THE PSEUDO-ISOTOPY THEOREM
FOR SIMPLY CONNECTED DIFFERENTIABLE MANIFOLDS *)

Jean Cerf

1. Statement of the theorem

Let V be a compact differentiable manifold. The group of pseudo-isotopies of V is the subgroup of $\mathrm{Diff}\,(V \times I)$ consisting of diffeomorphisms which keep $V \times \{0\}$ pointwise fixed. This group acts in a natural way on $\mathrm{Diff}\, V$ $(V = V \times 1)$. Two diffeomorphisms of V which are equivalent under that action are said to be pseudo-isotopic.

Theorem 1. If V is simply connected and dimension $V \geqq 5$, then the group of pseudo-isotopies of V is connected (for the C^∞-topology).

Corollary 1 (Same hypothesis on V). On $\mathrm{Diff}\, V$, the two relations isotopy and pseudo-isotopy coïncide.

Corollary 2. $\pi_0(\mathrm{Diff}\, D^n) = 0$ for $n \geqq 6$,
$\pi_0(\mathrm{Diff}\, S^n) \approx \Gamma_{n+1}$ for $n \geqq 5$.

2. Functional form of theorem 1

Let $\mathcal{F}$ be the space of C^∞ functions $V \times I \longrightarrow I$ which are equal to zero on $V \times \{0\}$ and to 1 on $V \times \{1\}$. Let $\mathcal{E}$ be the subspace of $\mathcal{F}$ consisting of functions which have no critical point. The group of pseudo-isotopies of V acts in a natural way on $\mathcal{F}$; it is easy to prove (using a Riemannian metric on $V \times I$) that $\mathcal{E}$ coïncides with the orbit of the natural projection $p: V \times I \longrightarrow I$ under that action, and that the map: $g \to g.p$ admits a global section over $\mathcal{E}$. The group of pseudo-isotopies of V (whose subgroup keeping p invariant is the group of isotopies of V, which is acyclic) has therefore the same homotopy type as $\mathcal{E}$. In particular, theorem 1 is equivalent to the following:

Theorem 1'. If V is simply connected and dimension $V \geqq 5$, then $\mathcal{E}$ is connected.

The general idea of the proof of theorem 1' is the following.
$\mathcal{E}$ is a union of connected components of the 0-stratum $\mathcal{F}^0$ of the natural stratification of $\mathcal{F}$ (see 3. below). Any two points in $\mathcal{E}$ can be joined by a path in $\mathcal{F}^0 \cup \mathcal{F}^1$. One has to prove that such a path can be deformed (with fixed end points) into a path in $\mathcal{E}$. The proof splits naturally into 3 parts:

---

*) Outline of [1].

- the local study of the natural stratification of $\mathcal{F}$ (up to codimension 2);
- the semi-local lemmas;
- the global part of the proof which makes use of Smale's filtration of $\mathcal{F}$ and of an algebraic lemma.

3. Local description of the stratification of $\mathcal{F}$ (in small codimensions)

Denote by W the manifold $V \times I$ and by G the group Diff $W \times$ Diff I; the group G acts in $\mathcal{F}$ according to the formula:

(1) $G \times \mathcal{F} \ni ((g, g'), f) \longmapsto g' \circ f \circ g^{-1}$

Definition 1 (Mather). The codimension of $f \in \mathcal{F}$ is the real codimension of the image of the linear tangent map to the map $p_f\colon G \longrightarrow \mathcal{F}$ obtained by fixing f in formula (1).

Denote by $\mathcal{F}^j$ the subspace of $\mathcal{F}$ defined by

$f \in \mathcal{F}^j \iff$ codimension $f = j$.

The sequence $\mathcal{F}^0, \mathcal{F}^1, \ldots, \mathcal{F}^j, \ldots, \mathcal{F}^\infty$ defines the natural stratification of $\mathcal{F}$.

