1. Setting up the counting question
2. Reduction to 2d TFT
3. Solving the theory
We know the classification of closed, oriented surfaces, so it seems reasonable to next classify maps between them, up to some notion of equivalence.

Degree theory tells us that for maps \( \pi : \Sigma' \to \Sigma \) of closed, oriented surfaces, only two things can happen:

1. Degree zero: \( \text{Im}(\pi) \) is a finite set (throw out)
2. Degree \( n \): away from a finite set, \( \pi \) is an \( |n| \)-fold covering map

In the second case, \( \pi \) is called a \textit{branched cover}. 

\[\]
Unbranched and branched covers
Our goal is to compute the number of branched covers, up to isomorphism of covers.

Fix additional data to obtain a finite number:

- The degree $n$ of the cover.
- The number of branch points $k$ (points where $\pi$ is not an actual cover).
- The ramification data around each point: in a circle of radius $\epsilon$ around each branch point, we have an $n$-fold cover of $S^1$. Which one?

An $n$-fold cover of $S^1$ is specified by a partition of $n$. 
Ramification data
Hurwitz numbers

Fixing $g := g(\Sigma)$, the degree $n$, the number of branch points $k$, and the ramification data $p_1, \ldots, p_k$, define the Hurwitz number

$$\mathcal{H}_{g,n,p_1,\ldots,p_k} := \sum_{\Sigma' \to \Sigma} \frac{1}{\text{Aut}(\Sigma' \to \Sigma)}.$$ 

That is: count the number of isomorphism classes of covers with that data, but weighted inversely by the number of automorphisms.

Fancy, succinct way to say this: there is a groupoid of branched covers with that data, and we are computing the groupoid cardinality.

Our goal: compute Hurwitz numbers using TFT.
The lay of the land

- Hurwitz set and solved this problem over a century ago, using representation theory
- More recently, it was realized that his arguments could be recast in TFT
- The TFT approach generalizes better:
  - Spin Hurwitz numbers (Gunningham), solving a problem in Gromov-Witten theory
  - Pin⁻ Hurwitz numbers? r-spin Hurwitz numbers?
Problem-solving strategy

1. First, convert from a question about covers to a question about principal $S_n$-bundles
2. Then, express that question in terms of finite gauge theory for $S_n$
3. 2d fully extended TFTs are “solved” (value on manifolds is understood in terms of algebraic data)
4. Import that algebraic data and conclude!
Exchanging covers and principal $S_n$-bundles

- There is a bijective correspondence between rank-$n$ vector bundles and principal $\text{GL}_n(\mathbb{R})$-bundles.

- Given a vector bundle $V \to M$, take the frame bundle $B(V) \to M$: the fiber at $x \in M$ is the $\text{GL}_n(\mathbb{R})$-torsor of bases of $V_x$.

- Given a principal $\text{GL}_n(\mathbb{R})$-bundle $P \to M$, take the associated vector bundle

  $$P \times_{\text{GL}_n(\mathbb{R})} \mathbb{R}^n := P \times \mathbb{R}^n / (p \cdot g, v) \sim (p, g \cdot v)$$

- Likewise, rank-$n$ complex vector bundles and principal $\text{GL}_n(\mathbb{C})$-bundles; oriented real vector bundles with a metric and principal $\text{SO}_n$-bundles, etc.
What we need is a discrete analogue of that correspondence: a bijective correspondence between $n$-fold covers of a space (not necessarily connected) and principal $S_n$-bundles.

Given an $n$-fold cover $M' \to M$, at $x \in M$, the fiber $M'_x$ has an $S_n$-torsor of total orderings; do this for all $x \in M$ and obtain a principal $S_n$-bundle $B(M') \to M$.

