Matroid Theory

Day two: Matroid operations

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July 6th, 2021

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Previous questions

Given a graphic matroid $M \cong M(G)$, construct a matrix for which it is also the vector matroid:

- Take $D(G)$ be any orientation of G.
- Let $A_{D(G)}$ be its incidence matrix $a_{ij} \in \{1, -1, 0\}$.
- $M \cong M(G) \cong M[A_{D(G)}]$ (use circuit viewpoint).

Previous questions

What is the relationship between matroids and flag varieties?

$$\text{REP.MATROIDS} \longleftrightarrow \text{torus orbits in Grassmanians}$$

- $\mathbb{T} = (\mathbb{C}^*)^n$ acts on $\mathbb{C}^n$.
- This induces an action on $\mathrm{Gr}_{r,n}$.
- Given $V \in \mathrm{Gr}_{r,n}$, consider the orbit $\mathbb{T} \cdot V$ and take the closure.
- $\overline{\mathbb{T} \cdot V} \to \text{matroid polytope}$ via the *moment map*.

Previous questions

What is the relationship between matroids and flag varieties?

$$\text{FLAG MATROIDS} \longleftrightarrow \text{torus orbits in flag varieties}$$

- A collection $\mathcal{F}$ of flags is a flag matroid if and only if
  - Every $M_i = \{F^i | F \in \mathcal{F}\}$ is a matroid.
  - For every pair (M_i, M_j), one is a quotient of the other.
  - Every flag $B_1 \subseteq \ldots \subseteq B_s$ with B_i a basis of M_i is in $\mathcal{F}$.
- The relation is obtained by considering flag matroid polytopes and lattice polytopes representing a toric variety.

... attempted to reduce Graph Theory to Linear Algebra. It showed that many graph-theoretical results could be generalized to algebraic theorems... I was discussing a theory of matrices in which elementary operations could be applied to rows but not columns.

W.T. Tutte

This is matroid theory.

Dan Younger, on the above

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The duality of ~~man~~ matroids

Definition of dual

Matroids come with a notion of duality.

Definition (Dual of a matroid)

Let $M = (E, \mathcal{B})$ be a matroid. The *dual* of M is

$$M^* := (E, \mathcal{B}^*)$$

where

$$\mathcal{B}^* := \{E - B : B \in \mathcal{B}\}$$

- $(M^*)^* = M$.
- $\mathrm{rank}(M) = |E| - \mathrm{rank}(M^*)$.
- Corank: $r^*(X) := r(E - X) + |X| - r(M)$ OR $r^*(X) = r(M) - r(X)$ $\leftarrow$ these are not equivalent.

Examples of dual

- $(U_n^r)^* = U_n^{n-r}$.
- If G is planar, $M^*(G) \cong M(G^*)$, where G^* is the geometric dual.
- If $M \cong M[I_r, |D]$, then $M^* \cong M[-D^T|I_{n-r}]$.

A brief detour on graphic matroids

We present a preview of Kuratowski's characterization of planar graphs.

Proof (Neither $M^*(K_5)$ nor $M^*(K_{3,3})$ is graphic):

First consider $M = M^*(K_5)$. Suppose that there exists a connected graph G such that $M \cong M(G)$. $M(K_5)$ has 10 elements and is rank 4, so M must have 10 elements and rank 6. For a tree T in a graph,

$$|\mathrm{Vert}(T)| = |\mathrm{Edges}(T)| + 1.$$

Taking a spanning tree B of G, i.e. a basis of M, we have

$$|V(G)| = |B| + 1 = \mathrm{rank}(M) + 1 = 7.$$

As M has 10 elements, G has 10 edges. So the average degree of a vertex is $2|\mathrm{Edges}(G)|/|\mathrm{Vert}(G)| = 20/7 < 3$ and there exists a vertex in G of degree at most 2, which implies M^* contains a loop or parallel element. But K_5 contains no such cycles, a contradiction. □

A brief detour on graphic matroids

Some useful results on graphic matroids:

- A graph G is planar if and only if $M^*(G)$ is graphic.
- If G is connected planar, $M(G^*) \cong M^*(G)$.
- Neither $M^*(K_5)$ nor $M^*(K_{3,3})$ is graphic.
- The class of matroids M for which both M and M^* are graphic is minor-closed.
- A regular (i.e. representable over every field) matroid is graphic if and only if it has no minor isomorphic to $M^*(K_5)$ or $M^*(K_{3,3})$.

