Bauer-Furuta III: John [Author Rmk: no guarantee of correctness]

Goal: How to get from a Fredholm map to a stable cohomotopy class?

Last Time

$$\frac{\{\text{fredholm } f = l + c : H \to H'\}}{\text{compact htpy rel } l} \longleftrightarrow \underset{\substack{\uparrow\\ \text{bad, since no canonical basepoint}}}{\pi^{ST}_{\text{ind}\, l}(S^0)}$$

↝ how to get invariants?

Today

1) View

$$\begin{tikzcd} E' \arrow[rr, "f = l + c"] \arrow[dr] & & E \arrow[dl] \\ & Y & \end{tikzcd} \rightsquigarrow \pi^{0}_{G,H,ST}(Y, \text{ind}\, l)$$

2) The monopole map μ is such a map

3) Recover SW

Part 1: Construction for general G-Equivariant Hilbert bundles

Flow
- I Get basepoint
- II Hilbert Bundles
- III G-bundles

I 1-pt compactification for Hilbert Spaces

Notation Domain: H' CoDomain H

$f: H' \to H$ fredholm map ↝ 1-pt completion $f: H'^+ \to H^+$

as close as smth co-dim can get to being cpt

where each fin.dim $V \subset H$ ↝ $\underset{\substack{\uparrow\\ \text{compact}}}{V^+} \subset H^+$

Lemma (Functional Analysis)

A fredholm map $f = l + c : H' \to H$ extends to a map from $H'^+ \to H^+$ if f satisfies the boundedness condition: preimages of bounded sets are bounded.

Goal Think of

$$\pi^{ST}_{\text{ind}(l)}(S^0) = \operatorname*{colim}_{V \subset H} \left[\overset{\overset{H'^+}{\cup}}{(l^{-1}(V))^+}, \overset{\overset{H^+}{\cup}}{V^+} \right]$$

V = fin.dim. ST − insert condition ✗ — (see next pg)

why is this dim ind(l)?

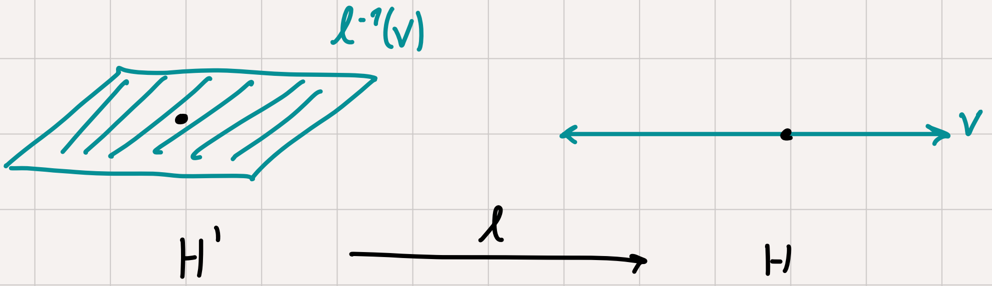
$$" = " \operatorname*{colim}_{V \subset H} \left[S^{\dim l^{-1}(V)}, S^{\dim V^+} \right]$$

