The Turaev-Viro-Barratt-Westbury state sum

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Outline

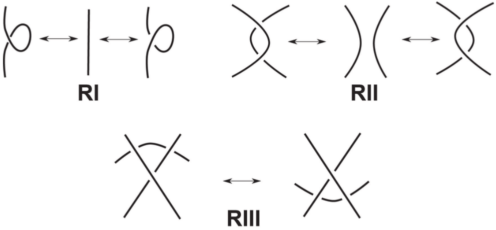
1. State sums: the what and the why
2. Monoidal categories, fusion categories, and spherical structures
3. Defining the TVBW model
4. (If time) A perspective on classifying 3d TFTs

The state-sum perspective on defining TFTs

- Common paradigm: make a definition by introducing extra structure, then checking the definition is independent of that structure
- In low-dimensional topology, it's extremely common for this structure to be combinatorial
- Then, one proves that a finite set of "moves" relates all equivalent combinatorial structures on the underlying object

Examples

- Knot invariants: choose a knot diagram (combinatorial structure), define an invariant of knot diagrams, and check that it is invariant under *Reidemeister moves*



- The Euler characteristic: choose a triangulation, count cells, and check that the answer does not depend on the triangulation

State-sum models

- The idea of state-sum TFTs is to do this for topological field theories, by using the combinatorial data of a triangulation (or something like it) to define an invariant, then checking that it is independent of the choice of triangulation and that it satisfies gluing (so it is a TFT)
- Typically, one begins with input data of some kind of category $\mathscr{C}$ with additional structure
  - Spherical fusion category: the TVBW model (3d TFT)
  - Ribbon fusion category: the Crane-Yetter model (4d TFT)
  - Many more (variants of Frobenius algebras for 2d oriented, spin, r-spin TFTs; Douglas-Reutter 4d state sum, ...)

Coloring

- Generally the category $\mathscr{C}$ has enough structure that we can make sense of simple objects (as in $\mathscr{R}ep_G$)
- A *coloring* of a triangulated manifold M is an assignment of a simple object of $\mathscr{C}$ for every 1-simplex of M
- The state-sum model fixes this and maybe additional data, then calculates the *weight* associated to each coloring, and the final invariant is a weighted sum using these weights
- To check invariance under change of triangulation, compute how it behaves under *Pachner moves*

The Turaev-Viro-Barratt-Westbury model: basic facts

- Input data is something called a *spherical fusion category* $\mathscr{C}$, which axiomatizes the behavior that particles ("anyons") in a 3d quantum system can have
- Turaev-Viro first defined this state sum in the special case where $\mathscr{C}$ is the category of modules for the Drinfeld double of $U_q(\mathfrak{sl}_2)$
- Then Barratt-Westbury generalized this to all spherical fusion categories
- Balsam-Kirillov made this model into a once-extended TFT, and studied its relationship with the *Levin-Wen model* in condensed-matter physics

Defining a fusion category

- Begin with a $\mathbb{C}$-linear category (i.e. Hom-sets are complex vector spaces; composition is linear)
- Add a monoidal structure, where $\otimes$ is bilinear on morphisms
- With this structure, we can define irreducible and indecomposable objects just as in representation theory
  - Irreducible (or simple): cannot be split as $x \cong y \oplus z$ in a nontrivial way
  - Indecomposable: cannot be written as an extension in a nontrivial way

Defining a fusion category

- $\mathscr{C}$ is *semisimple* if all objects are direct sums of simple ones
- Example: $\mathscr{R}ep_G$, G finite, char 0. Nonexample: $\mathscr{R}ep_G$, characteristic dividing $\#G$
- A *fusion category* is a semisimple $\mathbb{C}$-linear monoidal category with finitely many isomorphism classes of objects such that all objects have duals and the unit is simple

Defining a spherical fusion category

- In $\mathscr{V}ect_{\mathbb{C}}$, we have a natural isomorphism $\mathrm{id} \Rightarrow (-)^{**}$
- In a general fusion category, we only have a canonical trivialization of the *quadruple* dual (Radford; see Douglas–Schommer-Pries–Snyder)
- A *pivotal structure* is a natural isomorphism $\mathrm{id} \Rightarrow (-)^{**}$

