Bauer-Furuta Learning Seminar **Spring 2025**

Week 5: The Monopole Map and the Leray-Schauder-Schwarz Construction

Abhishek Shivkumar

Following Bauer and Furuta,[1] we will review and attempt to motivate the monopole map, and sketch the construction of an element in a stable homotopy group of spheres associated to a compact perturbation of a linear Fredholm map between separable Hilbert spaces.

[1] Bauer and Furuta, *A stable cohomotopy refinement of Seiberg-Witten invariants: I*.

Recap

All gauge theory talks to date have been given by Ian who is more comfortable with gauge theory than the rest of us put together, so I began with a recap of what our inputs and outputs are.

Seiberg-Witten Invariants

The input data for Seiberg-Witten theory is a (for our purposes) closed 4-manifold M with Riemannian metric g, and a spin^c structure $\mathfrak{s}$ which determines spinor bundles $W_\pm \otimes_{\mathbb{R}} L$. With these data, we seek pairs (d_A, ψ) of connections d_A on the line bundle L^2 (L itself may or may not actually exist as a line bundle) and sections $\psi \in \Gamma(W_+ \otimes L)$ that satisfy

$$D_A^+\psi = 0 \qquad F_A^+ = \sigma(\psi)$$

where D_A^+ is the Dirac operator, F_A^+ is the self-dual part of the curvature, and $\sigma : W_+ \otimes L \to \bigwedge_+^2 TM$ is some quadratic map. In the simply-connected case (the only case many of this seminar's attendees have studied in detail), the moduli space $\mathcal{M}_L$ of such solutions to the *Seiberg-Witten equations* is an oriented compact smooth manifold (provided $\phi \neq 0$) sitting inside of $\mathbb{CP}^\infty$. We can then obtain numerical invariants by integrating $\mathrm{PD}([\mathbb{CP}^1])^k$ or (equivalently) $c_1(\tau)^k$ against the homology class of $\mathcal{M}_L$ where τ is the tautological line bundle over $\mathbb{CP}^\infty$ (provided the moduli space is even dimensional; otherwise the Seiberg-Witten invariant is set to 0). The moduli space itself is not an invariant of the input data, but its cobordism class is (ignoring wall-crossing when $b_2^+ = 1$), so the numerical invariant is well-defined.

We ignore the perturbed equation $F_A^+ = \sigma(\psi) + \phi$ for ϕ a self-dual 2-form as it (interestingly) does not seem to make an appearance in Bauer-Furuta theory.

Perhaps the first nontrivial invariant we might want to extract from this is $\dim \mathcal{M}_L$; however, this gives us nothing new as the formal or virtual dimension of the moduli space is determined by classical algebraic data (e.g. $c_1(L^2)$, $\chi(M)$, $\sigma(M)$) via the Atiyah-Singer index theorem.

One thing to notice in all this is that we start with a cobordism class of smooth, compact, oriented manifolds, and all we manage to extract is a single number (which is 0 half the time for trivial reasons — there is nothing to integrate $[\mathcal{M}_L]$ against), so we are throwing away lots of information.

Picard Tori

The Picard group I'm familiar with classifies (complex) line bundles or divisors, I'm not exactly sure how this compares.

One wrinkle in our discussion last time was the introduction of the *Picard torus*, $\mathrm{Pic}^0(M) = H^1(M;\mathbb{R})/H^1(M;\mathbb{Z})$, which is trivial in the simply-connected case; the Picard torus plays a role in non-simply-connected Seiberg-Witten theory in the classification of unitary connections on L^2.

Theorem 1.1: Classification of Connections[2]

If $H^1(M;\mathbb{Z})$ is free abelian (which is automatic for compact oriented manifolds M), L a complex line bundle over M, then there is a bijection between the set of unitary connections modulo S^1-gauge equivalence and $C \times \mathrm{Pic}^0(M)$ where C is the set of closed 2-forms representing $c_1(L)$ given by

$$[A] \mapsto \left(\frac{1}{2\pi}F_A, (e^{i\theta_1(A)}, \cdots, e^{i\theta_{b_1(M)}(A)})\right)$$

The θ_i capture the holonomy of the connection. Explicitly, pick curves $\gamma_1, \cdots, \gamma_{b_1}$ representing generators for $H_1(M)$, all based at some fixed $m \in M$; then, parallel translation around the γ_i defines automorphisms $\tau_i : L_m \to L_m$. Since our connection is unitary, the τ_i are unitary, i.e., rotation by some angle θ_i.

