Cerf Theory Learning Seminar **Fall 2025**

Week 2: Stratifying Function Spaces

Columnated Ruins Domino

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This week, we'll cover roughly the first chapter of Cerf,[1] except for sections 4 and 5, which give some results on embeddings of submanifolds meeting other submanifolds (all of specified codimensions) that we won't have use for until much later. In particular, we'll give the basic definitions of stratifications and develop the notion of codimension for maps $f : M \to \mathbb{R}$. Whoever ends up talking about the pseudo-isotopy theorem will probably have to come back and cover those latter sections.

[1] Cerf, *La stratification naturelle des espaces de fonctions différentiables réelles et le théoreme de la pseudo-isotopie*.

Recap

Last week, Audrick told us about the basics of Morse theory.

Lemma 1.1: Morse Lemma

Let M be a smooth n-manifold, $p \in M$ a non-degenerate critical point (i.e., $df_p = 0$ and the Hessian is non-singular) of $f : M \to \mathbb{R}$. Then there exists a local coordinate chart $U \ni p$ such that

$$f|_U(x) = -x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$$

k is the *index* of the critical point p.

Proposition 1.2

Morse functions are open and dense in $C^\infty(M)$.

There's a nice (perhaps standard? but not to me) proof of this using parametric transversality (together with the fact that transversality is stable under small deformations).

A suggestive interpretation of these results is that we now understand the codimension zero or open locus in $C^\infty(M)$, and have a local model for the behavior of such functions near their critical points. A natural question is whether we can understand the positive codimension loci in a similar way, for some natural notion of codimension.

One reason to care about this is that Morse functions give us handle decompositions, and to compare two handle decompositions, we need to find a path between their Morse functions in $C^\infty(M)$. Such paths may unavoidably hit codimension one strata (put differently, a path between Morse functions is (spoilers) not always Morse for all time), so understanding what the codimension one locus looks like is key to understanding how handle decompositions differ.

Higher codimension loci will become relevant when studying diffeomorphisms, loops of diffeomorphisms, etc. which correspond roughly to paths of Morse functions, two-parameter families of Morse functions, etc. respectively.

Stratifications

In this chapter, Cerf works in greater generality than we will probably aspire to. I will try to tighten our focus whenever possible.

Definition 2.1: Stratifications

Let E be a topological space. A sequence $E^0, E^1, \cdots$ (not necessarily finite) of subsets of E is a *stratification* of E if the E^i partition E (i.e., $E = \cup_i E^i$ and the E^i are pairwise disjoint) and $\cup_{i=1}^n E^i$ is open for all $i \in \mathbb{N}$. E^i is called the i^{th} *stratum*. A *morphism of stratified spaces* $f : E \to F$ is one that respects the strata, i.e., $f(E^i) \subseteq E^i$.

Example 2.2: Triangulated Manifolds

Let X be a triangulated manifold of dimension n with k-skeleta X_k; then $X^k := X_{n-k} \setminus X_{n-k-1}$ defines a stratification of X. Moreover, $X^{k+1} \subseteq \overline{X^k}$ for all k, and X^k is a submanifold of codimension k in X.

Draw this with $X = S^2$ equipped with the octahedral triangulation.

Note that a subspace $A \subseteq E$ of a stratified space has a natural *induced* stratification given by $A^i := A \cap E^i$. Moreover, the product of stratified spaces E, F also has a natural *product* stratification given by

$$(E \times F)^i := \bigcup_{j+k=i} E^j \times E^k$$

These two notions are essential to the following definition:

Definition 2.3: Locally Trivial Stratification

Let E be a stratified space; we say its stratification is *locally trivial* if, for all $e \in E$, there exists the following data:

- a neighborhood $U \ni e$ of e
- a stratified space X with a point stratum $\{x\}$
- a space Y with basepoint y equipped with the trivial stratification
- an isomorphism of stratified spaces $\varphi : X \times Y \xrightarrow{\sim} U$ with $\varphi(x, y) = e$

In this setting we call X the *transverse model* of the stratification at x.