Defining a spherical fusion category

- Pivotal is data, spherical is condition
- Specifically, ask that

$$d_+(x) := 1 \xrightarrow{c} x \otimes x^* \xrightarrow{\psi \otimes \mathrm{id}} x^{**} \otimes x^* \xrightarrow{e} 1$$

$$d_-(x) := 1 \xrightarrow{c} x^* \otimes x^{**} \xrightarrow{\mathrm{id} \otimes \psi^{-1}} x^* \otimes x \xrightarrow{e} 1$$

coincide

Some notation

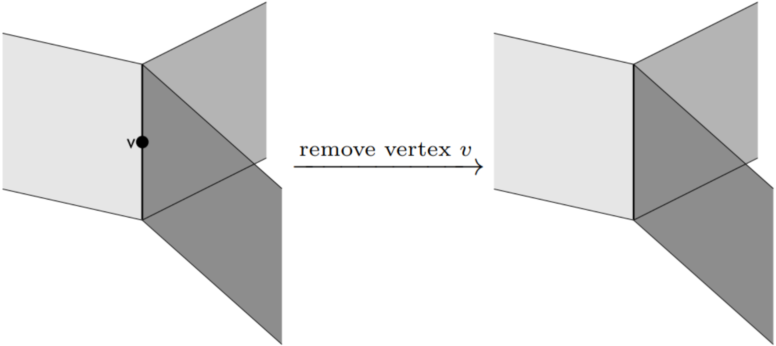
- Define $\langle x_1, \dots, x_n \rangle := \mathrm{Hom}_{\mathscr{C}}(1, x_1 \otimes \cdots \otimes x_n)$ for $x_1, \dots, x_n \in \mathscr{C}$
- The pivotal structure guarantees this "is invariant" under cyclic permutations (really: the pivotal structure provides a natural isomorphism between these functors)
- $S(\mathscr{C})$ denotes the finite set of isomorphism classes of simple objects
- $\dim(x) = d_+(x) = d_-(x)$ is the *dimension* of an object x
- The *global dimension* is

$$D := \sqrt{\sum_{x \in S(\mathscr{C})} (\dim x)^2}$$

Polytope decompositions

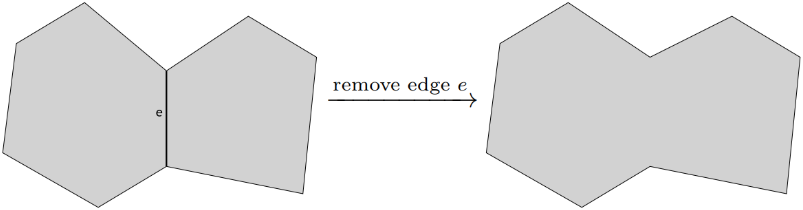
- This is a generalization of triangulations that allow for faces with more edges
- See Balsam-Kirillov
- We refer to a combinatorial manifold as a manifold with a polytope decomposition
- All combinatorial structures on the same underlying compact manifold are related by a finite series of Pachner moves (three in dimension 3)

Pachner moves



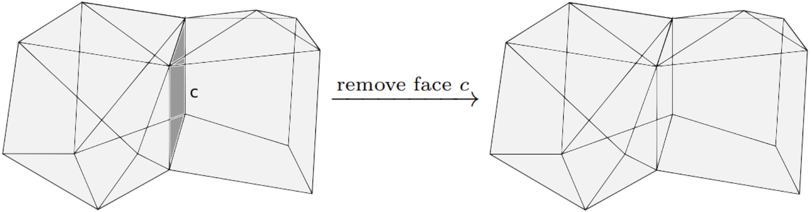
(Source: Balsam-Kirillov)

Pachner moves



(Source: Balsam-Kirillov)

Pachner moves



(Source: Balsam-Kirillov)

Colorings

- To every *oriented* edge e of a combinatorial manifold M, assign a simple object of $\mathscr{C}$
- Such that if you reverse the orientation of e, you get the dual object
- We will denote colorings $\ell : \{\text{oriented edges}\} \to S(\mathscr{C})$

Building the state space

- If f is an oriented 2-cell and ℓ is a coloring, let

$$H(f,\ell) := \langle \ell(e_1),\dots,\ell(e_k)\rangle,$$

  where $e_1,\dots,e_k$ are the edges of f in cyclic order
- Now for a closed, oriented, combinatorial 2-manifold Σ, let $H(\Sigma,\ell)$ be the direct sum of $H(f,\ell)$ over all 2-cells f of Σ, and let $H(\Sigma)$ be the direct sum of $H(\Sigma,\ell)$ over all colorings ℓ
- Not yet the state space!