[2] Moore, *Lectures on Seiberg-Witten invariants*, Theorem on Classification of Connections

The curvature of a connection on L represents $c_1(L)$ (up to scale) by Chern-Weil theory.

The punchline is that C classifies connections modulo gauge in the simply-connected case, and the Picard torus is required when we have nontrivial π_1 to keep track of the extra holonomy data. Without rehashing the entire construction of the Seiberg-Witten moduli space, since one of the input data for the equations is connections on L^2 modulo gauge, this replaces $\mathbb{CP}^\infty$ in the above discussion by a $\mathbb{CP}^\infty$-bundle over $\mathrm{Pic}^0(M)$.

Note that the holonomy isomorphisms τ_i (and hence the θ_i) are gauge invariant since $u : M \to S^1$ acts on a connection d_A by

$$d_A \mapsto u \circ d_A \circ u^{-1} = d_A + ud(u^{-1})$$

Integrating d_A and ud_Au^{-1} around γ_i we get conjugate endomorphisms from L_m to itself, but the trace is conjugation invariant (and the trace of a 1×1 matrix just reads off the sole entry) so θ_i (and therefore) τ_i are gauge invariant.

The other relevant fact to us about connections is that the space of connections on M is an affine space (or torsor) over $\Omega^1(M)$.

In general (for higher-dimensional bundles and gauge groups) the holonomy τ_i itself will not be gauge invariant, but its trace always will be by the above argument; this quantity is called the *Wilson loop operator*. The Wilson loop operators for $3D$ Chern-Simons theory along knots and links in S^3 give the Jones polynomial.

The Monopole Map

A seemingly trivial reframing of these input data is to repackage the equations into a map, called the *monopole map*: set

$$\tilde{\mathcal{A}} = \{A_0 + \ker d\} \times (\Gamma(W_+ \otimes L) \oplus \Omega^1(M))$$
$$\tilde{\mathcal{C}} = \{A_0 + \ker d\} \times (\Gamma(W_- \otimes L) \oplus \Omega^2_+(M) \oplus H^1(M;\mathbb{R}) \oplus \Omega^0(M)/\mathbb{R})$$

where A_0 is a base spin^c connection to trivialize the $\Omega^1(M)$-torsor of connections.

Define $\tilde{\mu} : \tilde{\mathcal{A}} \to \tilde{\mathcal{C}}$ by

$$(A, \psi, a) \mapsto (A, D^+_{A+ia}\psi, F^+_{A+ia} - \sigma(\psi), a_{\mathrm{harm}}, d^*a)$$

This ridiculous map is exactly like what all object-oriented programming looks like to me.

In the small amount of the literature I've looked through (e.g, what is cited in the references), there are nontrivial variations in the definition of μ, including additional input terms (such as a smooth function on M).

$\tilde{\mu}$ is a map of $\mathrm{Map}(M, S^1) = \mathcal{G}$-spaces, with the obvious $\mathcal{G}$-actions on each side (including no action at all when none is obvious). a_{harm} is the harmonic part of a, where, by Hodge theory, $a = a_{\text{harm}} + df + d^*\omega$ for some 0 and 2-forms f and ω. The idea of this map appears to be to just cram all the relevant inputs and outputs into a single map, and then, by conservation of information, $\tilde{\mu}$ should contain all the same information as our original setup. In particular, after modding out by gauge, we expect to be able to recover $\mathcal{M}_L$ from $\tilde{\mu}^{-1}(\{A_0 + \ker d\}, 0, 0, 0, 0)$.

An Aside on Asinine Gauge Slicing

I wasted a good amount of time trying to understand the choices made in the definition of this map.

The inputs $\{A_0 + \ker d\}$ and $\Omega^1(M)$ together give a redundant description of a single connection 1-form $A_0 + i\alpha + ia$ where $d\alpha = 0$; there are a few reasons for this:

- The action of $\mathcal{G}_0$ (the based gauge group, whose elements send a base-point $m \in M$ to $1 \in S^1$) fixes $\{A_0 + \ker d\}$ setwise and is is free, so we can take a quotient in the next section. In fact, the action of $\mathcal{G}_0$ on all of $\Omega^1(M)$ regarded as connections is also free, so our real motivation for restricting to $\{A_0 + \ker d\}$ is in the next line.

