Topological field theory: generalities

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Some logistics

- One hour lecture
- "Office hours" 3–4ish CDT
- I will post some exercises, which are mostly of the "interesting things to think about" variety

We regrettably won't cover…

- Extended TFT and the cobordism hypothesis
- Chern-Simons theory
- Connections with physics

Outline

1. Bordism
2. Bordisms with structure
3. Topological field theories
4. Duality

Some words that are different but mean the same thing

- Bordism vs cobordism

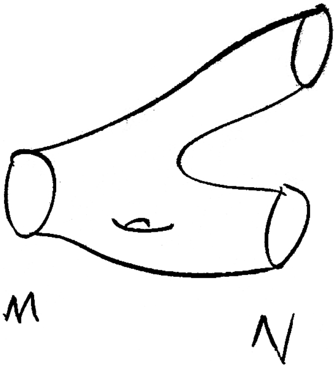
Some words that are different but mean the same thing

- Bordism vs cobordism
- Topological field theory (TFT) vs topological quantum field theory (TQFT)

Bordism

Definition

Let M_0 and M_1 be closed n-manifolds. A *bordism* from M_0 to M_1 is a compact $(n+1)$-manifold X, a partition $\partial X = Y_0 \amalg Y_1$, and diffeomorphisms $\theta_i \colon Y_i \xrightarrow{\cong} M_i$. If there is a bordism from M_0 to M_1, we say M_0 and M_1 are *bordant*.



Bordism is an equivalence relation

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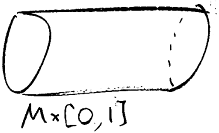

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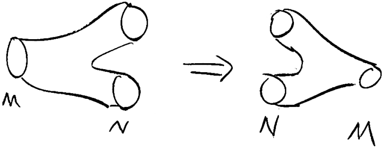

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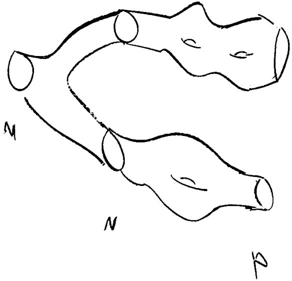


Algebraic structure

- Disjoint union descends to bordism classes, giving a commutative monoid Ω_n^{O} of bordism classes of closed n-manifolds
  - $\varnothing$ (which is a closed n-manifold!) is the unit
- It turns out this is an abelian group!
- Direct product turns $\Omega_*^{\mathrm{O}} := \bigoplus_n \Omega_n^{\mathrm{O}}$ into a $\mathbb{Z}$-graded ring

Tangential structures

- Goal: introduce variants of this notion which take into account additional *topological* information
- So stuff like orientations, spin structures, maps to a space
- Not geometric information (e.g. Riemannian metric or connection on a principal bundle)
- Information must be "local" (so nothing like a CW structure or a point inside the manifold)

Tangential structures

- Consider the *stable orthogonal group* $\mathrm{O} := \operatorname{colim}_n \mathrm{O}_n$. The classifying space $B\mathrm{O}$ is the classifying space for stable virtual vector bundles
  - "Virtual" means we allow formal differences $E - F$ for $E, F \to X$
  - "Stable" means we ignore the difference between E and $E \oplus \underline{\mathbb{R}}$
  - So $[M, B\mathrm{O}]$ is identified with stable isomorphism classes of virtual vector bundles
- A manifold has a canonical (homotopy class of) map $M \to B\mathrm{O}$ which classifies its tangent bundle

Tangential structures: definition

- Let $\xi : B \to B\mathrm{O}$ be a fibration. A ξ-*structure* on a manifold is a lift

$$\begin{tikzcd} & B \arrow[d, "\xi"] \\ M \arrow[ur, dashed] \arrow[r, "TM"'] & B\mathrm{O}. \end{tikzcd}$$

- Two ξ-structures are equivalent if they are homotopic through lifts of the tangent bundle map

Tangential structures: examples

- Given a family of maps $G_n \to \mathrm{O}_n$, obtain $\xi\colon BG \to B\mathrm{O}$
- In this case, a ξ-structure is a reduction of structure group for the frame bundle to G_n
- For example, for $B\mathrm{SO} \to B\mathrm{O}$, this is an orientation
- For $B\mathrm{Spin} \to B\mathrm{O}$, this is a spin structure

