Seiberg Witten Background

Lie gps

Recall for $n \geq 3$, $\pi_1(SO(3)) = \mathbb{Z}_2$.

Def. $Spin(n) =$ universal cover of $SO(n)$

$$Spin^c(n) = Spin(n) \times_{\mathbb{Z}_2} U(1) = Spin(n) \times U(1) / \langle(-1,-1)\rangle$$

non-id el in Spin that maps to id in $SO(n)$

Some important maps:

$$\begin{tikzcd} Spin(n) \arrow[d, "2"] \\ SO(n) \end{tikzcd} \qquad \begin{tikzcd} & Spin^c(n) = Spin(n) \times_{\mathbb{Z}/2} U(1) \arrow[dl] \arrow[dr, "\det\ (z \mapsto z^2)"] & \\ SO(n) & & U(1) \end{tikzcd}$$

Eg. $(n=3)$

unit quaternions

$$\begin{tikzcd} Spin(3) \cong SU(2) \cong Sp(1) \arrow[d, "2"] \\ SO(3) \end{tikzcd}$$

not obvious

$$\begin{tikzcd} & Spin^c(3) = SU(2) \times_{\mathbb{Z}/2} U(1) \cong U(2) \arrow[dl] \arrow[dr, "\det\ (\text{literally})"] & \\ SO(3) & & U(1) \end{tikzcd}$$

$(n=4)$

$$\begin{tikzcd} Spin(4) \cong SU(2) \times SU(2) \arrow[d] \\ SU(2) \times_{\mathbb{Z}/2} SU(2) \cong SO(4) \end{tikzcd}$$

$$\mathrm{Spin}^c(4) = (SU(2) \times SU(2)) \times_{\mathbb{Z}/2} U(1)$$
$$\cong U(2) \times_{U(1)} U(2) = U(2) \times U(2) / \langle (e^{i\theta}\mathbb{1}, e^{-i\theta}\mathbb{1}) \rangle$$
$$\cong \{ (A,B) \in U(2) \times U(2) \mid \det A = \det B \}$$

These all derive from Clifford algebras.

Let V be a v.s. over k, char $\neq 2$, with a norm $|\cdot|$.

in $k \subset k \oplus V \oplus V^{\otimes 2} \oplus \dots$

Def. The Clifford algebra $$Cl(V) = \underbrace{T^*V}_{\bigoplus_{k=0}^{\infty} V^{\otimes k}} / v \otimes v \sim |v|^2 \cdot 1$$

Because this relation associates elements in $V^{\otimes k}$ with something in k, this doesn't have a $\mathbb{Z}$-graded algebra structure, but it does have a $\mathbb{Z}_2$-graded algebra structure:

$$Cl(V) = Cl^o(V) \oplus Cl^e(V)$$

These are related to spin gps by:

$$\mathrm{Spin}(n) < Cl(\mathbb{R}^n)^\times$$

$$\mathrm{Spin}^c(n) < Cl(\mathbb{C}^n)^\times$$

Claim: there exist vector spaces $\$_n$, $\$_n^c$ and fundamental representations

$$\mathrm{Spin}(n) \to \mathrm{Aut}(\$_n), \qquad \mathrm{Spin}^c(n) \to \mathrm{Aut}(\$_n^c)$$

along with maps

$$\rho : \mathbb{R}^n \to \mathrm{End}(\$_n), \qquad \rho : \mathbb{C}^n \to \mathrm{End}(\$_n^c)$$

These are called the Clifford multiplication maps

Eg $(n=3)$

$\$_3 = \mathbb{H}$ as a quaternionic v.s. over itself.

$Spin(3) \curvearrowright \mathbb{H}$ acts as $Sp(1) \curvearrowright \mathbb{H}$.

$\$_3^{\mathbb{C}} = \mathbb{C}^2$ (same underlying space but different multiplication!)

$Spin^{\mathbb{C}}(3) \curvearrowright \mathbb{C}^2$ as $U(2) \curvearrowright \mathbb{C}^2$.

The real Clifford multiplication is

$$\rho : \mathbb{R}^3 \to End(\mathbb{H}) = M_{1\times 1}(\mathbb{H})$$
$$e_1 \mapsto i$$
$$e_2 \mapsto j$$
$$e_3 \mapsto k$$

The complex Clifford multiplication is

$$\rho : \mathbb{C}^3 \to End_{\mathbb{C}}(\mathbb{C}^2) = M_{2\times 2}(\mathbb{C})$$

$$e_1 \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \sigma_1$$

$$e_2 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sigma_2$$

$$e_3 \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \sigma_3$$

$(n=4)$

$\$_4 = \mathbb{H} \oplus \mathbb{H} \rightsquigarrow Spin(4) \cong Sp(1) \times Sp(1) \curvearrowright \mathbb{H} \oplus \mathbb{H}$

$\$_4^{\mathbb{C}} = \mathbb{C}^2 \oplus \mathbb{C}^2 \rightsquigarrow Spin^{\mathbb{C}}(4) \cong U(2) \times_{U(1)} U(2) \curvearrowright \mathbb{C}^2 \oplus \mathbb{C}^2$.

Real Clifford mult:

$$\rho : \mathbb{R}^4 \to End(\mathbb{H} \oplus \mathbb{H}) = M_{2\times 2}(\mathbb{H})$$
$$e_0 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$e_1 \mapsto \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$e_2 \mapsto \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$$
$$e_3 \mapsto \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}$$

Complex Clifford mult:

$$\rho : \mathbb{C}^4 \to \mathrm{End}(\mathbb{C}^2 \oplus \mathbb{C}^2) = M_{4\times 4}(\mathbb{C})$$

$$e_0 \longmapsto \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$e_i \longmapsto \begin{pmatrix} 0 & -\sigma_i^* \\ \sigma_i & 0 \end{pmatrix}$$

$*$ = conjugate transpose