In the other direction, take the associated covering to a principal $S_n$-bundle $P \to M$:

$$P \times_{S_n} \{1, \ldots, n\} := P \times \{1, \ldots, n\}/(p \cdot \sigma, m) \sim (p, \sigma \cdot m)$$

This defines an equivalence of groupoids from degree-$n$ covers (without branching) to principal $S_n$-bundles.
Allowing branching/singularities

- Let \( Y \subset X \), and \( Y' \to Y \) be an \( n \)-sheeted cover. The groupoid \( \text{Cov}_n^Y(X, Y) \) of relative \( n \)-sheeted covers has objects \( n \)-sheeted covers \( X' \to X \) together with data of an isomorphism \( \varphi : X'|_Y \cong Y' \).

- Morphisms in this category are isomorphisms of covers which commute with these maps \( \varphi \).

- In the same way, you can define a relative principal \( S_n \)-bundle given a specific principal \( S_n \)-bundle \( Q \to Y \), and obtain a groupoid \( \text{Bun}^Q_{S_n}(X, Y) \).

- The equivalence on the last slide extends to an equivalence of groupoids

\[
\text{Cov}_n^Y(X, Y) \cong \text{Bun}^Y_{S_n} \times_{S_n} \{1, \ldots, n\}(X, Y).
\]
A branched cover $\Sigma' \to \Sigma$ branched at $x_1, \ldots, x_k$ is equivalent to a genuine $n$-sheeted cover $\Sigma'' \to \Sigma \setminus \bigcup_i B_{\varepsilon}(x_i)$, together with the ramification data around each branch point.
On Wednesday, we used the finite path integral to “sum over principal $G$-bundles” for $G$ finite

Given a TFT of $\text{SO} \times G$-manifolds, obtain a TFT of oriented manifolds

Finite gauge theory $Z_G$: do this to the trivial theory

This counts principal $G$-bundles
Rephrasing the counting question in terms of finite gauge theory

- When there are zero branch points, the correspondence between $n$-sheeted coverings and principal $S_n$-bundles means we want to compute the partition function $Z_{S_n}(\Sigma_g)$.
- When there are $n$ branch points, we’re looking at $\Sigma_g$ minus $n$ small discs; call this $\Sigma_{g,n}$.
  - The ramification data defines an element of $Z(\partial \Sigma_{g,n})$: $Z_{S_n}(S^1)$ is free on $\pi_0(\mathcal{B}un_{S_n}(S^1))$, so we choose the $\delta$-function at our ramification data.
  - If you stare at the push-pull formula, it’s telling you that the Hurwitz number for this ramification data is the linear map $Z(\Sigma_{g,n}): Z(\partial \Sigma_{g,n}) \to \mathbb{C}$ applied to our $\delta$-function.
- So all the Hurwitz numbers are contained in this series of TFTs.
Solving 2d TFTs

- Folk theorem: 2d oriented TFTs are classified by commutative Frobenius algebras
- Fancier theorem: 2d fully extended TFTs $\mathcal{Bord}_2^{SO} \to \mathcal{A}lg_C$ are classified by semisimple Frobenius algebras
  - Fully dualizable implies semisimplicity
  - The Frobenius condition gets us from framed TFTs to oriented TFTs
$Z(S^1) \cong A \otimes_{A \otimes A^{\text{op}}} A$, which is the center of $A$
Let $e_1, \ldots, e_k$ be the primitive idempotents of the center of $A$

These are the elements such that $a^2 = a$, and which are not further factorizable into sums of other idempotents

The incoming disc sends $1 \mapsto \sum e_i$

The outgoing disc is the counit $\lambda$ (the adjoint of the unit map $\mathbb{C} \to A$ under the inner product)

Outgoing pair-of-pants is multiplication, and incoming pair-of-pants is sent to $A \to A \otimes A$ given by

$$e_i \mapsto \frac{1}{\lambda(e_i)} e_i \otimes e_i$$
Hence, if $\Sigma_g$ is a closed, connected, oriented surface of genus $g$,

$$Z(\Sigma_g) = \sum_{i=1}^{k} \lambda(e_k)^{1-g}$$

Prove by chopping $\Sigma_g$ into a sequence of pairs of pants, and an incoming and outgoing disc.
Finite gauge theory in this perspective