Note that $\mathcal{G} = \mathcal{G}_0 \times S^1$.

- The addition of a closed 1-form to a base connection does not change the curvature, so by Theorem 1.1, $\{A_0 + \ker d\}/\mathcal{G}_0 \cong \text{Pic}^0(M)$
- Although $A_0 + i\alpha$ is not a generic unitary connection, any connection 1-form can be written as $A_0 + i\alpha + ia$, but this decomposition can be redundant (i.e. we can write the same 1-form in multiple ways)
- To avoid this redundancy, in the next section we will set $a_{\text{harm}} = 0$, which, together with the Coulomb gauge condition $d^*a = 0$ will guarantee that $A_0 + i\alpha + ia$ is a unique decomposition for each relevant connection:

Lemma 2.1

Hodge theory gives us an orthogonal decomposition

$$\Omega^1(M) = d\Omega^0(M) \oplus d^*\Omega^2(M) \oplus \mathcal{H}^1(M) = \operatorname{im} d \oplus \operatorname{im} d^* \oplus \mathcal{H}^1(M)$$

where $\mathcal{H}^1(M)$ consists of harmonic 1-forms.

Thus, $a_{\text{harm}} = 0$ implies that $a = df + d^*\omega$ for a function f and 2-form ω; the Coulomb gauge condition $d^*a = 0$ then implies that $\Delta f = 0$. The harmonic functions on a connected manifold M are all constants so $df = 0$ and $a = d^*\omega$. Thus, a accounts for the $\operatorname{im} d^*$ component of $\Omega^1(M)$. Our full connection is $A_0 + i\alpha + ia$ with $d\alpha = 0$, thus $\alpha \in \ker d$, and $\ker d = (\operatorname{im} d^*)^\perp = \mathcal{H}^1(M) \oplus \operatorname{im} d$ using the fact that d and d^* are adjoint, so α and a together will uniquely specify a generic connection.

One thing I never quite figured out is why we want $d^*a = 0$ in $\Omega^0(M)/\mathbb{R}$ rather than just in $\Omega^0(M)$.

The action of $\mathcal{G}_0$ on $\Omega^1(M)$ is set to the trivial action, and all of this allows us to quotient by $\mathcal{G}_0$ and get the "right" answer (i.e. we will be able to recover the moduli space of monopoles).

TL;DR

We introduce redundancy and then eliminate it in a nontrivial way in order for $\mu = \tilde{\mu}/\mathcal{G}_0$ to have some nice properties.

A Better World was Possible

There is an alternative gauge slicing for the monopole map due to Khandawit[3] where the map looks much more reasonable:

$$\tilde{\mu} : \Omega^1(X) \oplus \Gamma(W_+ \otimes L) \to i\Omega^2_+(X) \oplus \Gamma(W_- \otimes L) \quad \tilde{\mu}(A_0 + ia, \psi) = (\frac{1}{2}F_A^+ - \sigma(\psi), D_A^+\psi)$$

[3] Khandhawit, *A new gauge slice for the relative Bauer–Furuta invariants*.

I don't really know what the $\frac{1}{2}$ or the $i\Omega^2_+(X)$ are all about but this map is essentially *just* the Seiberg-Witten equations, and can also be used to define the Bauer-Furuta invariants; the trade-off for the inputs and outputs being cleaner is that quotienting by the gauge group is a little fiddlier. I didn't switch to this paper for a source because we will be reading the sequel[4] to Bauer-Furuta[5] next, and that paper will use the monopole map as written here.

[4] Bauer, *A stable cohomotopy refinement of Seiberg-Witten invariants: II*.

[5] Bauer and Furuta, *A stable cohomotopy refinement of Seiberg-Witten invariants: I*.

Salient Properties of μ

Set $\mathcal{A} = \tilde{\mathcal{A}}/\mathcal{G}_0$ and $\mathcal{C} = \tilde{\mathcal{C}}/\mathcal{G}_0$, and let $\mu : \mathcal{A} \to \mathcal{C}$ be the induced map, which is well-defined since $\tilde{\mu}$ is $\mathcal{G}_0$-equivariant. There is a residual S^1-action on $\mathcal{A}$ and $\mathcal{C}$ given by a global phase; importantly, μ is equivariant with respect to this action. Thus, $\mu^{-1}(\{A_0 + \ker d\}, 0, 0, 0, 0)/S^1$ is precisely the moduli space $\mathcal{M}_L$ of Seiberg-Witten monopoles. Since $\{A_0 + \ker d\}/\mathcal{G}_0 = \mathrm{Pic}^0(M)$ appears as a factor of both $\mathcal{A}$ and $\mathcal{C}$ (and μ is the identity map on this factor), μ is manifestly a map of (trivial) Hilbert bundles over $\mathrm{Pic}^0(M)$.

