$$

Examples

- Choose $\alpha \in H^n(BG; \mathbb{R}/\mathbb{Z})$ and let $Z^{c\ell}$ be the bordism invariant defined by
$$(M, P) \mapsto \exp(2\pi i \int_M \alpha(P)).$$
Perform the finite path integral over principal G-bundles to obtain *Dijkgraaf-Witten theory*
- Replace BG with a space of finite X total homotopy and obtain *Quinn's finite homotopy TFT*
- If X has only two nonzero homotopy groups, this is also called the *Yetter model*

Bosonization and fermionization

- The Jordan-Wigner transform is a tool in the statistical mechanics of 1d systems: a formal change of variables from a bosonic system with a $\mathbb{Z}/2$ symmetry to a fermionic system
- Using the tools we've built so far, we can produce an analogue of this transform between 2d spin TFTs and 2d $\mathrm{SO} \times \mathbb{Z}/2$ TFTs
- This has various features of a Fourier transform

The Jordan-Wigner kernel

- First: recall that spin structures inducing a chosen orientation are an $H^1(-;\mathbb{Z}/2)$-torsor, and in fact given a spin structure s and a principal $\mathbb{Z}/2$-bundle P, there is a way to "tensor them together" into a new spin structure $s+P$
- Second: recall that there is an isomorphism $\mathrm{Arf}\colon \Omega_2^{\mathrm{Spin}} \to \{\pm 1\}$ given by the *Arf invariant*
- Third: recall that a bordism invariant lifts to a invertible TFT valued in $s\mathcal{V}ect_{\mathbb{C}}$
- So we define an invertible TFT $\alpha_{\mathrm{JW}}\colon \mathcal{B}ord_2^{\mathrm{Spin}\times\mathbb{Z}/2} \to s\mathcal{V}ect_{\mathbb{C}}$, called the *Jordan-Wigner kernel*, to lift the bordism invariant

$$(\Sigma, s, P) \mapsto \mathrm{Arf}(s+P)$$

Defining bosonization and fermionization

- Given a spin TFT $Z_f \colon \mathcal{B}ord_2^{\mathrm{Spin}} \to s\mathcal{V}ect_{\mathbb{C}}$, define its *bosonization* $Z_b \colon \mathcal{B}ord_2^{\mathrm{SO}\times\mathbb{Z}/2} \to s\mathcal{V}ect_{\mathbb{C}}$ as follows: tensor with α_{JW}, then perform the finite path integral over spin structures
- Conversely, given an $\mathrm{SO}\times\mathbb{Z}/2$ TFT Z_b, defines its *fermionization* by tensoring with α_{JW}, then performing the finite path integral over principal $\mathbb{Z}/2$-bundles
- These are *not quite inverses* — doing one, then the other, amounts to tensoring with an Euler theory
- This Euler theory is like the factor of 2π in the Fourier transform: harmless, and you can sweep it under the rug, but you cannot make it go away

Some interesting features

- The usual tensor product on TFTs (“pointwise multiplication”) is exchanged with a convolution-like operation
- If Z_f doesn’t depend on the spin structure (i.e. is really an oriented TFT), Z_b doesn’t depend on the principal $\mathbb{Z}/2$-bundle, and $Z_f \cong Z_b$; the vice versa statement is also true
- It is possible to soup this up to extended TFTs valued in the Morita 2-category of $\mathbb{C}$-superalgebras
- Also, α_{JW} extends to an invertible theory of pin$^-$ manifolds with a principal $\mathbb{Z}/2$-bundle, setting up a bosonization/fermionization duality between $\mathrm{O} \times \mathbb{Z}/2$ TFTs and pin$^-$ TFTs

Direct sums of TFTs

- In addition to the pointwise tensor product, there is a direct sum operation on TFTs
- If M is connected, $(Z_1 \oplus Z_2)(M) := Z_1(M) \oplus Z_2(M)$
- In order for symmetric monoidality to hold, must be different in general! On a disconnected manifold, do this on all connected components, then tensor those things together
- Likewise for bordisms: it's what you would call $\oplus$ on a connected bordism, and in general your hand is forced by symmetric monoidality

Direct sum as a finite path integral

- The space $\{1, 2\}$ certainly has finite total homotopy
- Given Z_1 and Z_2, build a TFT $Z_{1,2} \colon \mathcal{B}ord_n^{1,2} \to \mathcal{V}ect_{\mathbb{C}}$ as follows: the function to $\{1, 2\}$ is locally constant, so wherever it's equal to 1, assign $Z_1(–)$, and where it's equal to 2, assign $Z_2(–)$
- Then check that this is actually a TFT
- Now perform the finite path integral over maps to $\{1, 2\}$, and you get $Z_1 \oplus Z_2$
- Easier to generalize (e.g. to the extended or derived setting) than the by hand definition

Gauging and ungauging

- The finite path integral can be interpreted as gauging a G-symmetry
- If $G = A$ is finite abelian, you can “ungauge” using another finite path integral and end up back with the original theory!
- Similar Fourier-theoretic description as bosonization/fermionization, but now one side is $\mathrm{SO} \times \mathbb{Z}/2$ and the other is $\mathrm{SO} \times K(A^\vee, n-1)$
- On one side, a principal A-bundle; on the other, a “higher $A^\vee$-bundle” (representative of a degree $n-1$ $A^\vee$-valued cohomology class)
- $A^\vee = \mathrm{Hom}(A, \mathbb{C})$ (the character dual)