Invertible field theories

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Outline

1. Definition of invertible TFTs
2. Classifying invertible TFTs using Picard groupoids
3. Examples
4. The homotopy theory under the hood

Invertibility

- Symmetric monoidal categories are categorified commutative rings
- In a commutative ring A, an invertible element (a unit) x is an element such that there exists x^{-1} with $x \cdot x^{-1} = 1$
- In a symmetric monoidal category, we make the same definition
  - The bad news: x^{-1} and the isomorphism $x \otimes x^{-1} \xrightarrow{\cong} 1$ are now data!
  - The good news: this is a contractible choice, like duality data, so we continue to think of it as a condition
- Example: in $(\mathcal{V}ect_{\mathbb{C}}, \otimes)$, a vector space is invertible iff it is one-dimensional

Picard groupoids

- We let $\mathcal{C}^\times \subset \mathcal{C}$ denote the subcategory of $\otimes$-invertible objects and composition-invertible morphisms
- This is a *Picard groupoid*, i.e. a groupoid (all morphisms invertible) with a symmetric monoidal structure, such that all objects are invertible
- For example, $\mathcal{V}ect_{\mathbb{C}}^\times$ consists of lines (one-dimensional vector spaces) and nonzero linear maps between them

The symmetric monoidal category of TFTs

- The category of TFTs is the category of symmetric monoidal functors $\mathcal{B}ord_n^{\xi} \to \mathcal{V}ect_{\mathbb{C}}$ (so, symmetric monoidal functors and symmetric monoidal natural transformations)
- This has a symmetric monoidal structure given by "pointwise tensor product:"

$$(Z_1 \otimes Z_2)(M) := Z_1(M) \otimes Z_2(M)$$

- The unit is the trivial theory: the constant functor valued in $\mathbb{C}$ and $\mathrm{id}_{\mathbb{C}}$
- An *invertible TFT* is an invertible object in this symmetric monoidal category

Ok, but what is invertibility really?

- In a commutative ring of $\mathbb{C}$-valued functions, an element f is invertible iff $f(x) \in \mathbb{C}^\times$ for all x
- The tensor product of TFTs is completely analogous: a TFT Z is invertible iff for all closed $(n-1)$-manifolds M, $Z(M)$ is one-dimensional, and for all bordisms X, $Z(X) \neq 0$
- You can think of these as "nearly trivial TFTs"

Classifying invertible field theories

Theorem (Freed-Hopkins-Teleman)

The abelian group of invertible n-dimensional TFTs of ξ-manifolds valued in $s\mathcal{V}ect_{\mathbb{C}}$ is naturally isomorphic to the group of SKK ξ-bordism invariants $\mathrm{Hom}(\mathrm{SKK}_n^{\xi}, \mathbb{C}^{\times})$

- Idea: an invertible TFT is determined by its partition function, which is a bordism invariant (in a modified sense)

Proof sketch

1. Classifying invertible TFTs is purely a question about Picard groupoids
2. We can express Picard groupoids and maps between them with algebraic data
3. We know that data for $\mathcal{B}ord_n^{\xi}$ and $s\mathcal{V}ect_{\mathbb{C}}^{\times}$

Group completion

- Let $f : M \to N$ be a map of commutative monoids, and suppose $\mathrm{Im}(f) \subset N^\times$
- We can *group complete* M to an abelian group $\overline{M}$ by formally adjoining inverses, much like how $\mathbb{Z}$ is built from $\mathbb{N}$
- Then f extends to a map $\bar{f} : \overline{M} \to N^\times$ of abelian groups by setting $f(x^{-1}) := f(x)^{-1}$
- f and $\bar{f}$ determine each other, so the abelian group of invertible maps $M \to N$ is naturally isomorphic to the abelian group of all maps $\overline{M} \to N^\times$