- The semisimple Frobenius algebra is the group algebra \( \mathbb{C}[G] \)
- \( Z_G(S^1) = \mathbb{C}[G]^G \), the ring of characters, under pointwise multiplication
- Inner product:

\[
\langle \varphi, \psi \rangle = \frac{1}{\# G} \sum_{g \in G} \varphi(g) \psi(g^{-1})
\]
Conjugacy classes of $S_n$, as well as isomorphism classes of irreducible $S_n$-representations, are indexed by partitions of $n$.

For $V$ an irreducible representation of $S_n$,

$$e_V = \frac{\dim V}{n!} \sum_{\sigma \in S_n} \chi_V(\sigma)\sigma$$

and therefore

$$\lambda(e_V) = \left(\frac{\dim V}{n!}\right)^2$$

(because $e_V = e_V^2$, so $\lambda(e_V) = \langle e_V, e_V \rangle$)
We also need a “change of basis” formula

We want to evaluate $Z_{S_n}$ on a surface with boundary with a principal $S_n$-bundle; the restriction to a boundary $S^1$ is equivalent data to a conjugacy class $C$

We want to understand $\delta_C \in \mathbb{C}[S_n]^{S_n}$ in terms of the primitive idempotents $e_V$

$$\delta_C = |C| \sum_V \frac{\chi_V(C)}{\dim V} e_V$$
The number of elements of $S_n$ in the conjugacy class $C_P$ given by a partition $P$ is

$$|C_P| = \frac{n!}{\prod_{k=1}^{n} k^{p_k} p_k!}$$

If $V$ is the representation corresponding to $P$, $|p| = m$, $\ell_i = p_i + m - i$, and $\Delta = \prod_{i<j} (x_i - x_j)$, then

$$\dim V = \frac{n!}{\ell_1! \cdots \ell_m! \Delta(\ell_1, \ldots, \ell_m)}$$
Computing Hurwitz numbers

- $\Sigma_{g,m}$ decomposes as (1) a bunch of outgoing pairs of pants, followed by (2) incoming then outgoing pairs of pants, then (3) a disc

- (1) is multiplication and (3) \circ (2) sends $e_V \mapsto \lambda(e_V)^{1-g}$
Using the change-of-basis formula, (1) sends

\[ \delta_{p_1} \otimes \cdots \otimes \delta_{p_m} \mapsto \delta_{p_1} \cdots \delta_{p_m} \]

\[ = \prod_{j=1}^{m} |C_{p_j}| \sum_{V} \frac{\chi_V(p_j)}{\dim V} e_V \]

\[ = \sum_{V} \prod_{j=1}^{m} \left( \frac{|C_{p_j}| \chi_V(p_j)}{\dim V} \right) e_V \]
Computing Hurwitz numbers

Now we need to send \( e_V \mapsto \lambda(e_V)^{1-g} = (\dim V / n!)^{1-g} \):

\[
\sum \prod_{V, j=1}^{m} \left( \frac{|C_{p_j}| \chi(p_j)}{\dim V} \right) \left( \frac{\dim V}{n!} \right)^2
\]

This is Burnside’s formula for Hurwitz numbers!

We stated purely combinatorial formulas for \(|C_{p_j}|\) and \(\dim V\). Using them (and simplifying), we obtain

\[
\mathcal{H}_{g,n,p_1,\ldots,p_n} = \sum_q \left( \frac{\Delta(\ell_1, \ldots, \ell_m)}{\ell_1! \cdots \ell_m!} \right)^{2-2g-m} \prod_{j=1}^{m} \chi_q(C_{p_j}) \prod_{k=1}^{n} \frac{1}{k(p_j)_k (p_j)_k!}
\]

This is almost completely combinatorial: \(\chi_q(C_{p_j})\) can be determined with the Frobenius character formula, but is inexplicit and complicated.