EQUIVARIANT STABLE HOMOTOPY THEORY FOR BAUER FARUTA

JEMMA SCHRODER

1. Stable Homotopy Theory

1.1. Why stable homotopy theory?

Definition 1.1. The kth homotopy group of a pointed space (X, x_0) is givenby $\pi_k(X, x_0) = [(S^n, s_0) \to (X, x_0)]$ by homotopy classes of maps from the k-sphere into X, where the homotopy is through maps of this same form.

Lemma 1.2. *$\pi_k(X, x_0)$ has a group structure, where addition of two maps f, g is given by the composition $S^k \xrightarrow{c} S^k \vee S^k \xrightarrow{f \vee g} X$. The map $c : S^k \to S^k \vee S^k$ is defined as collapsing the equator $S^{k-1} \subset S^k$ to a point, where s_0 was chosen to lie in S^{k-1}.*

Definition 1.3. Given two based spaces $(X, x_0), (Y, y_0)$, the smash product of X and Y is

$$X \wedge Y = \frac{X \times Y}{X \vee Y}$$

Definition 1.4. For a based space (X, x_0), the reduced suspension ΣX is $X \wedge S^1$.

Proposition 1.5. $S^n \wedge S^k = S^{n+k}$

Remark 1.6. The case we'll be using the most is when $k = 1$. This means that $S^n \wedge S^1 = S^{n+1}$. But recall that $S^n \wedge S^1$ is exactly the definition of ΣS^n. So suspension acts on spheres by bumping up their dimension!

Remark 1.7. Sometimes the reduced suspension will be called the suspension. It is always the reduced suspension

Definition 1.8. Given a pointed space (X, x_0) and some integer $k \geq 0$, the suspension homomorphism is given by

$$\Sigma : \pi_k(X) \to \pi_k k + 1(\Sigma X), \ [f]_* \mapsto [f \wedge id_{S^1}]_*$$

The map $f \wedge id_{S^1} \in \pi_k k + 1(\Sigma X)$ is defined as

$$f \wedge id_{S^1} : S^{k+1} \cong \Sigma S^k \to \Sigma X$$

, where $x \wedge t \mapsto f(x) \wedge t$.

Alternatively, one can use the fact that $\pi_k(X) \cong \pi_{k-1}(\Omega X)$, and define a mep $X \to \Omega\Sigma X$ by $x \mapsto [\gamma : t \to x \wedge t]$. This induces a map $\pi_k(X) \to \pi_k(\Omega\Sigma X =) \cong \pi_{k+1}(\Sigma X)$. By repetedly suspending X, we get the following sequence of homotopy groups:

$$\pi_k(X) \xrightarrow{\Sigma} \pi_{k+1}(\Sigma X) \xrightarrow{\Sigma} \dots \xrightarrow{\Sigma} \pi_{k+r}(\Sigma^r X) \xrightarrow{\Sigma} \dots$$

which leads us to the following theorem.

Theorem 1.9 (Freudenthal Suspension). *Let X be an $(n-1)$ connected space for $n \geq 1$. Then the suspension homomorpism $\Sigma : \pi_k(X) \to \pi_{k+1}(\Sigma X)$ is a bijection for $k < 2n - 1$ and a surjection for $k = 2n - 1$*

This theorem is what motivates the definition of stable homotopy theory. If we view X as a CW complex, it is built from spheres of various dimensions. Since suspending a sphere changes it to a sphere of one dimension higher, repeatedly suspending some space X is akin to raising it's dimension without changing the internal structure.

1.2. **Stable Homotopy Groups.** Let X be a $(n-1)$ connected, ie, $\pi_k(X) = 0$ for all $0 < k \leq n-1$. Since $n - 1 < 2n - 1$, the Freudenthal Suspension theorem tell us that $\pi_k(X) \cong \pi_{k+1}(\Sigma X)$ for all $0 < k \leq n - 1$. In other words, $\pi_k(\Sigma X) = 0$ for all $0 < k \leq n$, so ΣX is n-connected. We can inductively repeat this process: Setting X now to be ΣX, we get that $\Sigma^2 X$ is $n + 1$ connected, and similarly, $\Sigma^r X$ to be $n + r$ connected. Applying Freudenthal again, we have that