Example: S^2 with equatorial S^1. Nonexample: S^2 with self-intersecting S^1.

Codimension

Let W be either a closed manifold or a cobordism, and let $\mathscr{F}$ denote $\mathscr{C}^\infty(W)$ or the set of C^∞ functions $(W, V, V') \to (I, 0, 1)$ without critical points on the boundary. Let $\mathscr{G}$ be the group $\operatorname{Diff} W \times \operatorname{Diff} \mathbb{R}$ in the closed case, or $\operatorname{Diff}(W, V, V') \times \operatorname{Diff}(I, 0, 1)$ in the cobordism case. In either case, $\mathscr{G}$ has a natural left action on $\mathscr{F}$ given by the following:

$$\mathscr{G} \times \mathscr{F} \ni ((g, g'), f) \mapsto g' \circ f \circ g^{-1} \in \mathscr{F}$$

We will now give a stratification of $\mathscr{F}$ by defining the codimension of a function $f \in \mathscr{F}$, whence $\mathscr{F}^i$ will consist of all codimension i functions.

Skip this in the talk.

Definition 3.1: Codimension (hard)

For $f \in \mathscr{F}$, consider the function $L_f : \mathscr{G} \to \mathscr{F}$ given by

$$(g, g') \mapsto g' \circ f \circ g^{-1}$$

Then the *codimension* of f is defined as

$$\operatorname{codim}((dL_f)_{(\mathrm{id}_W, x)} : T_{(\mathrm{id}_W, x)}\mathscr{G} \to T_f\mathscr{F})$$

Cerf attributes this definition to John Mather. I might refer to it as Mather's codimension.

This is too hard to work with. Instead, we will work with a particular subspace of $\mathscr{F}$, containing $\mathscr{F}^0, \mathscr{F}^1$, and $\mathscr{F}^2$ (to be defined below), and all Morse functions, on which we may define the codimension explicitly by a simple formula (which agrees with the preceding definition in codimension $0, 1$, and 2).

Incredibly unclear to me what this subspace is. Cerf says it contains $\mathscr{F}^0, \mathscr{F}^1$, and $\mathscr{F}^2$, and all Morse functions, but as far as I can tell that's all he says. Hatcher-Wagoner doesn't mention a subspace, they work in $\mathscr{F}^0 \cup \mathscr{F}^1 \cup \mathscr{F}^2 \cup \mathscr{F}^3$. But this certainly doesn't include all Morse functions, since you could have 50 critical points all with the same critical value.

Definition 3.2: Codimension of a critical point

For $f \in \mathscr{F}$, c a critical point of f, the *codimension* of c is the codimension of the $\mathbb{R}$-vector space generated by the germs of first partial derivatives of f at c in the vector space of germs of smooth functions from W to $\mathbb{R}$ that vanish at c.

The codimension of a vector subspace is also the dimension of the quotient.

Germs of functions at a point $x \in X$ are functions defined on some open neighborhood U of x modulo the equivalence relation generated by equality on a smaller open subset. Explicitly,

$$(f, U) \sim (g, U') \text{ if there exists } V \subseteq U \cap U' \text{ such that } f|_V = g|_V$$

Morally, this is local functions modulo zooming in. $\mathscr{O}_{W,c}$ denotes the ring of all germs at c, and $\mathfrak{m}_{W,c}$ its maximal ideal of germs vanishing at 0.

Lemma 3.3: Taxonomy of Low Codimension Critical Points

- Critical points of codimension zero are non-degenerate or Morse critical points, with local model

$$-x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$$

- Critical points of codimension one are called *birth* or *death* points, with local model

$$-x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^3$$

- Critical points of codimension two are called *dovetail* or *swallow's tail* points, with local model

$$-x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots \pm x_n^4$$

Index for all such critical points is defined as before (the number of minus signs in the local model).

There's a Dalí painting about the swallow's tail singularity, part of his series on Thom's work in catastrophe theory.