if V big enough, it contains coker

Since taking stable htpy, both sides blow up

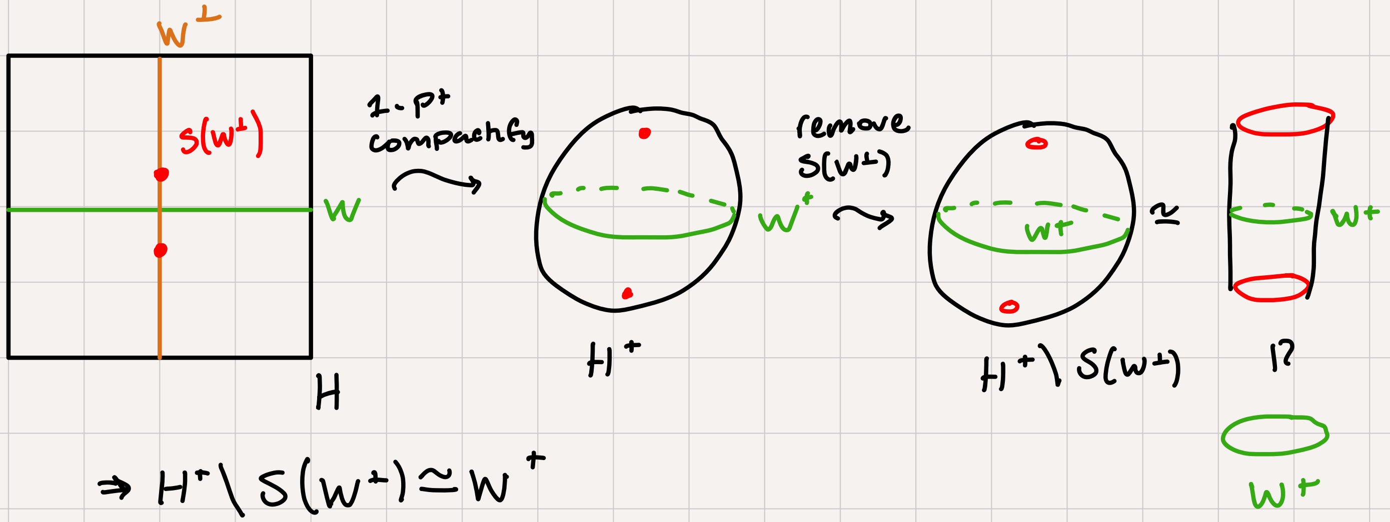
$$\dim(l^{-1}(V)) = \dim \ker(l) + (\cancel{\dim V} - \dim \operatorname{coker}(l))$$

↝ difference is $ind(\ell)$



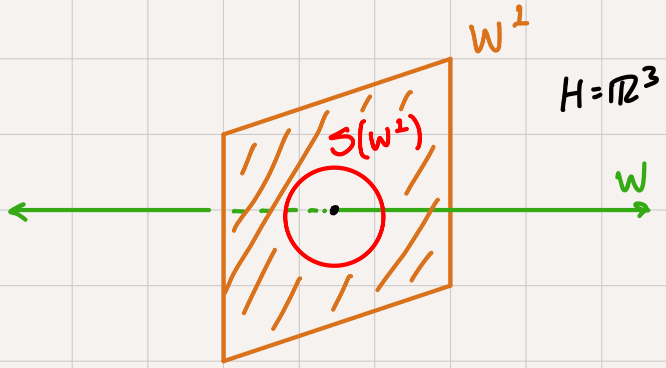
In diagrams,

2D:



$\Rightarrow H^+\setminus S(W^\perp) \simeq W^+$

3D:



Define $\rho_W : H^+\setminus S(W^\perp) \longrightarrow W^+$ to be this retraction map

$$\Rightarrow \pi^{ST}_{ind(\ell)}(S^0) = \underset{\substack{V\subset H \\ V \text{ f.d.}}}{\operatorname{colim}} \left[(\ell^{-1}(V))^+ ,\ H^+\setminus S(V^\perp) \right]$$

Prop $[f] := \underset{V\subset H}{\operatorname{colim}} \left[\left(f|_{\ell^{-1}(V)} \right)^+ \right]$ is contained in $\pi^{ST}_{ind(\ell)}(S^0)$

To prove, need ↓ lemma

★ Lemma There is some $V \subset H$ st ← when we take the colimit, we do so over all V that satisfy this lemma

1) $V \supset \operatorname{im} \ell^\perp$

2) $\forall\, W \supset V,\ W = \mathcal{U} \perp V,\ f|_{W'^+} : W'^+ \longrightarrow H^+$ misses $s(W^\perp)$