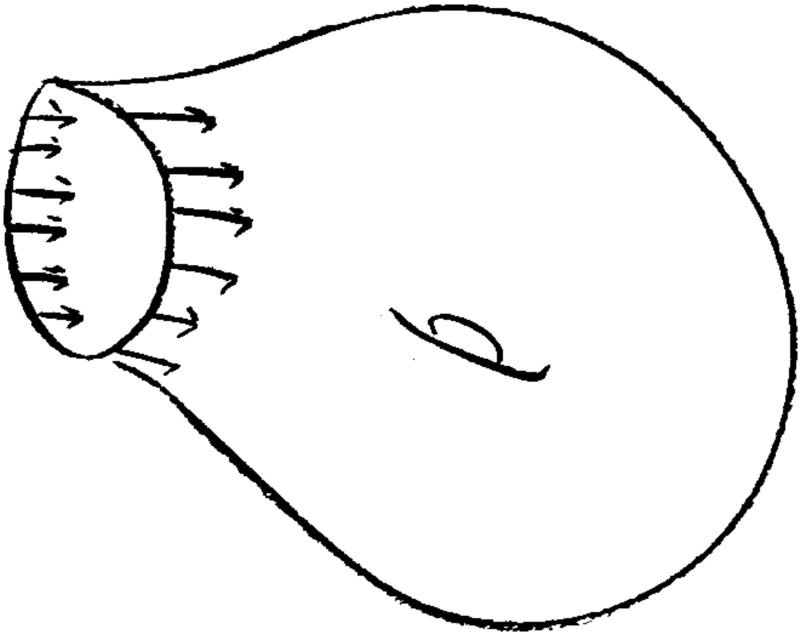
Tangential structures: examples

- $BO \times BG \to BO$: a principal G-bundle
- $BO \times X \to BO$: a map to X

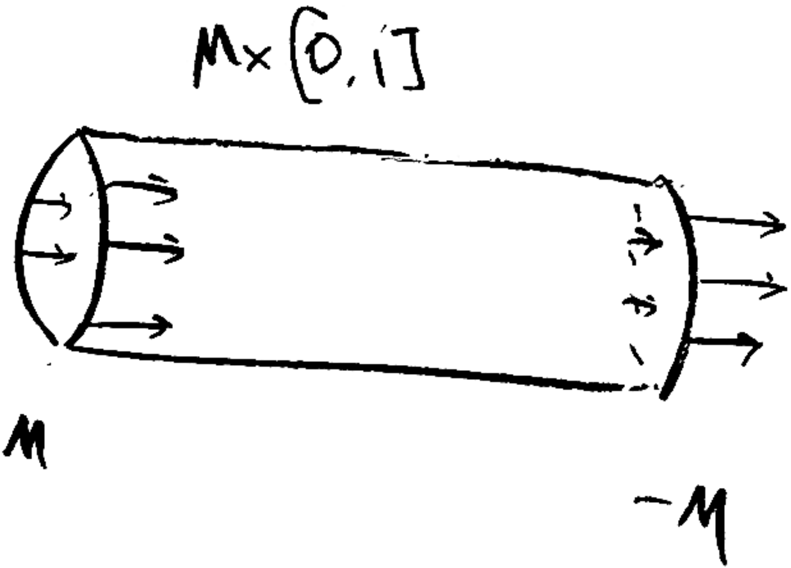
Induced structures on the boundary

- If M is a manifold with boundary, $T(\partial M) \oplus \nu \cong TM|_{\partial M}$
- Therefore as virtual stable vector bundles, $T(\partial M) \cong TM|_{\partial M} - \nu$
- ν is trivializable, but has two trivializations! (outward vs inward unit normal)
- A trivialization of ν and a ξ-structure on TM induce a ξ-structure on $T(\partial M)$, but the two ξ-structures may differ
- We let ∂M refer to ∂M with its ξ-structure via the inner unit normal, and $-\partial M$ for the ξ-structure via the outer unit normal