Defining a vector for a 3-cell

- Next goal: given a 3-cell f, define a vector $Z(f,\ell) \in H(\partial f, \ell)$
- Let Π be the polytope decomposition on $\partial f = S^2$ induced from that on M, and let $\Pi^\vee$ be the (Poincaré) dual polytope decomposition
- The labeling $\ell|_\Pi$ of Π induces a labeling $\ell^\vee$ of $\Pi^\vee$: $\ell^\vee(e^\vee) = \ell(e)$ if $e^\vee$ is given the “dual orientation”
- Precisely: e and $e^\vee$ meet at a single point, and $e^\vee$ should go from right to left from the point of view of e

Defining a vector for a 3-cell

- For every 2-cell C of Π (0-cell of $\Pi^\vee$) choose $\varphi_C \in H(C,\ell)^*$
- This defines a "string diagram" by removing a point of S^2 and then using the edges in $\Pi^\vee$ as strings joining the objects φ_C at the (dual) vertices
- This diagram can be evaluated: contract an edge to evaluate a vector and a covector
- The spherical structure is exactly what guarantees that it didn't matter how you removed a point to get from S^2 to the plane

Defining a vector for a 3-cell

- This procedure defines a function

$$\bigotimes_{C \in \Delta^2(\partial f)} H(C,\ell)^* = H(\partial f,\ell)^* \longrightarrow \mathbb{C},$$

  meaning it is an element of $H(f,\ell)^{**} = H(f,\ell)$
- $Z(f,\ell)$ is this element

Defining the TVBW model on a 3-manifold

- For M a closed, oriented 3-manifold, we have a map

$$\bigotimes_{f\in\Delta^3(M)} H(\partial f,\ell) = H(\partial M,\ell)\otimes \bigotimes_{c\in\Delta^2(M\setminus\partial M)} H(c,\ell)\otimes\underbrace{H(c,\ell)^*}_{=H(-c,\ell)} \xrightarrow{\mathrm{id}_{H(\partial M,\ell)}\otimes\bigotimes_c ev} H(\partial M,\ell),$$

  where $ev\colon V\otimes V^*\to\mathbb{C}$ is the evaluation map
- Apply this map to the vector

$$\bigotimes_{f\in\Delta^3(M)} Z(f,\ell)\in\bigotimes_{f\in\Delta^3(M)} H(\partial f,\ell)$$

  and call the result $Z(M,\ell)$
- Finally, sum this (with a normalization) over all weightings:

$$Z_{\mathscr{C}}(M) := \frac{1}{D^{2v(M)}}\sum_{\text{labelings } \ell} Z(M,\ell)\prod_{e\in\Delta^2(M)} \dim(\ell(e))^{n(e)}\in Z_{\mathscr{C}}(\partial M).$$

  Here D is the dimension of $\mathscr{C}$ as above; v is the number of interior edges of M plus $1/2$ the number of boundary edges; and $v(e)$ is 1 for interior edges and $1/2$ for boundary edges

Defining the state spaces

- Consider

$$Z_{\mathscr{C}}(\Sigma \times I) \in Z(\Sigma) \otimes Z(-\Sigma) = Z(\Sigma)^* \otimes Z(\Sigma) = \mathrm{Hom}(Z(\Sigma), Z(\Sigma))$$

- This is a projection, and we define $Z_{\mathscr{C}}(\Sigma)$ to be its kernel
- We had to triangulate the interval, but the answer turns out to not depend on this choice

