Hurwitz numbers and topological field theory

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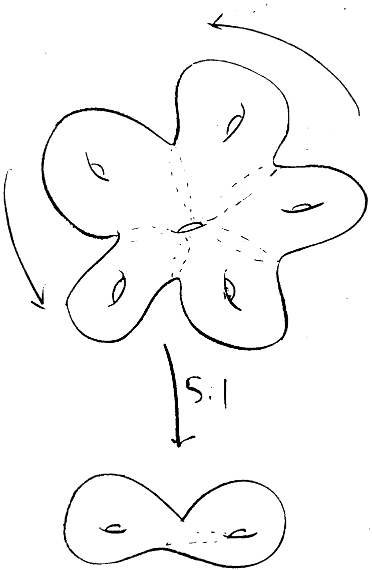
Outline

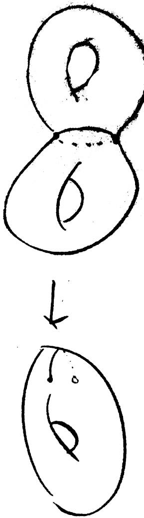
1. Setting up the counting question
2. Reduction to 2d TFT
3. Solving the theory

Classification of surfaces: morphisms

- We know the classification of closed, oriented surfaces, so it seems reasonable to next classify maps between them, up to some notion of equivalence
- Degree theory tells us that for maps $\pi\colon \Sigma' \to \Sigma$ of closed, oriented surfaces, only two things can happen:
  1. Degree zero: $\mathrm{Im}(\pi)$ is a finite set (throw out)
  2. Degree n: away from a finite set, π is an $|n|$-fold covering map
- In the second case, π is called a *branched cover*.

Unbranched and branched covers

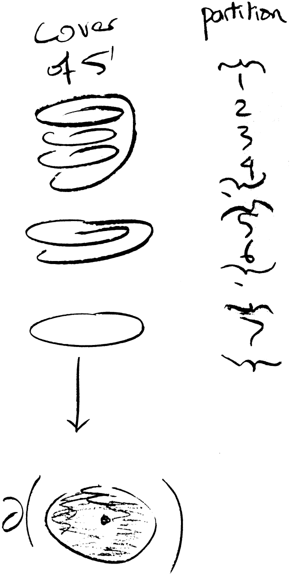




Classifying branched covers

- Our goal is to compute the number of branched covers, up to isomorphism of covers
- Fix additional data to obtain a finite number
  - The degree n of the cover
  - The number of *branch points* k (points where π is not an actual cover)
  - The *ramification data* around each point: in a circle of radius ε around each branch point, we have an n-fold cover of S^1. Which one?
- An n-fold cover of S^1 is specified by a partition of n

Ramification data



Hurwitz numbers

- Fixing $g := g(\Sigma)$, the degree n, the number of branch points k, and the ramification data $p_1, \ldots, p_k$, define the *Hurwitz number*
$$\mathscr{H}_{g,n,p_1,\ldots,p_k} := \sum_{\Sigma' \to \Sigma} \frac{1}{\mathrm{Aut}(\Sigma' \to \Sigma)}.$$
- That is: count the number of isomorphism classes of covers with that data, but weighted inversely by the number of automorphisms
- Fancy, succinct way to say this: there is a groupoid of branched covers with that data, and we are computing the *groupoid cardinality*
- Our goal: compute Hurwitz numbers using TFT

The lay of the land

- Hurwitz set and solved this problem over a century ago, using representation theory
- More recently, it was realized that his arguments could be recast in TFT
- The TFT approach generalizes better:
  - Spin Hurwitz numbers (Gunningham), solving a problem in Gromov-Witten theory
  - Pin^- Hurwitz numbers? r-spin Hurwitz numbers?

Problem-solving strategy

1. First, convert from a question about covers to a question about principal S_n-bundles
2. Then, express that question in terms of finite gauge theory for S_n
3. 2d fully extended TFTs are “solved” (value on manifolds is understood in terms of algebraic data)
4. Import that algebraic data and conclude!

Exchanging covers and principal S_n-bundles

- There is a bijective correspondence between rank-n vector bundles and principal $\mathrm{GL}_n(\mathbb{R})$-bundles
- Given a vector bundle $V \to M$, take the *frame bundle* $B(V) \to M$: the fiber at $x \in M$ is the $\mathrm{GL}_n(\mathbb{R})$-torsor of bases of V_x
- Given a principal $\mathrm{GL}_n(\mathbb{R})$-bundle $P \to M$, take the associated vector bundle

$$P \times_{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}^n := P \times \mathbb{R}^n / (p \cdot g, v) \sim (p, g \cdot v)$$

- Likewise, rank-n complex vector bundles and principal $\mathrm{GL}_n(\mathbb{C})$-bundles; oriented real vector bundles with a metric and principal SO_n-bundles, etc.