A brief detour on representability

Previously, we saw that U_4^2 is not graphic. We will now produce a matroid which is not representable.

The *Vamos matroid*:

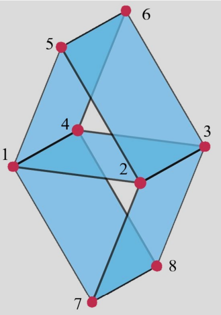

Figure 1: Geometric representation of the Vamos matroid V_8.

V_8 has eight elements, rank 4, all sets of three or fewer elements are independent, and has five 4-element circuits depicted as faces.

A brief detour on representability

Proof (V_8 is not representable):

Assume that V_8 is representable over $\mathbb{F}$ by a matrix A with columns labeled $[8]$. We write $W(i,j)$ to mean the span of the columns labeled by i,j.

- $\dim W([8]) = 4$ as rank(V_8)=4.

$\square$

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- $\dim[W(5,6) \cap W(1,2,3,4)] =$ $\dim W(5,6) + \dim W(1,2,3,4) - \dim\,[W(5,6) + W(1,2,3,4)] = 1$ by the usual formula for intersection.

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- $\dim[W(1,4,5,6) \cap W(1,2,3,4)] = \dim W(1,4,5,6) + \dim W(1,2,3,4) - \dim[W(1,4,5,6) + W(1,2,3,4)] = 2.$

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- $\dim[W(1,4,5,6) \cap W(1,2,3,4)] = \dim W(1,4,5,6) + \dim W(1,2,3,4) - \dim[W(1,4,5,6) + W(1,2,3,4)] = 2.$
- $\dim W(1,4) = 2$ and $W(1,4) \subseteq W(1,4,5,6) \cap W(1,2,3,4)$.

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- $\dim[W(1,4,5,6) \cap W(1,2,3,4)] = \dim W(1,4,5,6) + \dim W(1,2,3,4) - \dim[W(1,4,5,6) + W(1,2,3,4)] = 2.$
- $\dim W(1,4) = 2$ and $W(1,4) \subseteq W(1,4,5,6) \cap W(1,2,3,4)$.
- $W(1,4) = W(1,4,5,6) \cap W(1,2,3,4) \supseteq W(5,6) \cap W(1,2,3,4)$.

□

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- $W(1,4) = W(1,4,5,6) \cap W(1,2,3,4) \supseteq W(5,6) \cap W(1,2,3,4) = \langle v \rangle$.

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- Similarly, $W(2,3) \supseteq W(5,6) \cap W(1,2,3,4) = \langle v \rangle$.

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- Similarly, $W(2,3) \supseteq W(5,6) \cap W(1,2,3,4) = \langle v \rangle$.
- $\dim[W(1,4) \cap W(2,3)] =$ $\dim W(1,4) + \dim W(2,3) - \dim[W(1,4) + W(2,3)] = 1$.

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- Similarly, $W(2,3) \supseteq W(5,6) \cap W(1,2,3,4) = \langle v \rangle$.
- $\dim[W(1,4) \cap W(2,3)] =$
  $\dim W(1,4) + \dim W(2,3) - \dim[W(1,4) + W(2,3)] = 1$.
- Therefore, $\langle v \rangle = W(1,4) \cap W(2,3) = W(5,6) \cap W(1,2,3,4)$.