Equivariance with respect to the global phase is immediate after you think through what the action really does on each component; global phases act trivially on connections and forms and just rotate the spinor.

There are essentially two important properties of μ regarded as a map of Hilbert bundles:

1. μ is a "compact perturbation" of a linear Fredholm map, i.e., $\mu = l + c$ where l is linear Fredholm and c is a nonlinear compact operator. Specifically (and ignoring the $A \mapsto A$ part of μ),
$$c(\psi, a) = (0, F_A^+, 0, 0) + (ia \cdot \psi, -\sigma(\psi), 0, 0)$$
where $ia \cdot \psi$ is Clifford multiplication, and
$$l(\psi, a) = (D_A^+\psi, d^+(ia), a_{\text{harm}}, d^*a)$$
where $d^+\omega$ is the self-dual part of $d\omega$. In particular, the nonlinear component of μ is essentially quadratic.
2. The preimages of bounded sets are bounded — this is essentially equivalent both morally and mechanically to compactness of the moduli space in ordinary Seiberg-Witten theory. The punchline is that the Weitzenböck formula for D_A^+ gives us bounds on all the relevant input data.

Technically, we have to do some fiberwise Sobolev completions of $\mathcal{A}$ and $\mathcal{C}$ for the same reasons Sobolev completions come up in regular Seiberg-Witten theory (something something elliptic bootstrap), but the functional analysts have apparently figured all of that out, and we will therefore ignore it entirely.

Implicit in this decomposition are some mildly nontrivial claims about how D_A^+ and F_A^+ transform under perturbations of A.

Towards the Stable Cohomotopy Class

Functional Analysis

Definition 3.1: Fredholm Maps

Let $T : E_1 \to E_2$ be a continuous linear map of Hilbert spaces, then T is *Fredholm* if $\ker T$ and $\operatorname{coker} T$ are finite-dimensional, in which case we define the *index* of T as $\operatorname{ind}(T) := \dim\ker T - \dim\operatorname{coker} T$.

The usual definition of Fredholm includes the criterion that the range of T is closed; Wikipedia tells me that this condition is actually redundant.

The map we are interested in, however, is not linear Fredholm, but a compact perturbation thereof written $l + c$, between separable Hilbert spaces, and such that c is compact (these are confusingly also called Fredholm maps in Bauer-Furuta). The key result we will use about such maps is due to Schwarz:

Compactness for such a map c means that bounded sets get mapped to precompact sets, but we won't interact with this definition in any meaningful way.

Theorem 3.2: Schwarz

Compact homotopy classes of continuous Fredholm (in the above sense) maps relative to l are in bijection with the $\operatorname{ind}(l)^{\text{th}}$ stable homotopy group of spheres $\pi^{\text{st}}_{\operatorname{ind}(l)}$.

We need to define compact homotopy relative to l; say $\mu = l + c : \mathcal{A} \to \mathcal{C}$ is a Fredholm map between separable Hilbert spaces. Fix some disc $D \subseteq \mathcal{A}$, and restrict μ to ∂D. Then a *compact homotopy of μ relative to l* is of the form $\mu_t = l + c_t : \partial D \times I \to \mathcal{C} \setminus 0$ (i.e. only the compact part is changing) such that μ_t is non-vanishing on ∂D and for all time t.

There was some confusion about the requirement that μ_t is non-vanishing; in particular, what are we supposed to do if our original map $\mu_0 = l + c$ vanishes on ∂D? In particular if $c = 0$, and l has nontrivial kernel, we would seem to be in trouble. I'm not really sure what the answer is.