[A stratification of a topological space E is a family $E^0, E^1, \ldots, E^j, \ldots, E^\infty$ of subspaces such that

$E^j \cap E^{j'} = \emptyset$ for $j \neq j'$;

$\bigcup_{0 \leqq j} E^j = E$

$\bigcup_{0 \leqq j \leqq k} E^j$ is open for every $k \geqq 0$].

A locally trivial stratification of E is a stratification such that, for every $x \in E$, there exists a stratified space X with a punctual stratum {0}, a topological space Y with trivial stratification and a chosen point y, and a morphism (of stratified spaces) $\phi\colon X \times Y \longrightarrow E$ such that:

$\phi(0, y) = x$;

the image U of ϕ is a open subset of E;

ϕ defines an isomorphism between $X \times Y$ and U.

Concerning the natural stratification of $\mathcal{F}$, one can prove the following properties:

a) It is locally trivial (cf. Sergeraert, [2]);

b) for every j, $\mathcal{F}^j$ has codimension j in $\mathcal{F}$;

c) the stratification is invariant under the G-action; if $f \in \mathcal{F}^j$ and j is finite, the map p_f defines a locally trivial fibration $G \longrightarrow$ (orbit of f); if $j \leqq 5$, then $\mathcal{F}^j$ coincides in the neighborhood of f with the orbit of f.

Property b) implies that any map : $S^j \longrightarrow \mathcal{F}$ can be slightly deformed in order to avoid $\mathcal{F}^{j+1} \cup \ldots \cup \mathcal{F}^\infty$. Thus $\pi_1(\mathcal{F}, \mathcal{E}) \approx \pi_1(\mathcal{F}^0 \cup \mathcal{F}^1 \cup \mathcal{F}^2, \mathcal{E})$. In order to obtain a precise description of $\mathcal{F}^0 \cup \mathcal{F}^1 \cup \mathcal{F}^2$, one can replace Definition 1 by the following explicit formula, valid only if the codimension of f is very small ($j \leqq 5$):

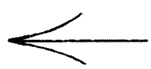
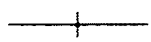
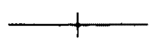
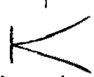
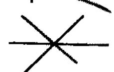
(2) codimension f = Σ (codimensions of the critical points of f)
$\qquad + \Sigma$ (codimensions of the critical values of f)

[the codimension of a critical point c is defined as the real codimension of the

ideal generated by the germs of the first partial derivatives in the ring of germs of C^∞-functions $W \longrightarrow \mathbb{R}$ with value zero at c.
The codimension of a critical value α is defined as being:
(number of critical points in $f^{-1}(\alpha)) - 1]$.
The classification of the critical points of codimension 0, 1 or 2 is the following:

codimension	canonical form	name
0	$-x_1^2 - \dots - x_i^2 + x_{i+1}^2 + \dots + x_n^2$	Morse point
1	$-x_1^2 - \dots - x_i^2 + x_{i+1}^2 + \dots + x_{n-1}^2 + x_n^3$	birth point
2	$-x_1^2 - \dots - x_i^2 + x_{i+1}^2 + \dots + x_{n-1}^2 - x_n^4$	swallow's tail point

(The integer i is called the index of the critical point).
The classification of the functions of codimension 0, 1 or 2 follows immediately, using formula (2):

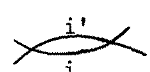
codimension	type	transversal model of the stratification	Name
0	(0; 0)	(one point)	excellent function
1	(1; 0)		birth function
1	(0; 1)		crossing function
2	(2; 0)		swallow's tail function
2	(1+1; 0)		two births
2	(1; 1)		birth + crossing
2	(1; 1)		birth at a critical level
2	(0; 2)		triple valued function
2	(0; 1+1)		two crossings

4. The semi-local lemmas

These lemmas give some homotopic information about the union of "cocells" (a cocell is a connected component of a stratum) whose closure contains a given cocell (of codimension 1 or 2). In terms of "graphics", one can say that the lemmas concern the possibility of deforming a path in $\tilde{\mathcal{F}}$ in order to obtain some given deformation of its graphic.
[Let $(f_t)_{t \in I}$ be a path in $\tilde{\mathcal{F}}$; the corresponding graphic Γ is the subset of $I \times I$ defined by

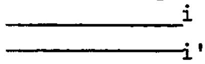
$(t, \alpha) \in \Gamma \Longleftrightarrow \alpha$ is a critical value of $f_t]$.