□

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- Similarly, $W(2,3) \supseteq W(5,6) \cap W(1,2,3,4) = \langle v \rangle$.
- $\dim[W(1,4) \cap W(2,3)] = \dim W(1,4) + \dim W(2,3) - \dim[W(1,4) + W(2,3)] = 1$.
- Therefore, $\langle v \rangle = W(1,4) \cap W(2,3) = W(5,6) \cap W(1,2,3,4)$.
- By symmetry, we can replace $\{5,6\}$ with $\{7,8\}$ in the above.

□

A brief detour on representability

Proof (V_8 is not representable):

Assume that V_8 is representable over $\mathbb{F}$ by a matrix A with columns labeled [8]. We write $W(i, j)$ to mean the span of the columns labeled by i, j.

- In total:

$$\langle v \rangle = W(1,4) \cap W(2,3) = W(5,6) \cap W(1,2,3,4)$$

$$= W(7,8) \cap W(1,2,3,4).$$

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- So, $\langle v\rangle \subseteq W(5,6)\cap W(7,8)$.

$\square$

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$$= W(7,8) \cap W(1,2,3,4).$$

- So, $\langle v \rangle \subseteq W(5,6) \cap W(7,8)$.
- But then,

$$1 \leq \dim[W(5,6) \cap W(7,8)]$$

$$= \dim W(5,6) + W(7,8) - \dim[W(5,6) + W(7,8)] = 2 + 2 - 4.$$

□

A brief detour on representability

The Vamos matroid is the smallest non-representable matroid. It is self-dual.

Minors, extensions, and quotients

Deletion

Definition (Matroid deletion)

Let $M = (E, \mathcal{I})$ be a matroid given by independent sets and let $X \subseteq E$. The *deletion* of X from M is denoted $M \backslash X$ and is given by the ground set $E - X$ and independent sets

$$\mathcal{I}|(E - X) := \{I \subseteq E - X : I \in \mathcal{I}\}.$$

This is sometimes called the *restriction* of M to $E - X$.

Contraction

Definition (Matroid contraction)

Let $M = (E, \mathcal{I})$ be a matroid given by independent sets and let $X \subseteq E$. The *contraction* of X from M is defined as

$$M/X := (M^* \backslash X)^*.$$

- Take the dual.
- Delete.
- Take the dual again.

Examples of deletion and contraction

- Uniform matroids:
  - $U_n^r \backslash X \cong \begin{cases} U_{n-|X|}^{n-|X|} & \text{if } n \geq |X| \geq n-r, \\ U_{n-|X|}^{r} & \text{if } |X| < n-r. \end{cases}$
  - $U_n^r / X \cong \begin{cases} U_{n-|X|}^{0} & \text{if } n \geq |X| \geq r, \\ U_{n-|X|}^{r-|X|} & \text{if } |X| < r. \end{cases}$
- Graphic matroids:
  - Graph theoretic deletion and contraction on any underlying graph.
- Regular matroids:
  - Deletion is removing columns
  - Contraction: (1) pick a single column label to contract, (2) adjust the matrix so that the column contains only a single non-zero entry, (3) delete the row and column.

Matroid minors

Definition (Matroid minor)

A matroid N is called a *minor* of M if it is obtained from M via a sequence of contractions and deletions.

- Any minor can be written in the form $M \backslash X / Y$ for disjoint X, Y.
- N is a minor of M if and only if N^* is a minor of M^*.
- Many desirable classes of matroids are minor (and dual) closed.
- Any minor can be written as a sequence of single element contractions and deletions.
- Deletion and contraction are commutative operations.
- (Scum Theorem): The formation of a minor can be thought of as a two-step process- contraction to obtain the rank and deletion to remove excess elements. In the loop-free case, one can always contract by a flat.

Single-element extension

Definition (Single-element extension)

A matroid M is called a single-element extension of N if N is a minor of M obtained by deleting a single element.

There are three boring extensions:

- Add a loop, this does not increase the rank.
- Add a coloop (i.e. an element present in every basis), this increases the rank.
- Add a parallel element.