Exchanging covers and principal S_n-bundles

- What we need is a discrete analogue of that correspondence: a bijective correspondence between n-fold covers of a space (not necessarily connected) and principal S_n-bundles
- Given an n-fold cover $M' \to M$, at $x \in M$, the fiber M'_x has an S_n-torsor of total orderings; do this for all $x \in M$ and obtain a principal S_n-bundle $B(M') \to M$
- In the other direction, take the associated covering to a principal S_n-bundle $P \to M$:

$$P \times_{S_n} \{1, \ldots, n\} := P \times \{1, \ldots, n\} / (p \cdot \sigma, m) \sim (p, \sigma \cdot m)$$

- This defines an equivalence of groupoids from degree-n covers (*without branching*) to principal S_n-bundles

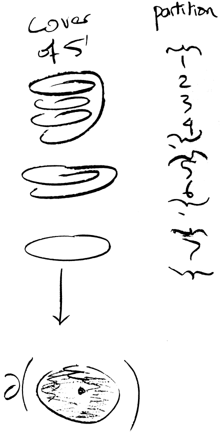
Allowing branching/singularities

- Let $Y \subset X$, and $Y' \to Y$ be an n-sheeted cover. The groupoid $\mathscr{C}ov_n^{Y'}(X,Y)$ of *relative n-sheeted covers* has objects n-sheeted covers $X' \to X$ together with data of an isomorphism $\varphi : X'|_Y \xrightarrow{\cong} Y'$
- Morphisms in this category are isomorphisms of covers which commute with these maps φ
- In the same way, you can define a *relative principal S_n-bundle* given a specific principal S_n-bundle $Q \to Y$, and obtain a groupoid $\mathscr{B}un_{S_n}^{Q}(X,Y)$
- The equivalence on the last slide extends to an equivalence of groupoids

$$\mathscr{C}ov_n^{Y'}(X,Y) \xrightarrow{\simeq} \mathscr{B}un_{S_n}^{Y' \times_{S_n} \{1,\dots,n\}}(X,Y).$$

From ramification data to relative principal S_n-bundles

- A branched cover $\Sigma' \to \Sigma$ branched at $x_1, \dots, x_k$ is equivalent to a genuine n-sheeted cover $\Sigma'' \to \Sigma \setminus \bigcup_i B_\varepsilon(x_i)$, together with the ramification data around each branch point



Finite gauge theory

- On Wednesday, we used the finite path integral to "sum over principal G-bundles" for G finite
- Given a TFT of $\mathrm{SO} \times G$-manifolds, obtain a TFT of oriented manifolds
- Finite gauge theory Z_G: do this to the trivial theory
- This counts principal G-bundles

Rephrasing the counting question in terms of finite gauge theory

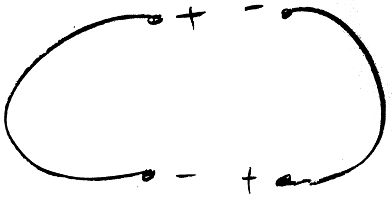
- When there are zero branch points, the correspondence between n-sheeted coverings and principal S_n-bundles means we want to compute the partition function $Z_{S_n}(\Sigma_g)$
- When there are n branch points, we're looking at Σ_g minus n small discs: call this $\Sigma_{g,n}$
  - The ramification data defines an element of $Z(\partial\Sigma_{g,n})$: $Z_{S_n}(S^1)$ is free on $\pi_0(\mathscr{B}un_{S_n}(S^1))$, so we choose the δ-function at our ramification data
  - If you stare at the push-pull formula, it's telling you that the Hurwitz number for this ramification data is the linear map $Z(\Sigma_{g,n})\colon Z(\partial\Sigma_{g,n}) \to \mathbb{C}$ applied to our δ-function
- So all the Hurwitz numbers are contained in this series of TFTs

Solving 2d TFTs

- Folk theorem: 2d oriented TFTs are classified by commutative Frobenius algebras
- Fancier theorem: 2d fully extended TFTs $\mathscr{B}ord_2^{\mathrm{SO}} \to \mathscr{A}lg_{\mathbb{C}}$ are classified by semisimple Frobenius algebras
  - Fully dualizable implies semisimplicity
  - The Frobenius condition gets us from framed TFTs to oriented TFTs

From semisimple Frobenius algebras to 2d TFTs

- $Z(S^1) \cong A \otimes_{A\otimes A^{\mathrm{op}}} A$, which is the center of A



$${}_{\mathbb{C}}A_{A\otimes A^{op}} \otimes {}_{A\otimes A^{op}}A_{\mathbb{C}}$$

$$\emptyset \longrightarrow pt_+ \sqcup pt_- \longrightarrow \emptyset$$

From semisimple Frobenius algebras to 2d TFTs

- Let $e_1, \ldots, e_k$ be the primitive idempotents of the center of A
- These are the elements such that $a^2 = a$, and which are not further factorizable into sums of other idempotents
- The incoming disc sends $1 \mapsto \sum e_i$
- The outgoing disc is the counit λ (the adjoint of the unit map $\mathbb{C} \to A$ under the inner product)
- Outgoing pair-of-pants is multiplication, and incoming pair-of-pants is sent to $A \to A \otimes A$ given by

$$e_i \mapsto \frac{1}{\lambda(e_i)} e_i \otimes e_i$$

From semisimple Frobenius algebras to 2d TFTs

- Hence, if Σ_g is a closed, connected, oriented surface of genus g,

$$Z(\Sigma_g) = \sum_{i=1}^{k} \lambda(e_k)^{1-g}$$

  Prove by chopping Σ_g into a sequence of pairs of pants, and an incoming and outgoing disc