What we have so far

- We already have defined a vector space associated to a closed, oriented surface (with a triangulation – though it turns out to not depend on the triangulation)
- If M is a closed 3-manifold, $Z_{\mathscr{C}}(M)$ is an element of $Z_{\mathscr{C}}(\varnothing)$, which is isomorphic to $\mathbb{C}$ kind of vacuously. So we get a partition function
- More generally if M is an oriented bordism from Σ_0 to Σ_1, it comes with an identification $\partial M \cong -\Sigma_0 \amalg \Sigma_1$, and therefore

  $$Z_{\mathscr{C}}(M) \in Z_{\mathscr{C}}(\partial M) = Z_{\mathscr{C}}(\Sigma_0)^* \otimes Z_{\mathscr{C}}(\Sigma_1) = \mathrm{Hom}(Z_{\mathscr{C}}(\Sigma_0), Z_{\mathscr{C}}(\Sigma_1)).$$

  That is, this construction defines a map $Z_{\mathscr{C}}(\Sigma_0) \to Z_{\mathscr{C}}(\Sigma_1)$, as we wanted

Many things yet left to check

- Does disjoint union map to tensor product? (Yes, and this isn't that hard.)
- Why are all of these invariant under the Pachner moves?
- Why does a cylinder act by the identity?
- Why does gluing of bordisms correspond to composition?

Finite gauge theory

- There is a spherical fusion category structure on $\mathcal{V}ect_G$, the category of G-graded vector bundles
- Monoidal structure is convolution:

$$(V \otimes W)_g = \bigoplus_{h \in G} V_h \otimes W_{h^{-1}g}$$

- Upshot: this is a fusion category
- Use the standard pivotal structure on $\mathcal{V}ect$ to define one on $\mathcal{V}ect_G$, giving a spherical structure
- The TVBW model for this is finite gauge theory (aka untwisted Dijkgraaf-Witten theory)

Dijkgraaf-Witten theory

- Choose a cocycle $\alpha \in Z^n(BG; \mathbb{C}^\times)$, concretely a map $\alpha\colon G \times G \times G \to \mathbb{C}^\times$
- Use α to twist the associator

$$\alpha\colon (U_g \otimes V_h) \otimes W_k \Longrightarrow U_g \otimes (V_h \otimes W_k)$$

- Specifically, we have the standard associator coming from $\mathscr{V}ect$; multiply it by $\alpha(g,h,k) \in \mathbb{C}^\times$
- Cocycle implies this satisfies the pentagon identity; cohomologous cocycles define isomorphic theories: Dijkgraaf-Witten theory for $[\alpha] \in H^n(BG; \mathbb{C}^\times)$

Additional examples: quantum groups

- Given a semisimple Lie algebra $\mathfrak{g}$ and $q \in \mathbb{C}^\times$, one can deform the universal enveloping algebra $U(\mathfrak{g})$ to a "quantum group" $U_q(\mathfrak{g})$
- For q a root of unity, it is possible to extract a spherical fusion category from the category of representations of $U_q(\mathfrak{g})$
- Closely related to Chern-Simons theories for the simply connected group with Lie algebra $\mathfrak{g}$ — not the same, but related

Tambara-Yamagami TFTs

- Choose a finite abelian group A, a square root z of $1/\#A$, and a nondegenerate symmetric bilinear form $\chi : A \times A \to \mathbb{C}^\times$; this data defines a spherical fusion category $\mathcal{TY}(A, z, \chi)$ called a *Tambara-Yamagami category*
- Simple objects are indexed by elements of A plus an additional object m
- Monoidal structure: $a \otimes b = ab$ $(a, b \in A)$, $a \otimes m = m \otimes a = m$, and $m \otimes m = \bigoplus_{a \in A} a$. The unit is $1 \in A$

Tambara-Yamagami TFTs

- Morphisms are as in Schur's lemma: for simple objects, $\mathrm{Hom}(x,y) = \mathbb{C}$ if $x \cong y$, and 0 if otherwise
- Defining the associator:

$$\alpha(a,m,b) := \chi(a,b) \cdot \mathrm{id}_m : m \to m$$
$$\alpha(m,a,m) := (\chi(a,x)\delta_{xy}\mathrm{id}_x) : \bigoplus_{x\in A} x \to \bigoplus_{y\in A} y$$
$$\alpha(m,m,m) := z\chi(x,y)^{-1}\mathrm{id}_m : \bigoplus_{x\in A} m \to \bigoplus_{y\in A} m$$

The remaining ones are all the identity