Modular cuts

The more interesting extensions are known as *modular cuts*:

- Begin with a matroid $M = (E, \mathcal{F})$ given by its flats.
- Create a new set $\mathcal{M} \subseteq \mathcal{F}$ satisfying

(Conferred) If $F \in \mathcal{M}$ and $F \subseteq F'$, then $F' \in \mathcal{M}$.

(Modular) If $F_1, F_2 \in \mathcal{M}$ satisfy

$$r(F_1) + r(F_2) = r(F_1 \cup F_2) + r(F_1 \cap F_2)$$

then $F_1 \cap F_2 \in \mathcal{M}$.

- Then $\{F \cup e : F \in \mathcal{M}\} \cup \{F : F \notin \mathcal{M}\}$ is the set of flats of a single-element extension of M.

SINGLE ELEMENT EXT. $\longleftrightarrow$ MODULAR CUTS

Matroid quotients

Definition (Matroid quotient)

A matroid Q is a quotient of M if there exists a matroid N on ground set E_N and some $X \subseteq E_N$ such that

$$M = N \backslash X$$

and

$$Q = N/X$$

Matroid quotients

If M_1, M_2 are matroids with rank functions r_1, r_2 on common ground set E, then following are all equivalent:

- M_2 is a quotient of M_1.
- Every flat of M_2 is a flat of M_1.
- If $X \subseteq Y \subseteq E$, then $r_1(Y) - r_1(X) \geq r_2(Y) - r_2(X)$.
- Every circuit of M_1 is a union of circuits of M_2
- If $X \subseteq E$, then $\mathrm{cl}_1(X) \subseteq \mathrm{cl}_2(X)$.

Elementary quotients

The previous concepts come together in the notion of an *elementary quotients*.

Let $M +_{\mathcal{M}} e$ be the extension of M by e with respect to a modular cut $\mathcal{M}$. Then

$$(M +_{\mathcal{M}} e)/e$$

is the *elementary quotient* of M with respect to $\mathcal{M}$.

All quotients can be formed by taking a sequence of elementary quotients.

Algebraic structures on all matroids

Matroid union and intersection

Definition (Matroid union)

Given a set of matroids $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$, their union is the matroid $M_1 \vee M_2$ on the ground set $E_1 \cup E_2$ and set of independent sets given by

$$\mathcal{I} = \{I_1 \cup I_2 : I_i \in \mathcal{I}_i\}$$

- This does give rise to a matroid and the ground sets need not be disjoint.
- Note that M, N are quotients of $M \vee N$.
- When their ground sets are disjoint, we write $M_1 \oplus M_2$ and call it the direct sum.

Matroid union and intersection

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Definition (Matroid intersection)

Let M, N be matroids on a ground set E. The intersection of M and N is defined as

$$M \wedge N := (M^* \vee N^*)^*.$$

- $M \wedge N$ is a quotient of both M and N.
- There is another definition of matroid intersection, but it does not always yield a matroid.
- Contraction distributes over intersection.

Notation

For the remainder of this talk:

- M is a matroid on $[n]$.
- The rank of M is $r > 1$.
- M is loopfree.
- $\mathfrak{C}_{r,n}$ is the set of all proper chains of length r on $[n]$, i.e.

$$\emptyset \subset F_1 \subset ... \subset F_r = [n].$$

- $V_{r,n}$ is the free $\mathbb{Z}$-module whose coordinates are indexed by elements of $\mathfrak{C}_{r,n}$.
- $\mathbb{M}_{r,n}^{free}$ is the free $\mathbb{Z}$-module with generators the set of all loopfree matroids of rank r on $[n]$.