$$\Sigma : \pi_{k+r}(\Sigma^r X) \to \pi_{k+r+1}(\Sigma^{r+1} X)$$

is an isomorphism for $k + r < 2(n + r) - 1$. Writing this in terms of r, for a fixed n and k, the suspension map is an isomorphism when $r > k - 2n + 1$. Since n and k are fixed, this means that for r large enough, $\pi_{k+r}(\Sigma^r X) \cong \pi_{k+r+1}(\Sigma^{r+1} X)$, so the homomorphisms in the sequence

$$\pi_k(X) \xrightarrow{\Sigma} \pi_{k+1}(\Sigma X) \xrightarrow{\Sigma} \dots \xrightarrow{\Sigma} \pi_{k+r}(\Sigma^r X) \xrightarrow{\Sigma} \dots$$

eventually all become isomorphisms, ie, the sequence stabilizes.

Definition 1.10. Let X be a $n - 1$ connected pointed space For $k \geq 0$, the kth stable homotopy group of X is defined as the $\pi_k^S(X) = \mathrm{colim}_r\, \pi_{k+r}(\Sigma^r X)$

Remark 1.11. This definition of stable homotopy groups works for any pointed space X, not necessarily the ones that are $n - 1$ connected.

Definition 1.12. The k-th stable homotopy group of the spheres is

$$\pi_k^S := \pi_k^S(S^0) = \mathrm{colim}_r\, \pi_{k+r}(S^r) = \pi_{k+n}(S^n)$$

where $n > k + 1$.

2. G-spaces and G-CW complexes

Definition 2.1. A G-space is a topolgical space X and a group G with a continuous action $G \times X \to X$ such that $ex = x$ and $g_1(g_2 x) = (g_1 g_2)x$.

Definition 2.2. A G map (or G-equivariant map) is a continuous map $f : X \to Y$ such trhat $f(gx) = gf(x)$.

If X is based, we can define a based G-space by a space with that the basepoint x_0 is a fixed point of G.
Given some space X, we would like a way to approximate it akin to usual non-equivariant CW complexes.

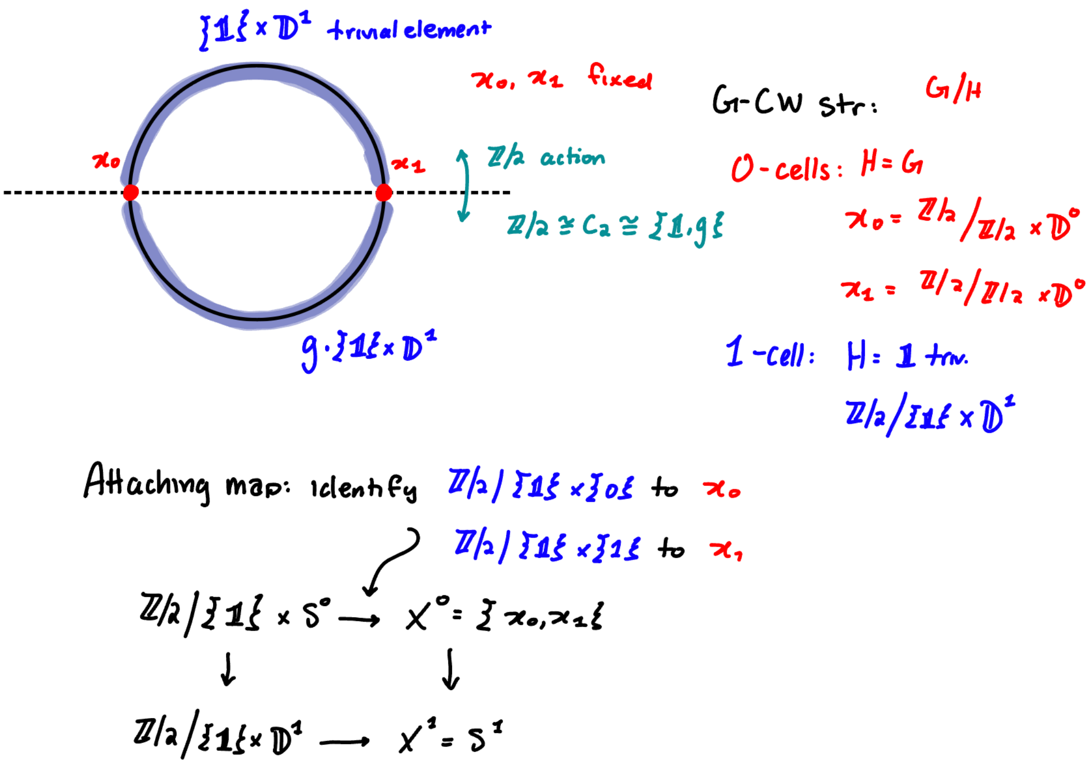
Definition 2.3. A G-CW complex X is the union of sub G-spaces $X = \cup_{i=d}^{n} X^n$ such that