Picard groupoid completion

- There is a entirely analogous story for Picard groupoids: given a symmetric monoidal category $\mathcal{C}$ and a map f to a Picard groupoid $\mathcal{D}^\times$, the map factors through the *Picard groupoid completion* $\overline{f}\colon \overline{\mathcal{C}} \to \mathcal{D}^\times$
- Construct $\overline{\mathcal{C}}$ by formally inverting all objects and morphisms in $\mathcal{C}$
- Construct $\overline{f}$ by the formula $\overline{f}(x^{-1}) := f(x)^{-1}$
- Again f and $\overline{f}$ determine each other
- Letting $\mathcal{C} = \mathcal{B}ord_n^\xi$, this means that to classify invertible TFTs, we need to understand Picard groupoid maps $\overline{\mathcal{B}ord_n^\xi} \to \mathcal{V}ect_{\mathbb{C}}^\times$

Picard groupoids as algebraic data

- $\pi_0(\mathcal{C})$ is the abelian group of isomorphism classes of objects under tensor product
- $\pi_1(\mathcal{C}) := \mathrm{Aut}_{\mathcal{C}}(1)$ (Eckmann-Hilton implies this is abelian)
- Using $- \otimes \mathrm{id}_x \colon \mathrm{Aut}_{\mathcal{C}}(1) \to \mathrm{Aut}_{\mathcal{C}}(1 \otimes x) = \mathrm{Aut}_{\mathcal{C}}(x)$, $\pi_1(\mathcal{C})$ is canonically identified with $\mathrm{Aut}_{\mathcal{C}}(x)$ for all $x \in \mathcal{C}$
- The *k-invariant* $k \colon \pi_0(\mathcal{C}) \otimes \mathbb{Z}/2 \to \pi_1(\mathcal{C})$: given $x \in \pi_0(\mathcal{C})$, take the class of the symmetry map $\sigma \in \mathrm{Aut}_{\mathcal{C}}(x \otimes x) = \pi_1(\mathcal{C})$

Skeletonizing the homotopy category of Picard groupoids

- It is a theorem of Hoàng that (π_0, π_1, k) determine a Picard groupoid up to equivalence
- Moreover, homotopy classes of morphisms between Picard groupoids $f\colon \mathcal{C} \to \mathcal{D}$ are naturally identified with the abelian group of pairs of maps $f_0\colon \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$ and $f_1\colon \pi_1(\mathcal{C}) \to \pi_1(\mathcal{D})$ which commute with the k-invariant
- Upshot: to prove the theorem, we should determine (π_0, π_1, k) for $\overline{\mathcal{B}ord_n^{\xi}}$ and $s\mathcal{V}ect_{\mathbb{C}}^{\times}$

So, what are (π_0, π_1, k) for our friends?

- For $\mathcal{V}ect_{\mathbb{C}}$, you can do this directly: $\pi_0 = 0$ and $\pi_1 = \mathbb{C}^\times$. $k = 0$
- For $s\mathcal{V}ect_{\mathbb{C}}$, you can also do this directly: $\pi_0 = \mathbb{Z}/2$ (even and odd lines), $\pi_1 = \mathbb{C}^\times$, and the k-invariant is the nontrivial map
- For $\mathcal{B}ord_n^\xi$, this is a major theorem! Due to Galatius-Madsen-Tillmann-Weiss, Nguyen
  - $\pi_0 = \Omega_{n-1}^\xi$, and $\pi_1 = \mathrm{SKK}_n^\xi$, the *SKK bordism group*
  - The k-invariant is taking the product with S^1 (ξ-structure induced by nonbounding framing)

Proving the theorem: the last step

- Based on what we've seen, invertible TFTs valued in $s\mathcal{V}ect_{\mathbb{C}}$ are identified with pairs of maps $f_0\colon \Omega^{\xi}_{n-1} \to \mathbb{Z}/2$ and $f_1\colon \mathrm{SKK}^{\xi}_n \to \mathbb{C}^\times$ intertwining the k-invariant
- Crucially, the k-invariant for $s\mathcal{V}ect^{\times}_{\mathbb{C}}$ is injective, so f_1 uniquely determines f_0 (if such an f_0 can exist)
- The k-invariant tensors with $\mathbb{Z}/2$, so the image of $f_1 \circ k_{\mathcal{B}ord^{\xi}_n}$ is contained in $\mathbb{Z}/2 = \{\pm 1\} \subset \mathbb{C}^\times$, so such an f_0 must exist

Right, but what is the SKK group?