All deformations of paths we will consider have to be understood with both end points fixed.

Classification of crossings. Suppose a path (f_t) has graphic 

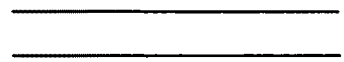
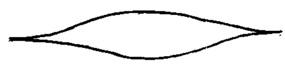
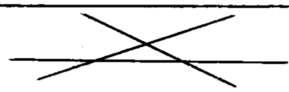
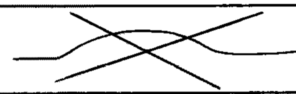
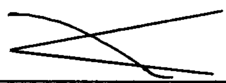
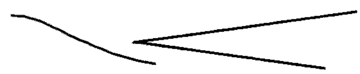
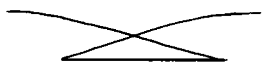
(i' and i are the indices of the corresponding critical points).
Then:

1) if $i' < i$, it is possible to deform (f_t) on such a way that the graphic becomes



2) if $i' = i$, there is an obstruction with values in $\mathbb{Z}_+$ to do the preceeding.

Other lemmas:

	It is possible to deform a path with that graphic ...	... into a path with this graphic
Uniqueness of births		
Uniqueness of deaths		nothing
Triangle		
Beak		
Swallow's tail		

For the precise conditions for these lemmas, see [1].

5. Global part of the proof. [The dimension of V will be denoted by $n-1$]
Smale has defined a decreasing filtration of $\mathcal{F}$ which stops at $\mathcal{E}$. In particular, an important intermediary space between $\mathcal{E}$ and $\mathcal{F}$ is the space $\mathcal{F}_i$ ($0 \leq i \leq n-1$; in fact, the interesting cases occur when i is close to $\frac{n}{2}$)

Definition of $\mathcal{F}_{i,q}$, $\mathcal{F}_{i,q;\alpha}$ and $\mathcal{F}_i$ [q denotes any positive integer].

$f \in \mathcal{F}_{i,q} \iff$ f has $2q$ critical points; each of them is a Morse point; q of them have index $i+1$, q have index i; if c has index $i+1$ and c' has index i, then $f(c) > f(c')$.

$f \in \mathcal{F}_{i,q;\alpha} \iff$ f has the singularities of a function of $\mathcal{F}_{i,q}$, and in addition a birth point (with index i) at an intermediate level.

$$\mathcal{F}_i = \bigcup_{q \geq 0} (\mathcal{F}_{i,q} \cup \mathcal{F}_{i,q;\alpha})$$

Notice that $\mathcal{F}_{i,0} = \mathcal{E}$.

It is easy, using the semi-local lemmas, to prove the following results:

Proposition 1. If $2 \leq i \leq n-3$, then $\mathcal{F}_i$ is connected.

Proposition 2. Let us denote by $\emptyset_{i,q}$ (resp. $\emptyset_i$) the nerve of $\mathcal{F}_{i,q}$ (resp. $\mathcal{F}_i$). There exists a sequence of natural injections

$$\emptyset_{i,0} \hookrightarrow \dots \hookrightarrow \emptyset_{i,q} \hookrightarrow \emptyset_{i,q+1} \hookrightarrow \dots$$

and $\emptyset_i$ is isomorphic to the telescopic limit of this sequence.