Finite gauge theory in this perspective

- The semisimple Frobenius algebra is the group algebra $\mathbb{C}[G]$
- $Z_G(S^1) = \mathbb{C}[G]^G$, the ring of characters, under pointwise multiplication
- Inner product:

$$\langle \varphi, \psi \rangle = \frac{1}{\#G} \sum_{g \in G} \varphi(g)\psi(g^{-1})$$

Importing facts from the representation theory of S_n

- Conjugacy classes of S_n, as well as isomorphism classes of irreducible S_n-representations, are indexed by partitions of n
- For V an irreducible representation of S_n,

$$e_V = \frac{\dim V}{n!} \sum_{\sigma \in S_n} \chi_V(\sigma)\sigma$$

and therefore

$$\lambda(e_V) = \left(\frac{\dim V}{n!}\right)^2$$

(because $e_V = e_V^2$, so $\lambda(e_V) = \langle e_V, e_V \rangle$)

Importing facts from the representation theory of S_n

- We also need a “change of basis” formula
- We want to evaluate Z_{S_n} on a surface with boundary with a principal S_n-bundle; the restriction to a boundary S^1 is equivalent data to a conjugacy class C
- We want to understand $\delta_C \in \mathbb{C}[S_n]^{S_n}$ in terms of the primitive idempotents e_V

$$\delta_C = |C| \sum_V \frac{\chi_V(C)}{\dim V} e_V$$

Importing facts from the representation theory of S_n

- The number of elements of S_n in the conjugacy class C_P given by a partition P is

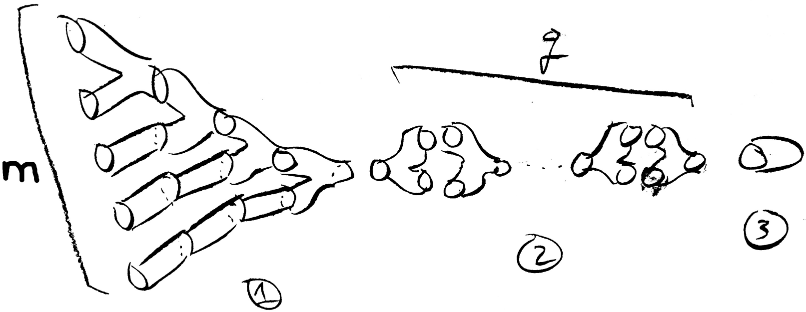
$$|C_P| = \frac{n!}{\prod_{k=1}^{n} k^{p_k} p_k!}$$

- If V is the representation corresponding to P, $|p| = m$, $\ell_i = p_i + m - i$, and $\Delta = \prod_{i<j}(x_i - x_j)$, then

$$\dim V = \frac{n!}{\ell_1! \cdots \ell_m!} \Delta(\ell_1, \ldots, \ell_m)$$

Computing Hurwitz numbers

- $\Sigma_{g,m}$ decomposes as (1) a bunch of outgoing pairs of pants, followed by (2) incoming then outgoing pairs of pants, then (3) a disc



- (1) is multiplication and (3) $\circ$ (2) sends $e_V \mapsto \lambda(e_V)^{1-g}$

Computing Hurwitz numbers

- Using the change-of-basis formula, (1) sends

$$\begin{aligned} \delta_{p_1} \otimes \cdots \otimes \delta_{p_m} \mapsto & \delta_{p_1} \cdots \delta_{p_m} \\ &= \prod_{j=1}^{m} |C_{p_j}| \sum_{V} \frac{\chi_V(p_j)}{\dim V} e_V \\ &= \sum_{V} \prod_{j=1}^{m} \left(\frac{|C_{p_j}| \chi_V(p_j)}{\dim V} \right) e_V \end{aligned}$$

Computing Hurwitz numbers

- Now we need to send $e_V \mapsto \lambda(e_V)^{1-g} = (\dim V/n!)^{1-g}$:

$$\sum_V \prod_{j=1}^{m} \left(\frac{|C_{p_j}| \chi_V(p_j)}{\dim V} \right) \left(\frac{\dim V}{n!} \right)^2$$

  This is Burnside's formula for Hurwitz numbers!

- We stated purely combinatorial formulas for $|C_{p_j}|$ and $\dim V$. Using them (and simplifying), we obtain

$$\mathscr{H}_{g,n,p_1,\ldots,p_n} = \sum_q \left(\frac{\Delta(\ell_1, \ldots, \ell_m)}{\ell_1! \cdots \ell_m!|} \right)^{2-2g-m} \prod_{j=1}^{m} \chi_q(C_{p_j}) \prod_{k=1}^{n} \frac{1}{k^{(p_j)_k} (p_j)_k!}$$

  This is almost completely combinatorial: $\chi_q(C_{p_j})$ can be determined with the Frobenius character formula, but is inexplicit and complicated