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Goal: Define key objects & the monopole map

SW Invariants of 4-mflds

$(X^4, \mathfrak{s}) \rightsquigarrow SW(X^4, \mathfrak{s}) \in \mathbb{Z}$ (or $\mathbb{Z}/2$)

refine: $\rightsquigarrow BF^{S^1}(X^4; \mathfrak{s}) \in \{\underbrace{(Th(Pic^0(X)), ind(\not{D}))}, S^{b_2^+(X)}\}$

S^1 equiv. stable map between ___ and sphere

⇒ map to sphere ⇒ cohomotopy!

Rmk: what is the gp. str?

$X \to S^a$, $X \to S^b$ $\rightsquigarrow X \to S^a \vee S^b \hookrightarrow S^{a+b}$ Well-defined in a stable range

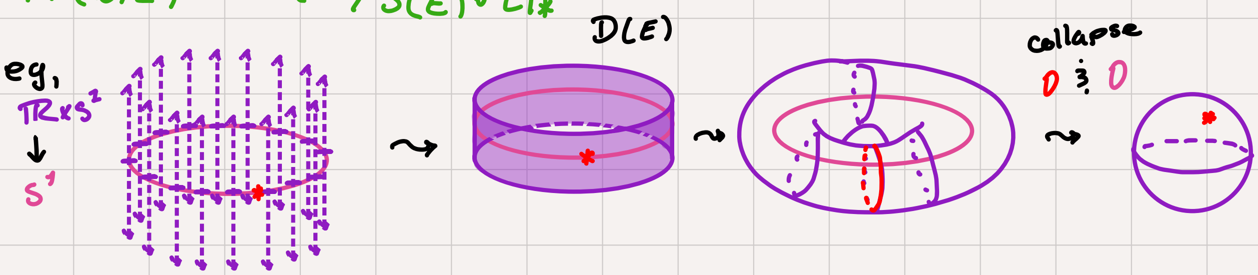
Defining the characters:

$Pic^0(X) := H^1(X;\mathbb{R}) / H^1(X, 2\pi i\,\mathbb{Z}) \cong T^{b_1(X)}$

$Ind(\not{D})$: $\not{D}_X$ induces a family of Dirac operators $\not{D}$ over $Pic^0(X)$. Fredholm ⇒ virtual index bundle

$ind(\not{D}) := ker(\not{D}) \ominus coker(\not{D})$

$Th(B,E) = D(E)/S(E) \vee E|_*$



Def $BF^{S^1}(X^4; \mathfrak{s}) \in \{(Th(Pic^0(X)), ind(\not{D})), S^{b_2^+(X)}\}$

$= \pi^{b_2^+(X)}_{S^1, st.}(Th(Pic^0(X)), ind(\not{D}))$

Why can we do this?

→ Nice Properties of SW eq's

up to finite order

Virtual fundamental domain, $[0,1]$ is FD for $\mathbb{R}/\mathbb{Z}$

1) ∃ a "global Coulomb slice" for the Gauge gp action

• instead of taking quotients, we'd rather look at a subspace.

hard to work w/ equiv. classes

Proj onto subspace = Modding out by gauge

This subspace is exactly the Coulomb slice

Even still, we're left with a finite amt. of Gauge symmetries, exactly $b_1(X)$ of them. Identifying these gives $T^{b_1(X)}$

2) SW moduli space is compact

$\mathcal{H}^1 \cong H^1$, $\mathcal{H}^1$ is space of harmonic 1-forms

Subsp of const. fxns in 0-forms of X $\Leftrightarrow$ $(\ker(d))^{\perp} \subset \Omega^0(X)$

Monopole Map

$$\tilde{\mu} : \underset{\uparrow}{\mathrm{Conn}} \times (\Gamma(S^+) \oplus \underset{\Omega^1(X; i\mathbb{R})}{\underline{\quad}})) \to \mathrm{Conn} \times (\Gamma(S^-) \oplus \Omega^2_+(X) \oplus \mathcal{H}^1(X, i\mathbb{R}) \oplus \Omega^0(X)/\mathbb{R})$$

Spin^c connections, affine space over $\Omega^1(X, i\mathbb{R})$

$$(A, \phi, a) \longmapsto (A, \underbrace{D_{A+a}\phi, \; F^+_{A+a} - \rho^{-1}(\phi \otimes \phi^*)_0}_{\text{Ordinary SW equ}^{\text{ns}}}, \underbrace{a_{\text{harm}}, \; d^*a}_{\text{gauge fixing}})$$

fix A, since really a family of maps over every spin^c str.

Coulomb slice

Pick a basepoint $b \in X$

$$\mathcal{G}_0 = \{ u : X \to S^1 \mid u(b) = 1 \} < \mathcal{G}$$

based gauge gp — normal gauge gp.

Still have residual $S^1 = \mathcal{G}/\mathcal{G}_0$ action, given by global twisting

$\rightsquigarrow$ mod out by $\mathcal{G}_0$. Define $\mathcal{A}$, $\mathcal{C}$ for notational convenience as:

Fix $A \in \mathrm{Conn}$, $\ker d \subset \Omega^1(X, i\mathbb{R})$

$$\mathcal{A} = \Big(\underline{(A + \ker d)} \times (\Gamma(S^+) \oplus \Omega^1(X, i\mathbb{R})) \Big) / \mathcal{G}_0$$

restrict to affine slice

$$\mathcal{C} = \Big((A + \ker d) \times \big((\underbrace{\Gamma(S^+)}_{\text{Spinors}} \oplus \underbrace{\Omega^2_+(X, i\mathbb{R})}_{\text{Self dual im val. 2-forms}}) \oplus \underbrace{\mathcal{H}^1(X, i\mathbb{R})}_{\text{harm im val 1-forms}} \oplus \ker d^{\perp} \big) \Big) / \mathcal{G}_0$$

$\Rightarrow \mu := \tilde{\mu} / \mathcal{G}_0 : \mathcal{A} \to \mathcal{C}$

monopole map

Claim $\mathcal{A}$, $\mathcal{C}$ are both ∞-dim bundles over $\mathrm{Pic}^0(X)$

$$\mathrm{Pic}^0(X) = \frac{\mathcal{H}^1(X, i\mathbb{R})}{\mathcal{H}^1(X, \underbrace{2\pi i\mathbb{Z}})}$$

↑ from Gauge $\mathcal{C}$ on ker d

Idea First mod out by $G_0^{harm^\perp} := \{ u \in G_0 \mid u_{harm} = 0 \} < G_0$

$\rightsquigarrow$ ∞-dim v-spaces$/G_0^{harm}$

$$G_0^{harm} \Leftrightarrow \mathcal{H}^1(X, 2\pi i \mathbb{Z})$$

Projecting onto a coordinate $\rightsquigarrow$ get $\mathrm{Pic} \subset \mathcal{A}$

— " — a_{harm} coord $\rightsquigarrow$ get $\mathrm{Pic} \subset \mathcal{C}$