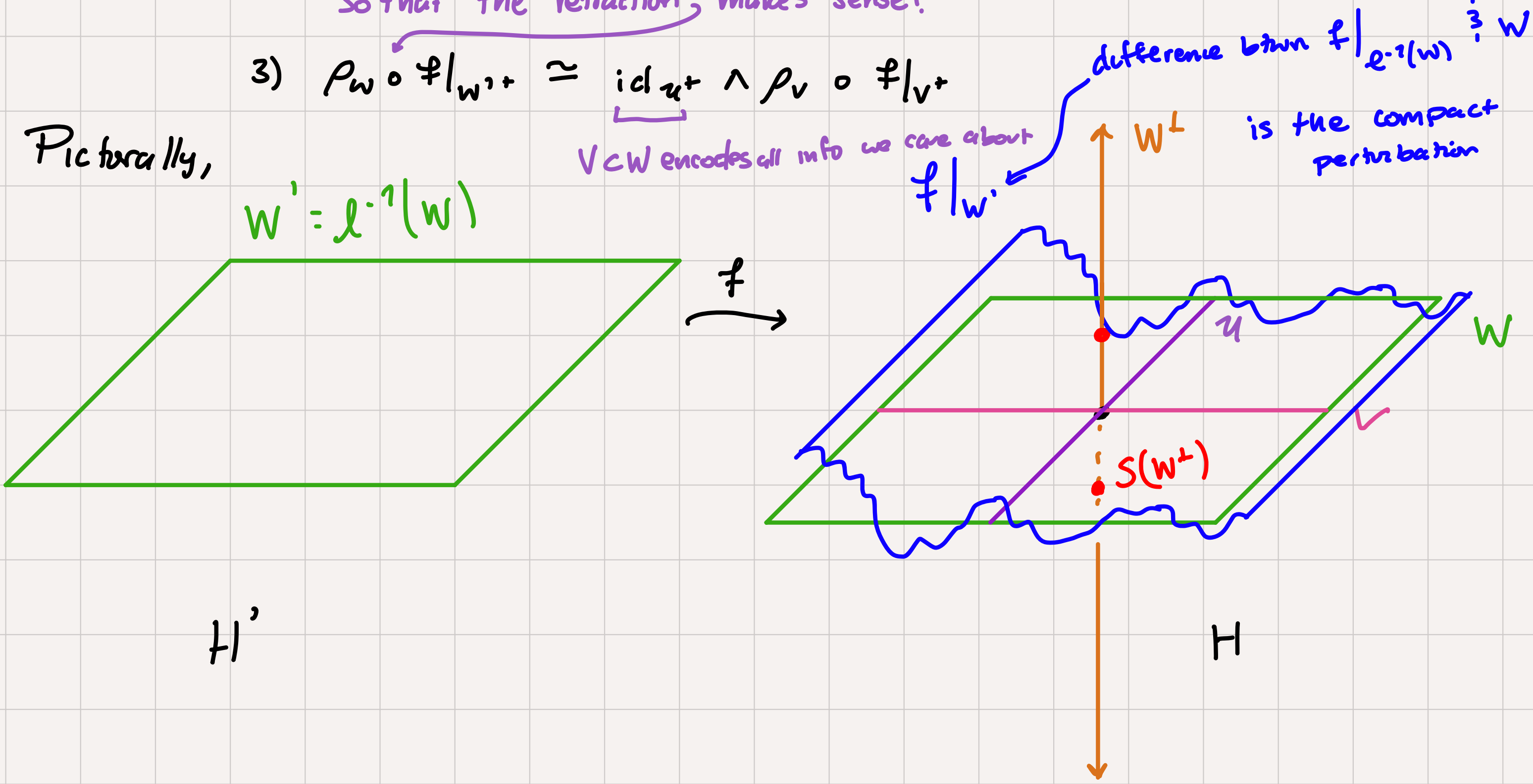
ie, all things V is contained in also miss $S(\text{thing's}^\perp)$

So that the retraction makes sense!

3) $\rho_W \circ f|_{W'^+} \simeq \underbrace{\mathrm{id}_{\mathcal{U}^+}}_{} \wedge \rho_V \circ f|_{V^+}$

V⊂W encodes all info we care about

Pictorally,



II Hilbert Bundles

As before, we want to get a basepoint:

$$\begin{tikzcd} E' \arrow[rr, "f: \ell + c"] \arrow[dr] & & E \arrow[dl] \\ & Y & \end{tikzcd}$$

E, E' : Hilbert Bundles over Y

ℓ : fibrewise-linear Fredholm

c: fibrewise compact

By Kuipers Thm, all Hilbert bundles are trivial.

However, trivialization looses the data of how f maps.

So we only want to trivialize one side.

Which side? in codomain. this is why we get cohomotopy.

Now, we compactify!

We want to reproduce I, but fibrewise.

Let $T(E')$, $T(E)$ be the Thom Spaces.