(1) $X^n \subseteq X^{n+1}$ for all n
(2) X^0 is the disjoint union of orbits G/H, for $H \leqslant G$ any subgroup
(3) X^{n+1} is obtained from X^n by attaching G-cells $G/H \times D^{n+1}$ along attaching G-maps $G/H \times S^n \to X^n$

This gives the following equivariant pushout diagram:

$$\begin{array}{ccc} \coprod_\alpha G/H_\alpha \times S^n & \longrightarrow & X^n \\ \downarrow & & \downarrow \\ \coprod_\alpha G/H_\alpha \times D^{n+1} & \longrightarrow & X^{n+1} \end{array}$$

Example 2.4. Consider $X = S^1$ with the action of $\mathbb{Z}/2$ given by reflection across the x-axis



3. Some Representation Theory

In this section we will take G to be a compact Lie group. For our purposes (Bauer-Faruta), $G = U(1)$. If X is a G space, then when we suspend like usual, G can also act on this extra copy of S^1. Therefore we need a way to keep track of these group actions. To do so, instead of indexing the suspensions by $\mathbb{N}$, we index them by a new set based on the representations of G.

Definition 3.1. A representation of G on a vector space V over some field k is a group homomorphism

$$\rho : G \to Gl(V).$$

V is called the representation space, but we'll sometimes call V itself the representation.

Definition 3.2. Given a representation $\rho : G \to Gl(V)$, a linear subspace $W \subseteq V$ is G-invariant if $\rho(g)w \in W$ for all $g \in G$ and all $w \in W$. We call the restriction of ρ to any such subspace W a subrepresentation.

Remark 3.3. Going forward, we will restrict the codomain $GL(V)$ to the orthogonal matrices $O(V)$ via Gram-Schmidt. These representations are called orthogonal representations.

Example 3.4.

(1) The trivial representation is the map $\rho : G \to GL(V)$ that sends $g \mapsto 1$ for every $g \in G$.
(2) The regular representation is a representation on the vector space V generated by the elements of G. This is $k[G] = k^{\oplus|G|}$ the group ring, where k is again our base field.

Definition 3.5. A representation V is irreducible if its only subrepresentations are the trivial representations.

Proposition 3.6. *For any representation V of G, there exists an integer n such that V embeds in ρ_R^n, where ρ_R is the regular representation.*

Proof. The proof essentially follows from the following two facts.

(1) Any representation admits a decomposition into the direct sum of irreducible representations.
(2) Any irreducible representation embeds into the regular representation.

□

Definition 3.7. The representation ring $R(G)$ is the free abelian group generated by all isomorphism classes of representations of G, where $V + W \sim V \oplus W$. Equivalently, $R(G)$ can be defined as the Grothendieck completion of the set of all irreducible representations of G.

Remark 3.8. Since we are working with orthogonal representations, we will denote the representation ring of this retraction as $RO(G)$.

Definition 3.9. A G -universe $\mathcal{U}$ is a countable direct sum of representations such that $\mathcal{U}$ contains each of its subrepresentations and the trivial representation infinitely often.

Definition 3.10. A G-universe $\mathcal{U}$ is complete if it contains every irreducible representations up to isomorphism.

4. Borel (co)homology

Definition 4.1. Let X be a G-space. If G acts on X freely, then we define

$$H^*_G(X; R) := H^*(X/G; R),$$

where $H^*(-)$ is just regular cohomology.

This definition must be modified is G does not act on X freely. Recall that for G a compact Lie group, there exists a contractible space EG on which G acts freely. This is the universal principal G bundle, and the quotient $EG/G = BG$ is the classifying space. Recall that there is a bijection between the set of principal G bundles over X and $[X, BG]$. Therefore $EG \times X$ is homotopy equivalent to X, and we can use it to define G-equivariant cohomology on X

Definition 4.2. For X a G-space, we define the G-equivariant cohomology groups of X as

$$H^*_G(X; R) := H^*((EG \times X)/G; R) = H^*(EG \times_G X; R)$$

Remark 4.3. Note that we are modding out by G at the level of homology, not at the level of cochains. An alternative (later) construction by Bredon (called Bredon cohomology, fittingly) does exactly, this, using an equivariant cell structure on X that encodes more data.