No proof of this is given in Cerf, but we can sort of see the vision by thinking about representations of germs. In particular, Hadamard's lemma implies that $\mathfrak{m}_{W,c} = (x_1, \cdots, x_n)$ for some local coordinates x_i on a chart centered at c. Less algebraically, this is just the claim that any function f vanishing at c can be written as

$$f(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, \cdots, x_n)$$

It's then pretty believable that the only codimension one ideals in $\mathfrak{m}_{W,c}$ are of the form $(x_1, \cdots, x_i^2, x_{i+1}, \cdots, x_n)$ (which are of codimension one since the quotient $\mathfrak{m}_{W,c}/(x_1, \cdots, x_i^2, x_{i+1}, \cdots, x_n) = (x_i)$ is generated by a single element), which is precisely the ideal generated by the first partials of the birth model above.

Codimension two is similar, with ideals of the form $(x_1^3, x_2, \cdots, x_n)$ giving rise to quotients $\mathfrak{m}_{W,c}/(x_1^3, x_2, \cdots, x_n) = \langle x_1, x_1^2 \rangle$ of rank two. The sign data of the local models above is totally lost when passing to the germs of first partials, so there is still some extra work needed to prove the above lemma.

I'm pretty unclear on why there's an extra $\pm$ for dovetail singularities. In Cerf, the quartic term is given a minus sign, which is already weird, but in Hatcher-Wagoner it's allowed to have either sign, while the cubic term in birth singularities is always positive.

If we had a singularity modeled on

$$\pm x_1^3 \pm x_2^3 \pm x_3^2 \pm \cdots$$

with the remaining terms quadratic, we'd end up with the ideal

$$(x_1^2, x_2^2, x_3, \cdots, x_n)$$

which is of codimension 3, since the corresponding quotient is spanned by x_1, x_2, and $x_1 x_2$. Thankfully, codimension 3 phenomena are just barely outside of our purview.

Definition 3.4: Codimension (easy)

For $f \in \mathscr{F}$, let $v_1(f)$ be the sum of the codimensions of the critical points of f, and $v_2(f)$ the sum of the codimensions of the critical *values* of f (which for a given critical value α is defined as $|f^{-1}(\alpha)| - 1$, i.e., the number of times the critical value is repeated). Then

$$\mathrm{codim}(f) := v_1(f) + v_2(f)$$

Table 1.1: Taxonomy of low codimension strata

Stratum	$(v_1(f), v_2(f))$	Description
$\mathscr{F}^0$	$(0,0)$	Morse functions with distinct critical values (excellent functions, in Thom's terminology).
$\mathscr{F}^1_\alpha$	$(1,0)$	One birth point, all other critical points non-degenerate, and distinct critical values.
$\mathscr{F}^1_\beta$	$(0,1)$	All non-degenerate critical points, one repeated critical value.
$\mathscr{F}^2_\alpha$	$(2,0)$	One dovetail point and all critical values distinct.
$\mathscr{F}^2_\beta$	$(2,0)$	Two birth/death points and all critical values distinct.
$\mathscr{F}^2_\gamma$	$(1,1)$	One birth point, one double critical value for two non-degenerate critical points.
$\mathscr{F}^2_\delta$	$(1,1)$	A birth point and a non-degenerate critical point with the same critical value.
$\mathscr{F}^2_\epsilon$	$(0,2)$	Three non-degenerate points have the same critical value.
$\mathscr{F}^2_\zeta$	$(0,2)$	Two double critical values among four non-degenerate critical points.

Theorem 3.5: Cerf, Hatcher-Wagoner[2]

The above notion of codimension gives a locally trivial stratification of $\mathscr{F}^0 \cup \cdots \cup \mathscr{F}^k$ for $0 \leq k \leq 3$, which is open in $\mathscr{F}$.