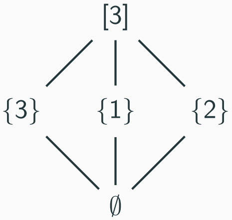
Indicator vector of chains of flats

For each (r, n) pair, there is a homomorphism

$$\Phi_{r,n} : \mathbb{M}^{free}_{r,n} \to V_{r,n}$$

which takes our matroid M and maps it to a vector v_M where, for each chain indexing v_M,

$$(v_M)_C := \begin{cases} 1, & C \text{ is a chain of flats in } M \\ 0, & \text{otherwise} \end{cases}$$



$\mathcal{L}(M(G))$ for a triangle graph, $r(M) = 2$

$v_M = (1, 1, 1, 0, 0, 0)$

Chains:

1: $\emptyset \subseteq \{1\} \subseteq [3]$

2: $\emptyset \subseteq \{2\} \subseteq [3]$

3: $\emptyset \subseteq \{3\} \subseteq [3]$

4: $\emptyset \subseteq \{2, 3\} \subseteq [3]$

5: $\emptyset \subseteq \{1, 3\} \subseteq [3]$

6: $\emptyset \subseteq \{1, 2\} \subseteq [3]$

The intersection ring of matroids

The *intersection ring of matroids on* $[n]$ is the $\mathbb{Z}$-module

$$\mathbb{M}_n := \oplus_{r=1}^{n} \mathbb{M}_{r,n}$$

where $\mathbb{M}_{r,n} = \mathbb{M}_{r,n}^{free} / ker(\Phi_{r,n})$.

It is a true ring with product

$$M \cdot N := \begin{cases} M \wedge N, & \text{if } M \wedge N \text{ is loopfree} \\ 0, & \text{otherwise} \end{cases}$$

This corresponds to the tropical intersection product.

Matroid homology

There are two sets of natural $\mathbb{Z}$-module homomorphisms, one from deletion and one from contraction:

$$d_i(M) := \begin{cases} M\backslash i, & \text{if } i \text{ is not a coloop of } M \\ 0, & \text{otherwise} \end{cases}$$

$$c_i(M) := \begin{cases} M/i, & \text{if } \mathrm{cl}_M(\{i\}) = \{i\} \\ 0, & \text{otherwise} \end{cases}$$

In the tropical case, these correspond to coordinate projection and intersection products with a hyperplane at $x_i = \infty$.

Matroid homology

Let $\mathbb{M} = \oplus_n \mathbb{M}_n$. Then there are two natural boundary maps:

$$\partial_d : \mathbb{M} \to \mathbb{M} \quad , \quad M \mapsto \sum (-1)^i d_i(M)$$

$$\partial_c : \mathbb{M} \to \mathbb{M} \quad , \quad M \mapsto \sum (-1)^i c_i(M).$$

These are in fact differentials and give rise to homology groups on minor-closed classes.

Open questions

- Kontsevich homology
- Zero-conjecture
- $\ker \Phi_{r,n}$
- Matroid polytope of matroids
- Other minor-closed classes

Hopf algebra of matroids

Definition

The *Hopf algebra of matroids* $\mathbb{M}^{Hopf}$ is the free $\mathbb{Z}$-module generated by matroids modulo isomorphisms with product and coproduct

$$\cdot : \mathbb{M}^{Hopf} \otimes \mathbb{M}^{Hopf} \to \mathbb{M}^{Hopf} \quad , \quad M \cdot N := M \oplus N$$

$$\Delta : \mathbb{M}^{Hopf} \to \mathbb{M}^{Hopf} \otimes \mathbb{M}^{Hopf} \quad , \quad \Delta(M) := \sum_{S \subseteq E} (M/(E-S)) \otimes (M/S)$$

The antipode of the Hopf algebra of matroids is given by

$$S(M) = \sum_{P_N \text{ face of } P_M} (-1)^{c(N)} N$$

where P_M is the matroid polytope and $c(N)$ is the number of connected components of N.

NB! A connected component of a matroid M is an equivalence class of the relation $\sim$ given by $e \sim f$ if $e = f$ or $\{e, f\} \subseteq C$ a circuit of M.

Sources

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Questions?