$\mathrm{Spin}^{(c)}$ structures

Def. Let (M^n, g) be a Riem mfd. Then we have the diagram:

$$\begin{tikzcd} \mathrm{Spin}^{(c)}(n) \arrow[r, hook] \arrow[d] & P \arrow[d] \\ SO(n) \arrow[r, hook] & Fr(M) \arrow[d] \\ & M \end{tikzcd}$$

Frame bundles are classified by maps $M \to BSO(n)$. Accordingly, a $\mathrm{Spin}^{(c)}$-str is a lift

$$\begin{tikzcd} & B\mathrm{Spin}^{(c)}(n) \arrow[d] \\ M \arrow[r] \arrow[ur, dashed] & BSO(n) \end{tikzcd}$$

To understand these lifts, it helps to know the structure of $B\mathrm{Spin}^{(c)}(n)$:

$$\begin{tikzcd} \mathbb{RP}^\infty = K(\mathbb{Z}_2, 1) \arrow[r, hook] & B\mathrm{Spin}(n) \arrow[d] \\ & BSO(n) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{CP}^\infty = K(\mathbb{Z}, 2) \arrow[r, hook] & B\mathrm{Spin}^c(n) \arrow[d] \\ & BSO(n) \end{tikzcd}$$

$\Rightarrow \quad \mathrm{Spin}(M) \leftrightarrow H^1(M; \mathbb{Z}_2) \qquad \mathrm{Spin}^c(M) \leftrightarrow H^2(M; \mathbb{Z})$

We get from these $\mathrm{spin}^{(c)}$-structures spinor bundles by taking the associated bundle to the fundamental representations.

spin: $\$ = P \times_{\mathrm{Spin}(n)} \$_n$

spin^c: $\$ = P \times_{\mathrm{Spin}^c(n)} \$_n^c$

Concretely, for $n=3$,

$$Spin(3) = SU(2) \curvearrowright \mathbb{H} \quad \Rightarrow \quad \begin{array}{c} \mathbb{H} \hookrightarrow \$ \\ \downarrow \\ M \end{array}$$

$$Spin^c(3) = U(2) \curvearrowright \mathbb{C}^2 \quad \Rightarrow \quad \begin{array}{c} \mathbb{C}^2 \hookrightarrow \$ \\ \downarrow \\ M \end{array}$$

For $n=4$,

$$\$_n \cong \mathbb{H} \oplus \mathbb{H} = \$_n^+ \oplus \$_n^-, \text{ acted on by } Spin(4) = SU(2) \times SU(2)$$

$$\begin{array}{c} \mathbb{H} \hookrightarrow \$^\pm \\ \downarrow \\ M \end{array}$$

$$\$_n^c = \mathbb{C}^2 \oplus \mathbb{C}^2 = \$_n^{+c} \oplus \$_n^{-c}, \text{ acted on by } Spin^c(4) \;\; U(2) \times_{U(1)} U(2)$$

So really spinor bundles are just $\mathbb{H}\oplus\mathbb{H}$- or $\mathbb{C}^2 \oplus \mathbb{C}^2$-bundles with a Clifford multiplication structure.

<u>Specify to $Spin^c(4)$:</u>

The Clifford mult ρ looks like

$$\rho : \underbrace{\Lambda^1 M}_{T^*M} \otimes \mathbb{C} \hookrightarrow \underbrace{\underline{\mathfrak{sl}(\$)}}$$

"traceless endomorphisms of $\$$", meaning in charts, they can be rep'd by traceless matrices

We can extend this to a multiplication on complexified 2-forms using a Leibniz rule:

$$\rho_\pm : \Lambda^2_\pm \otimes \mathbb{C} \xrightarrow{\cong} \mathfrak{sl}(\$^\pm, \$^\mp) \subset Hom(\$^\pm, \$^\mp)$$

where if g is the metric, the associated Hodge star $*_g$ gives a splitting

$$\underset{SD}{\underset{\uparrow}{\Lambda^2_+ M}} \oplus \underset{ASD}{\underset{\uparrow}{\Lambda^2_- M}}$$

Connections

Let $E \to M$ be a v.b. A connection ∇ is a map

$$\nabla : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$$

In the special case that $E = TM$, $\exists!$ connection compatible with g:

$$\underbrace{\nabla^{LC}}_{\text{Levi-Civita}} : \Gamma(TM) \longrightarrow \Gamma(T^*M \otimes TM)$$