- Consider the notion of bordism where we say “bounding” means M bounds W *and* the outward normal vector field on M extends to a nonvanishing vector field on W
- This defines only a commutative monoid under disjoint union, so we group-complete to obtain the *SKK group* SKK_n^ξ
- References: Jänich, Karras-Kreck-Neumann-Ossa, Reinhardt, Madsen-Tillmann
- aka: vector field bordism, Reinhardt bordism, Madsen-Tillmann bordism, Lorentz bordism

SKK bordism invariants

- Ordinary bordism invariants are SKK invariants
- The Euler characteristic is an SKK invariant! (Euler char of W vanishes by Poincaré-Hopf, then use gluing formula)
- The *Kervaire semicharacteristic* in dimension $4k+1$ ($\xi = \mathrm{SO}$)

$$\kappa(M) := \sum_{i=0}^{2k} b_i(M) \bmod 2.$$

- That's about it

Many examples of ordinary bordism invariants

- General idea: integrating canonical cohomology classes defined on manifolds with ξ-structures gives bordism invariants of ξ-manifolds
  - This is quite general: you can use generalized cohomology, twisted cohomology, ...
- For example, suppose we want to study oriented 6-manifolds with principal U_1-bundles. There is an invariant $\Omega_6^{\mathrm{SO}}(B\mathrm{U}_1) \to \mathbb{Z}$ given by $M, P \mapsto \int p_1(M)c_1(P)$
  - $p_1 \in H^4(M;\mathbb{Z})$ is the first Pontrjagin class; $c_1 \in H^2(M;\mathbb{Z})$ is the first Chern class
  - Why? Both p_1 and c_1 admit de Rham models as closed forms; then use Stokes' theorem (in general, see Milnor-Stasheff)

The homotopy theory in the background

- The proof of the theorem of Freed-Hopkins-Teleman classifying invertible TFTs actually passes through stable homotopy theory
- Idea: given a Picard groupoid $\mathcal{C}$, take the geometric realization of the nerve, which gives you a pointed CW complex with $\pi_i = 0$ for $i \geq 2$
- Picard groupoid $\Longrightarrow$ this is a grouplike E_∞-space, so it defines a spectrum with $\pi_i = 0$ for $i \neq 0, 1$
- Called the “classifying spectrum” of $\mathcal{C}$

The one-dimensional stable homotopy hypothesis

- The 1-*dimensional stable homotopy hypothesis* conjectures that taking the classifying spectrum defines an equivalence of homotopy theories from the category of Picard groupoids to the category of spectra with only π_0 and π_1 nontrivial
- This was a folk theorem proven by many people (Bullejos-Carrasco-Cegarra, Hopkins-Singer, Drinfeld, Patel, Johnson-Osorno, Ganter-Kapranov)
- Then, Postnikov theory tells you how to determine homotopy classes of maps between such spectra using the *k*-invariant (also folklore; see Johnson-Osorno)
- What Galatius-Madsen-Tillmann-Weiss and Nguyen did was identify the homotopy type of the classifying spectrum of $\mathcal{B}ord_n^\xi$, stated in a homotopical way

Extended invertible TFTs

- Why all this homotopy theory? We're often interested in classifying extended TFTs (formulated in terms of bordism higher categories), and the homotopical approach generalizes *much* better
- Invertible extended TFTs are classified by maps between Picard n-groupoids
- The n-dimensional stable homotopy hypothesis says "geometric realization of nerve" defines an equivalence between the homotopy theory of Picard n-groupoids and that of spectra with homotopy groups concentrated in degrees $[1, n]$
  - this is a recent theorem of Moser-Ozornova-Paoli-Sarazola-Verdugo
- Schommer-Pries computes the homotopy type of the bordism n-category
- Upshot: depending on target $\mathcal{C}$, invertible extended TFTs are classified in terms of homotopy or cohomology groups of *Madsen-Tillmann spectra* (again giving SKK groups)