[The nerve of a stratification $E^0 \cup E^1 \cup \dots$ is the ordered simplicial complex defined on $\pi_0(E^0) \cup \pi_0(E^1) \cup \dots$ by the relation $A \subset \bar{B}$.
The telescopic limit of a sequence of maps is the inductive limit of the sequence defined by the corresponding mapping cylinders].
The idea is now to associate to every $f \in \mathfrak{F}_{i,q}$ an algebraic invariant ("invariant" means that this element depends only on the image of f in $\emptyset_{i,q}$), and to organize all these invariants in order to obtain a simplicial complex $\mathcal{A}_q$ and a morphism $\bar{\omega}_q: \emptyset_{i,q} \longrightarrow \mathcal{A}_q$, in such a way that there exists a commutative diagram:

$$\begin{array}{ccccccccc} \emptyset_{i,0} & \hookrightarrow & \dots & \hookrightarrow & \emptyset_{i,q} & \hookrightarrow & \emptyset_{i,q+1} & \hookrightarrow & \dots \\ \downarrow{\scriptstyle \bar{\omega}_0} & & & & \downarrow{\scriptstyle \bar{\omega}_q} & & \downarrow{\scriptstyle \bar{\omega}_{q+1}} & & \\ \mathcal{A}_0 & \hookrightarrow & \dots & \hookrightarrow & \mathcal{A}_q & \hookrightarrow & \mathcal{A}_{q+1} & \hookrightarrow & \dots \end{array}$$

Moreover, one proves (using essentially the classification of crossings) that each $\bar{\omega}_q$ is a covering map. Suppose now that we know a family of generators of $\pi_1(\mathcal{A}_q)$, and that each of these generators can be lifted into a loop in $\emptyset_{i,q}$; then we conclude that $\emptyset_{i,q}$ is a product-covering of $\mathcal{A}_q$. If this is true for every q, then $\emptyset_i$ is a product covering of the telescopic limit $\mathcal{A}$ of the $\mathcal{A}_q$. By Proposition 1, $\emptyset_i$ is connected; therefore $\emptyset_i$ is isomorphic to $\mathcal{A}$, and $\emptyset_{i,q}$ is isomorphic to $\mathcal{A}_q$ for every q. In particular, $\emptyset_{i,0}$ is isomorphic to $\mathcal{A}_0$, which consists of a single point; this proves theorem 1'.

Sketch of the definitions of $\mathcal{A}_q$ and $\bar{\omega}_q$

Let us denote by M an intermediary manifold of f (that is a level manifold of f which separates the critical points of index $i+1$ from those of index i) and by W^+ and W^- the closure of the two parts in which M cuts W. It is known by Morse theory that

$$H_{i+1}(W^+, M) \approx H_{n-i}(W^-, M) \approx \mathbb{Z}^q$$

Choose two dual basis $\phi_0: \mathbb{Z}^q \xrightarrow{\approx} H_{i+1}(W^+, M)$

and $\phi_0': \mathbb{Z}^q \xrightarrow{\approx} H_{n-i}(W^-, M)$

("dual" means that ϕ_0 and ϕ_0' take the natural pairing of $\mathbb{Z}^q$ with itself to the pairing of $H_{i+1}(W^+, M)$ and $H_{n-i}(W^-, M)$ given by Poincaré duality). To every basis ϕ of $H_{i+1}(W^+, M)$ one associates the matrix $g = \phi_0^{-1} \circ \phi$ (and to every ϕ', one associates $g' = \phi_0'^{-1} \circ \phi'$); the matrix ${}^t g.g'$ is called the intersection matrix of ϕ and ϕ'.

Definition. A basis $\phi: \mathbb{Z}^q \longrightarrow H_{i+1}(W^+, M)$ is said to be adapted to f if there exists a decreasing order $c_1, \dots, c_q$ of the critical set of $f \mid W^+$, and a family of disjoint descending manifolds $D_1, \dots, D_q$ relative to $c_1, \dots, c_q$, such that (for every $j = 1, \dots, q$) $\phi(\varepsilon_j)$ is equal to the image of the fundamental class of D_j [$\varepsilon_1, \dots, \varepsilon_q$ is the natural basis of $\mathbb{Z}^q$].
A matrix $g \in GL(q, \mathbb{Z})$ is said to be adapted to $f \mid W^+$ if $\phi_0 \circ g$ is an adapted basis of $H_{i+1}(W^+, M)$.