For $E = \$$, we have Spin^c connections satisfying

$$\underbrace{\nabla}_{\text{connection}}{}_{\underbrace{v}_{\text{tangent vectors}}} \rho(\omega) \cdot \underbrace{\varphi}_{\in \Gamma(\$)} = \rho(\omega) \nabla_v \varphi + \rho(\nabla_v^{LC} \omega) \varphi$$

The set of Spin^c-connections ($\$$) then is affine over $\Omega^1(M; i\mathbb{R})$, and so we can write it as

$$\text{Spin}^c \text{ conns on } \$ \leftrightarrow \underbrace{\{A_0\}}_{\text{some base Spin}^c \text{ conn}} + \Omega^1(M; i\mathbb{R})$$

Dirac operators

Let $A \in \{\text{Spin}^c \text{ conns on } \$\}$. Define the Dirac operator

$$\not{D}_A : \Gamma(\$) \to \Gamma(\$)$$

splitting as

$$\not{D}_A^{\pm} : \Gamma(\$^{\pm}) \to \Gamma(\$^{\mp})$$

by

$$\begin{tikzcd} \Gamma(\$^{\pm}) \arrow[r, "\nabla_A"] \arrow[rr, bend left, dashed, "\not{D}_A^{\pm}"] & \Gamma(T^*M \otimes \$^{\pm}) \arrow[r, "\rho"] & \Gamma(\$^{\mp}) \end{tikzcd}$$

SW eqns

Let X^4 be closed, oriented, and $\$ \to X$ a spinor bundle. We want to consider pairs

$$(A, \varphi) \in \mathcal{A} = \mathrm{Conn}_{\mathrm{Spin}^c}(X) \oplus \Gamma(\$^+) \cong \Omega^1(X, i\mathbb{R}) \oplus \Gamma(\$^+).$$

We would like to look for solutions to

$$\begin{cases} F_A^+ = \rho^{-1}(\varphi \otimes \varphi^*)_0 \\ \not{D}_A^+ \varphi = 0 \end{cases}$$

In detail:

F_A^+ = self-dual part of the curvature F_A of A, where $A = \overset{\text{flat}}{A_0} + a$, $a \in \Omega^1(X, i\mathbb{R})$

$$\Rightarrow F_A \in \Omega^2(X, i\mathbb{R})$$

$$\Rightarrow F_A^+ \in \Omega^2_+(X, i\mathbb{R})$$

φ = section of $\$^+$

φ^* = section of $(\$^-)^*$

$\Big\} \in \Gamma(\$^+ \otimes (\$^-)^*) \cong \Gamma(\mathrm{Hom}(\$^+, \$^-))$

$-_0$ = traceless part $\Rightarrow (\varphi \otimes \varphi^*)_0 \in \Gamma(\mathrm{Hom}(\$^+, \$^-))_0$

$$\Omega^2_+(X, i\mathbb{R}) \xrightarrow[\cong]{\rho} \mathfrak{sl}(\$^+, \$^-)$$

We can interpret the SW eqns as a map b/w ∞-dim v.s.'s. Fixing A_0, the sol set to the SW eqns are the 0 set of

$$sw(A, \varphi) = (F_A^+ - \rho^{-1}(\varphi \otimes \varphi^*)_0 , \not{D}_A^+)$$

The problem is that the solution space $\mathcal{M}$ is ∞ dim'l and has a bunch of extra symmetries.

The gauge gp is $\mathcal{G} = \mathrm{Maps}_{C^\infty}(X, S^1)$ for the SW eqns. $\mathcal{G}$ acts on $\mathcal{M}$ by:

$u \in \mathcal{G} \to u.(A, \varphi) = (A - u^{-1}du, u \cdot \varphi)$

When we mod out by this gp, we get the SW moduli space:

$$\mathcal{M}_{SW}(X, \mathfrak{s}, g) = \mathcal{M}/\mathcal{G}$$

We want $\mathcal{M}_{SW}$ to be a mfd and be topologically invt (independent of g). Proving this requires us to (a) Banach-ify (b) perturb the eqns in a way that requires $b_2^+ > 0$.