The construction for associating such classes of maps to elements in the $\operatorname{ind}(l)$-stem is as follows:

- approximate the compact part c on ∂D uniformly by maps c_n with finite dimensional range $\operatorname{im}(l)^{\perp} \subseteq V_n \subseteq \mathcal{C}$ (one can always do this according to the functional analysts) where $\operatorname{im}(l)^{\perp}$ is finite-dimensional since l is Fredholm

- Let $\mu_n = l + c_n$ restricted to $l^{-1}(V_n) \cap \partial D$; this is the intersection of a finite-dimensional vector space and the sphere S^∞, hence a finite-dimensional sphere. Dimension counting finds that $\dim(l^{-1}(V_n) \cap \partial D) = \operatorname{ind}(l) + \dim V_n - 1$

- Since μ_n is non-vanishing, our map to a sphere is defined as $\frac{\mu_n}{\|\mu_n\|}$, and the target sphere is $(\dim V_n - 1)$-dimensional, thus gives us an element of $\pi_{\dim V_n + \operatorname{ind}(l) - 1}(S^{\dim V_n - 1})$

- One (but not us) can show that the limit of these maps is a well-defined element in the $\operatorname{ind}(l)$-stem, and that compact homotopy classes of μ give the same stable homotopy class. Moreover, the class we get in $\pi^{\text{st}}_{\text{ind}(l)}$ is invariant under compact homotopy relative to l.

Remark 3.3: Leray-Schauder Degree[6]

For $l = \text{id}$, this construction associates to any compact operator an element of the 0-stem, $\pi^{\text{st}}_0 = \mathbb{Z}$, so, an integer. This integer is called the *Leray-Schauder degree*, and can be used together with the Leray-Schauder fixed point theorem to prove existence of solutions to PDEs.

[6] Berger, *Nonlinearity and functional analysis: lectures on nonlinear problems in mathematical analysis*, Chapter 5.3

The above citation also gives a good discussion of the general construction sketched above for $\text{ind}(l) \neq 0$.

Hilbert Bundles

The monopole map is not merely a compact perturbation of a linear Fredholm map, but rather, a bundle's worth of such maps over $\text{Pic}^0(M)$, so we need to lift the above construction to bundles of Hilbert spaces. Reiterating our bad terminology from the above discussion, we will define *Fredholm maps over* Y to be continuous maps of Hilbert bundles over a compact base Y of the form $f : l + c$ where l is linear Fredholm and where c is fiber-preserving, compact, and whose preimages of bounded sets are bounded.

Theorem 3.4: Kuiper

Any Hilbert bundle with infinite dimensional fiber is trivial.

A priori, I suppose there could be problems with simultaneously trivializing both $\mathcal{A}$ and $\mathcal{C}$; however, they seem to be constructed as manifestly trivial Hilbert bundles, so I'm not even really sure why we have to invoke Kuiper here. I think we probably don't have to, and it only shows up in Bauer-Furuta when they are being more general than they have to, and explaining how to associate cohomotopy classes to maps of Hilbert bundles in general.

Thus, looking at the monopole map, we may trivialize the codomain as $\mathcal{C} = \text{Pic}^0(M) \times (\Gamma(W_- \otimes L) \oplus \Omega^2_+(M) \oplus H^1(M;\mathbb{R}) \oplus \Omega^0(M)/\mathbb{R}$; call the fiber of this trivialized bundle C. We could also trivialize the domain, but, instead, we look at the map

$$\widehat{\mu} : \text{Th}(\mathcal{A}) \to C^+$$

where C^+ is the Hilbert sphere (one point compactification), and $\text{Th}(\mathcal{A})$ is the Thom space of $\mathcal{A}$. This is well-defined by our boundedness criteria.

Recall that the Thom space of a bundle is defined as

$$\text{Th}(E) = D(E)/(S(E) \cup E_b)$$

where E_b is the full fiber above the basepoint b; this makes $\text{Th}(E)$ into a based space. The Thom space of a trivial bundle is just the smash product with a sphere of appropriate dimension, or, equivalently, the iterated (reduced) suspension. Therefore, we can think of the Thom space as being a kind of "twisted" suspension.

One reason for doing so is that the target C^+ has "forgotten" the input information from $\text{Pic}^0(M)$ which is part of the data of a connection, and we need to remember that information somewhere (if we trivialized $\mathcal{A}$ and $\mathcal{C}$ simultaneously and just considered the map between compactified fibers, we would lose all holonomy data). We want to extend the association of μ to a stable homotopy class above, to $\widehat{\mu}$. However, now, since only the target is a sphere, we will get a (stable) cohomotopy class. This construction (and its S^1-equivariant refinement) will be taken up next week.

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