[2] Hatcher and Wagoner, *Pseudo-isotopies of compact manifolds*

Apparently[3] if you go out to codimension 7, this stratification by codimension stops being locally trivial. Perhaps this is not the case for Mather's definition of codimension. Also, apparently[4] the combinatorial definition of codimension given here can be extended to arbitrary codimension in a way that agrees with Mather (ours is only guaranteed to work up to codimension 2).

[3] Hatcher and Wagoner, *Pseudo-isotopies of compact manifolds*, page 22.

[4] Mata-Lorenzo, *A note on the codimension of a map*.

Remark 3.6

This is not a finite stratification; for example, a constant function is of infinite codimension.

Elementary Paths

Reference the S^2 example as one of these.

An important class of stratifications for our study are the *codimension one* stratifications, i.e., $E = E^0 \cup E^1$. For example, take E any manifold (of finite or infinite dimension), and E^1 any submanifold of codimension 1.

Cerf's codimension one stratifications also satisfy a technical condition (they are conical and points in E^1 have transverse models which are cones on finite sets), the relevance of which I am unsure of.

In such a stratified space, for any $y \in E^1$, let $\mathscr{L}_y$ be the space of paths starting at y and ending in E^0 (i.e., paths from y to the generic locus).

Definition 4.1: Good Paths

For $\gamma : I \to E$, t_0 an isolated point of $\gamma^{-1}(E^1)$, the germs of γ at $y = \gamma(t_0)$ define a pair of path components of $\mathscr{L}_y$. If these elements are distinct, we say that γ *traverses* E^1 at y. We say that γ is a *good* path if γ traverses E^1 finitely many times. A good path crossing E^1 a single time is called a *crossing path*.

Draw a traversing and non-traversing path. Germs being equal is like tangency.

Lemma 4.2: Lemma of Elementary Paths

Let E be a locally path connected space equipped with a codimension one stratification, and a good* action of a topological group G. Let $\mathscr{C}$ be the space of paths crossing E^1, $\mathscr{C}'$ some union of components of $\mathscr{C}$, and $\mathscr{C}''$ the subset of $\mathscr{C}$ that is stable under the action of G. The elements of $\mathscr{C}''$ are called *elementary paths*.

If for all $\gamma \in \mathscr{C}'$, there exists an elementary path with the same intersection with E^1 and crossing direction as γ, then all $\gamma \in \mathscr{C}'$ are homotopic in $\mathscr{C}'$ to elementary paths, via a homotopy fixing the basepoint of the path.

The proof is algebraic and phrased in the language of simplicial sets (e.g. invoking Kan complexes and Kan fibrations). Therefore I did not read it.

* By good, here, we mean firstly that the G respects the stratification (i.e. the orbit of E^i is contained in E^i for all i), but also that for all $e \in E^0$ (i.e., for generic points), the stratum of e coincides in the neighborhood of e with the orbit of e. In other words, G is large enough that its action exhausts all directions of local perturbation. Additionally, we require that the map $G \to E$ given by $g \mapsto g \cdot e$ for all $e \in E$ is a locally trivial fibration onto the orbit of e.

Apparently, this lemma will be used constantly throughout Cerf's text.

In our setting, $\mathscr{G} = \operatorname{Diff} W \times \operatorname{Diff} \mathbb{R}$, and given a path $f_t : I \to \mathscr{F}$, the induced action on paths is

$$f_t \mapsto g' \circ f_t \circ g^{-1}$$

Question 4.3

What does an element of $\mathscr{C}'$ unstable under $\mathscr{G}$ look like? What is a non-elementary crossing path? Why do we need this lemma?

References

Jean Cerf. *La stratification naturelle des espaces de fonctions différentiables réelles et le théoreme de la pseudo-isotopie.* In: *Publications Mathématiques de l'IHÉS* 39 (1970), pp. 5–173.

Allen E. Hatcher and John B. Wagoner. *Pseudo-isotopies of compact manifolds*. In: *Astérisque* 6 (1973).

Luis E Mata-Lorenzo. *A note on the codimension of a map*. In: *Journal of mathematical analysis and applications* 137.1 (1989), pp. 37–45.

John B? As in Sloop John B?