This definition can be modified to hold for based spaces:

Definition 4.4. For (X, x_0) a based G-space, the G-equivariant reduced cohomology groups of X are

$$\tilde{H}^*_G(X; R) := \tilde{H}^*(EG_+ \times_G X; R)$$

Example 4.5. Recall that for $G = S^1$, $ES^1 = S^\infty$ and $BS^1 = \mathbb{C}P^\infty$. This can be seen by recalling that $EU(n) = \lim_{k\to\infty} Fr_{\mathbb{C}}(n, k)$, where $Fr_{\mathbb{C}}(n, k)$ is the space of orthonormal frames of n vectors in $\mathbb{C}^k$. Quotienting by $U(n)$, we get that $BU(n) = \lim_{k\to\infty} Gr_{\mathbb{C}}(n, k)$. Then set $n = 1$, so $U(1) = S^1$

5. Equivariant Stable Homotopy Theory

Definition 5.1. Given a G-representation V, define the G-sphere S^V as the one point compactification of V. Since G acts trivially at ∞, we choose this as the base point.

Remark 5.2. One can also view S^V as $D(V)/S(V)$ where D and S are the unit disk/sphere of V.

Definition 5.3. The equivariant suspension of a G-space X is $\Sigma^V X = S^V \wedge X$.

From this point on, we'll ignore some of the subleties of G-equivariant spectra in favor of brevity.

Definition 5.4 (Equivariant Freudenthal Suspension)**.** There exists some representation $V \in \mathcal{U}$ such that the suspension homomorphism

$$\Sigma^V : [X, Y]_G \to [\Sigma^V X, \Sigma^V Y]$$

is a bijection for V and all representations in $\mathcal{U}$ larger than V.

This allows us to define the unstable and stable homotopy groups of G-spaces:

Definition 5.5. Take V a G-representation. For any subgroup $H \leqslant G$, V can be thought of as an H-representation by restriction. The RO(G)-graded homotopy groups of X are thus defined to be

$$\pi^H_V(X) := [S^V, X]_H \cong [G_+ \wedge_H S^V, X]_G.$$

Remark 5.6. Note that these sets aren't actually groups–the base point doesn't behave well under the equivariant collapse map $S^V \to S^V \wedge S^V$. This technicality is resolved in the stable case, which is what we care about.

When we stabilize and take a colimit, RO(G) as before does not encode all possible information. Instead, we index over $\mathcal{U}$ a complete G-universe.

Definition 5.7. A stable G-map between a finite based G-CW complex X and any based G-space Y is

$$\{X, Y\}_G = \mathrm{colim}_{V\in\mathcal{U}}[\Sigma^V X, \Sigma^V Y]_G$$

where the colimit is taken over

$$[\Sigma^V X, \Sigma^V Y]_G \xrightarrow{-\wedge S^{W-V}} [\Sigma^W X, \Sigma^W Y]_G$$

for any $V \subseteq W$.

Taking $X = S^V$, we can define the equivariant stable homotopy groups.

Definition 5.8.

$$\pi^{\mathrm{Stab},G}_V(Y) = \{S^V, Y\}_G = \mathrm{colim}_{W\in\mathcal{U}}[\Sigma^W S^V, \Sigma^W Y]$$

6. The Bauer Faruta Case

Recall that we care about $\mathrm{U}(1) \cong S^1$ equivariant.

Proposition 6.1. *The ring of orthogonal representations of* S^1 *is*

$$\mathrm{RO}(S^1) = \langle \mathbb{R}, \mathbb{R}^2_k \rangle,$$

where $\mathbb{R}^2_k$ *is the representation given by the rotation*

$$R_k : \mathbb{R}^2 \to \mathbb{R}^2$$
$$\theta \mapsto \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

However, for Bauer Faruta, we will not be using a complete representation universe, which will turn out not to matter. Instead, we will use the universe $\mathcal{U}^{BF}_{S^1} = \langle \mathbb{R}, \mathbb{C} \rangle = \langle \mathbb{R}, \mathbb{R}^2_1 \rangle$.

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