The boundedness condition on f becomes:

f can be extended $\Longleftrightarrow$ preimages of bounded disk $^{(\text{sub})}$bundles are disk $^{(\text{sub})}$bundles

$\Rightarrow$ we get a map $Tf : T(E') \longrightarrow T(E)$

$\rightsquigarrow$

$$\begin{tikzcd}
E' \arrow[r, "f"] \arrow[dr, "pr_H \circ f"'] & E \arrow[d, "pr_H"] \arrow[r, phantom, "\overset{T}{\Longrightarrow}"] & TE' \arrow[r, "f"] \arrow[dr, "(pr_H \circ f)^+"'] & TE \arrow[d, "pr_H^+"] \\
 & H & & H^+
\end{tikzcd}$$

Where do our invariants live?

Set $\lambda = F_0 - F_1 \left(\in KO^\circ(Y) \right)$

where $F_0 \searrow Y \swarrow F_1$ $\;F_1 \cong Y \times V$, $V \subset H$ fin. dim subspace. ,

cohomotopy w/ coeffs:

$$\pi^n_H(Y;\lambda) = \underset{\substack{\mathcal{U} \subset V^\perp \\ \text{f.d.}}}{\operatorname{colim}} \left[\mathcal{U}^+ \wedge TF_0 ,\ \mathcal{U}^+ \wedge V^+ \wedge S^n \right]$$

$\uparrow$ defined to be the stable cohomotopy of Thom bundle

III G-Equivariant Hilbert Bundles

Notation H = G-Universe = Space containing any $n\mathbb{R}$, $m\mathbb{C}$ as subreps $\forall\, n,m \in \mathbb{Z}$

where $\mathbb{R} \longleftrightarrow$ triv. 1-dim $\mathcal{U}(1)$ rep

$\mathbb{C} \longleftrightarrow$ unit speed rotation 2-dim $\mathcal{U}(1)$ rep $\quad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \curvearrowright \mathbb{C}$

(or whatever the right rotation matrix is)

V = finite dim. G-rep ; $\mathcal{U}$ fin. dim subrep

$\lambda = F_0 - F_1$, F_i = finite dim. equiv. vector bundle over Y

Y = G-CW complex

sphere ↓ sphere ↓ , Smash of Spheres is larger sphere

$$\rightarrow \pi^{n}_{\substack{H,G\\ st}}(Y,\lambda) = \operatorname*{colim}_{\mathcal{U}\subset V^{\perp}} \{\overset{\text{sphere}}{\mathcal{U}'^{+}} \wedge TF_0 , \underbrace{\mathcal{U}^{+}\wedge \overset{\text{sphere}}{V^{+}} \wedge S^{n}}_{\text{Sphere of dim} > n}\}^{G}$$

Prop $$[f] := \operatorname*{colim}_{\mathcal{U}\subset V^{\perp}} \left[\left((pr_H \circ f)\big|_{\mathcal{U}'^{+}\wedge TF_0}\right)^{+}\right] \in \pi^{0}_{\substack{H,G\\ st}}(Y,\lambda)$$

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Parts 2 & 3: Applying Part 1 to the Monopole map + recovering the normal SW invariant

Prop Monopole map is Fredholm & satisfies appropriate boundedness condition

$$\mu : \mathcal{A} \to \mathcal{C}$$

μ admits decomposition into linear + compact

$$\ell = D_A \oplus d^{+} \oplus Pr_{harm} \oplus d^{*}$$

$$c : (\phi, a) \mapsto (0, F_A^{+}, 0, 0) + (a\cdot\phi, -\sigma(\phi), 0, 0)$$

$$[\mu] \in \pi^{0}_{\substack{S^1;H\\ st}}(Pic^{0}(X), \lambda) = \pi^{b_2^{+}}_{\substack{S^1;H\\ st}}(Pic^{0}(X), \operatorname{ind}(\mathcal{D}))$$

H: Sobolev completion $\Gamma(S^{-} \oplus \Lambda^{2}_{+}(T^{*}X))$

$$\lambda = \operatorname{ind}\mathcal{D} \ominus \underset{\substack{\uparrow\\ \text{from coker } d^{+}}}{\mathcal{H}^{2}_{+}(X)} \times Pic^{0}(X)$$

Thm X^4 closed & homology oriented

$$\rightsquigarrow t : \pi^{b_2^{+}}_{\substack{S^1;H\\ st}}(Pic^{0}(X), \operatorname{ind}\mathcal{D}) \to \mathbb{Z}$$

$$[\mu] \longmapsto SW(X)$$