Induced structures on the boundary



Induced structures on the boundary



Structured bordisms

- We can now define bordisms of ξ-manifolds in much the same way, except that we ask for an identification of manifolds with ξ-structure from ∂X to $M \amalg -N$
- This is again an equivalence relation compatible with disjoint union, giving us bordism groups Ω_n^{ξ}
  - (This is a good thing to think through if you're seeing this stuff for the first time!)
- It is not always true that we get a graded ring

Bordism categories

- We want to upgrade or categorify this structure
- Define a bordism category $\mathcal{B}ord_n^{\xi}$ whose objects are closed $(n-1)$-dimensional ξ-manifolds, and whose morphisms are* ξ-structured bordisms between them
- Composition is gluing of bordisms
- *: we need to take diffeomorphism classes rel boundary of bordisms in order for composition to be associative

Bordism categories: extra structure

- $(\amalg, \varnothing)$ induce a "categorical commutative monoid" structure on $\mathcal{B}ord_n^\xi$, the structure of a *symmetric monoidal category*
- This is a unit and a "tensor product" $\amalg$ which has data enforcing associativity and commutativity up to natural isomorphism, etc.
- Example: $(\mathcal{V}ect_{\mathbb{C}}, \otimes)$
- Also a notion of symmetric monoidal functors and symmetric monoidal natural transformations

Topological field theories

- A *topological field theory* is a symmetric monoidal functor $Z\colon \mathcal{B}ord_n^{\xi} \to \mathcal{V}ect_{\mathbb{C}}$
- n is called the *(spacetime) dimension* of the theory; n is the *space dimension*
- For every closed $(n-1)$-manifold M, we get a vector space $Z(M)$ called the *state space*
- A bordism $X\colon M \to N$ defines a linear map $Z(X)\colon Z(M) \to Z(N)$; gluing goes to composition
- $Z(\varnothing) = \mathbb{C}$. Therefore a closed n-manifold X, as a bordism $\varnothing \to \varnothing$, defines a linear map $\mathbb{C} \to \mathbb{C}$; the image of 1 is called the *partition function* of X

Example: the Euler TFT

- Assign to every closed $(n-1)$-manifold the state space $\mathbb{C}$
- Assign to every bordism $X\colon M \to N$ the quantity $\lambda^{\chi(X,N)}$ ($\lambda \in \mathbb{C}^{\times}$ fixed; $\chi(X,n)$ is the relative Euler characteristic)
- Gluing and symmetric monoidality hold because of formulas for χ

A first theorem

Theorem

Let $Z\colon \mathcal{B}ord_n^{\xi} \to \mathcal{V}ect_{\mathbb{C}}$ be a TFT and M be a closed $(n-1)$-dimensional ξ-manifold. Then the vector space $Z(M)$ is finite-dimensional.

We will prove this by defining a generalization of "finite-dimensional" in arbitrary symmetric monoidal categories, preserved by symmetric monoidal functors; then showing all objects in $\mathcal{B}ord_n^{\xi}$ are "finite-dimensional"

Duality in symmetric monoidal categories

- Let $\mathcal{C}$ be a symmetric monoidal category and $x \in \mathcal{C}$. *Duality data* for x is an object $x^\vee \in \mathcal{C}$ and morphisms $e\colon x \otimes e^\vee \to 1$ and $c\colon 1 \to x \otimes x^\vee$ such that the following maps compose to the identity:

$$x \xrightarrow{c \otimes \mathrm{id}_x} x \otimes x^\vee \otimes x \xrightarrow{\mathrm{id}_x \otimes e} x \tag{1a}$$

$$x^\vee \xrightarrow{\mathrm{id}_{x^\vee} \otimes c} x^\vee \otimes x \otimes x^\vee \xrightarrow{e \otimes \mathrm{id}_{x^\vee}} x^\vee. \tag{1b}$$

If duality data exists for x, we call x *dualizable*, $x^\vee$ the *dual* of x, e *evaluation*, and c *coevaluation*.

Visualizing dualizability

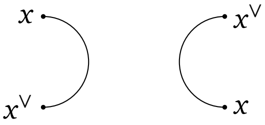


Figure: Evaluation (on left) and coevaluation (on right).