a) Suppose f is "excellent" (in other words, f has codimension zero). Then $\phi(\varepsilon_q)$ is well defined up to sign; $\phi(\varepsilon_{q-1})$ is well defined up to sign and the addition of any integer multiple of $\phi(\varepsilon_q)$; etc. Eventually: <u>given any matrix</u> g <u>adapted to</u> $f \mid W^+$, <u>the family of all adapted matrices is</u> $g.T_q$ [T_q designates the subgroup of $GL(q, \mathbb{Z})$ consisting of lower triangular matrices]. Therefore if g is adapted to $f \mid W^+$ and g' to $f \mid W^-$, then the image of the intersection matrix ${}^t g.g'$ in the double-coset space $\check{T}_q \setminus GL(q, \mathbb{Z}) / T_q$ is independent of the particular choice of g and g'; that image is the algebraic invariant of f.

b) Suppose all critical values of the same index of f are equal. Then T_q has to be replaced by the symmetric group S_q.

c) In the general case, let J be the subset of $\{1, \ldots, q-1\}$ defined by the condition: $j \in J \iff f(c_j) = f(c_{j+1})$; then the set of matrices which are adapted to $f \mid W^+$ is gS_JT_J, where $S_J \subset S_q$ is the subgroup generated by the transpositions s_j, $j \in J$, and $T_J \subset T_q$ is the subgroup **consisting of** the matrices (a_{kl}) whose off-diagonal a_{kl} are zero for $(k-1, l) \in J \times J$.

<u>The algebraic lemma</u>

Let us denote by t_j the elementary matrix (a_{kl}) such that $a_{j+1,j} = 1$. For $g \in GL(q, \mathbb{Z})$, $\ddot{g}$ will denote the image of g in $\check{T}_q \setminus GL(q, \mathbb{Z}) / T_q$.

The group $\pi_1(\mathcal{A}_q; \ddot{e})$ is generated by the following loops:

$$\gamma_{j,\lambda} = [\ddot{e}, \ddot{\overline{t_j^\lambda s_j}}, \ddot{\overline{s_j t_j^\lambda s_j}}] \qquad 1 \leq j \leq q-1; \quad \lambda \in \mathbb{Z};$$

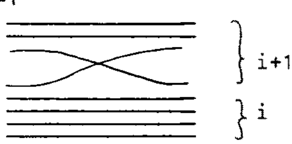
$$\zeta_j = [\ddot{e}, \ddot{s}_j, \ddot{\overline{s_j t_j s_j}}] \qquad 1 \leq j \leq q-1;$$

$$\delta_j = [\ddot{e}, \ddot{\overline{t_j s_j}}] \qquad 1 \leq j \leq q-1.$$

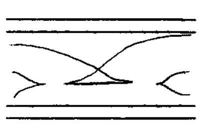
[s_j denotes the transposition which permutes j with $j+1$].

<u>Lifting of the generators</u>

Consider for example the generator δ_{q-1}. The graphic of a path α in $\mathcal{F}_{i,q}$ which represents δ_{q-1} is the following:



By Smale's cancellation lemma, α can be deformed into a path α' with graphic



Using the swallow's tail lemma and the uniqueness of births, one can deform α' into a path with "trivial" graphic; this proves that the end points of α can be joined by a path in the space $\mathcal{F}^0_{i,q}$ (subspace of $\mathcal{F}_{i,q}$ consisting of functions of codimension zero). In other words, the image of α in $\emptyset_{i,q}$ is a loop.

The proofs of the same property for the generators $\gamma_{j,\lambda}$ and ζ_j are analogous.

REFERENCES

[1] J. Cerf - La Stratification naturelle des espaces de fonctions différentiables réelles et le theorème de la pseudo-isotopie, to appear in the Publications Scientifiques de l' I.H.E.S., vol. 39.

[2] F. Sergeraert - Comptes Rendus Acad. Sc. Paris. Série A, t. 271, pp. 453-56.

Département de Mathématique
Faculté des Sciences
ORSAY