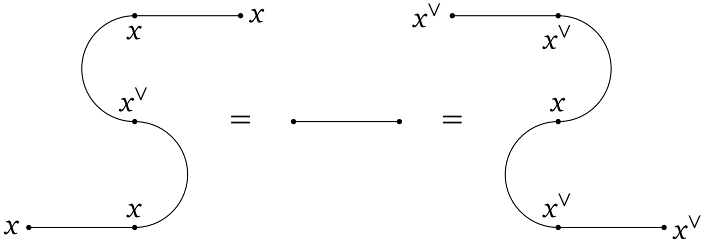


Figure: Left: the S-diagram, encoding (1a). Right: the Z-diagram, encoding (1b). These equalities are the conditions on duality data.

Dualizability in $\mathcal{V}ect_k$

- Say V is dualizable with duality data $(V^\vee, c, e)$, and let $c(1) = \sum v^i \otimes v_i$. Crucially, this is a finite sum!
- Apply the Z-diagram to compute that for any $x \in V$,

$$x = \sum_i e(x, v^i) v_i,$$

  i.e. the finite set $\{v_i\}$ spans V.
- Conversely, given a finite-dimensional vector space, let $V^\vee := \mathrm{Hom}(V, \mathbb{C})$, e be evaluation, and c send $1 \mapsto \sum e^i \otimes e_i$ ($\{e_i\}$ a basis, $\{e^i\}$ the dual basis)

Every object is dualizable in $\mathcal{B}ord_n^{\xi}$

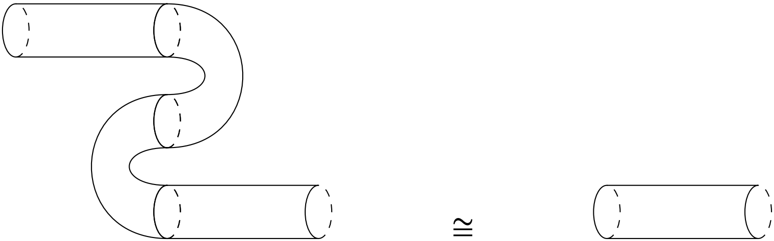


Figure: "Zorro's lemma," that these two bordisms are equivalent, shows that all objects of $\mathcal{B}ord_n^{\xi}$ are dualizable.

Proof of the main theorem

- If $f\colon \mathcal{C} \to \mathcal{D}$ is a symmetric monoidal functor and $x \in \mathcal{C}$ is dualizable, the image of the duality data for x under f is duality data for $f(x)$, so $f(x)$ is dualizable
- So if $Z\colon \mathcal{B}ord_n^\xi \to \mathcal{V}ect_{\mathbb{C}}$ is a TFT and M is a closed $(n-1)$-dimensional ξ-manifold, then M is dualizable in $\mathcal{B}ord_n^\xi$, so $Z(M)$ is dualizable in $\mathcal{V}ect_{\mathbb{C}}$, i.e. finite-dimensional.

Mapping class group actions

- Another general feature of TFTs which is occasionally useful
  - Working with general tangential structures requires some care, but O, SO, etc., are fine
- Idea: $\mathrm{Diff}(M)$ acts on the state space $Z(M)$ by mapping cylinders: if $\varphi \in \mathrm{Diff}(M)$, then it defines a bordism $M \to M$ by $[0,1] \times M$, where at 0 we attach by id, and at 1 we attach by φ
- If φ, φ' are isotopic, their mapping cylinders are diffeomorphic rel boundary, so they define the same morphism in $\mathcal{B}ord_n$. So the $\mathrm{Diff}(M)$-action factors through the action of the *mapping class group* $\mathrm{MCG}(M) := \mathrm{Diff}(M)/\mathrm{Diff}_0(M)$

Traces

- The *mapping torus* of $\varphi \in \mathrm{Diff}(M)$ is $M_\varphi := [0,1] \times M/(0,x) \sim (1,\varphi(x))$.
- One can show that if Z is a TFT, the partition function $Z(M_\varphi)$ is the trace of the action of φ on the state space $Z(M)$
- Special case: $Z(M \times S^1) = \dim Z(M)$