1 Vector Bundles

Intuitively, vector bundles are families of vector spaces parametrized by a particular base space. This leads to the natural definition:

Definition 1.1. (Quasi-vector bundles) Let k be $\mathbb{R}$ or $\mathbb{C}$. A k-quasi-vector bundle over X a topological space is the data $\pi : E \to X$ a surjection such that each of the fibers $E_x := \pi^{-1}(x)$ carries a k-vector space structure for all $x \in X$.

Now as is, the fibers of π need not be related to one another. They could jump in dimension and in general be rather ill-behaved. In order to have a more tractable definition, we might require that such a map be locally trivial.

Definition 1.2. (Vector bundles) A k-quasi-vector bundle is a k-vector bundle if for each $x \in X$ there exists an open neighborhood U of x and a homeomorphism $\varphi : \pi^{-1}(U) \to U \times E_x$ which makes the following diagram commutative.

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times E_x \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & U & \end{array}$$

and $\varphi_{E_y} : E_y \to \{y\} \times E_x$ is a linear isomorphism. These maps φ are called a local trivialization of E over U.

Remark. The fact that we require local triviality now means that the fibers of vector bundles over connected components (if the space is locally path connected) is now constant, i.e. if $[x] = [y] \in \pi_0(X)$ then $\dim E_x = \dim E_y$.

Example 1. If V is a finite dimensional vector space then for X a topological space $p_1 : X \times V \to X$ is a vector bundle. This is called the trivial vector bundle since it is the constant (trivia) family with value V over every point.

Example 2. Consider the $\mathbb{Z}$ action on $\mathbb{R} \times \mathbb{R}$ given by $a(x, y) = (x + a, (-1)^a y)$. The quotient space $(\mathbb{R} \times \mathbb{R})/\mathbb{Z}$ has a natural vector bundle structure over $\mathbb{R}/\mathbb{Z}$ coming from the trivial structure on $\mathbb{R} \times \mathbb{R}$.The space $\pi : (\mathbb{R} \times \mathbb{R})/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is the familiar Möbius bundle. Because $\mathbb{Z}$ acts linearly, freely, and properly discontinuously $q : \mathbb{R} \times \mathbb{R} \to (\mathbb{R} \times \mathbb{R})/\mathbb{Z}$ is a covering map and for $p \in \mathbb{R}$ there is a small enough neighborhood say $(p - \epsilon, p + \epsilon) \times \mathbb{R}$ which projects homeomorphically onto $q((p - \epsilon, p + \epsilon) \times \mathbb{R})$. Let $\tilde{q} : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ by the quotient map. We then have a local trivialization $\pi^{-1}(\tilde{q}(p - \epsilon, p + \epsilon)) \to (p - \epsilon, p + \epsilon) \times \mathbb{R}$.

One natural way to get a vector bundle from another one is the *pullback bundle*. Let $f : X \to Y$ be a continuous map and $\pi : E \to Y$ a vector bundle. We can form the pullback bundle of E by f, $f^*\pi : f^*E \to X$ by defining $f^*E_x := E_{f(x)}$. To see that this is locally trivial, suppose $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ is a collection of local trivializations of E which cover $f(X)$. Since f is continuous $f^{-1}(U_\alpha)$ is an open cover of X. We can then create maps $\psi_\alpha : f^*\pi^{-1}(f^{-1}(U_\alpha)) \to f^{-1}(U_\alpha) \times \mathbb{R}^n$ by $u \mapsto (\varphi_\alpha(u), f^*\pi(u))$. The pullback bundle fits into the following pullback diagram

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ {\scriptstyle f^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & Y \end{array}$$

As is often the case, we may probe objects by looking at maps to and from them. In thinking of vector bundles as families of vector spaces, this implies that the correct notion of morphism between vector bundles is a continuously varying family of morphisms.

Definition 1.3. Let $\pi : E \to X$ and $\pi' : E' \to X$ be vector bundles over X. A morphism of vector bundles $F : (\pi, E, X) \to (\pi', E', X)$ is a map $F : E \to E'$ which makes the following diagram commute:

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi'} & \\ & X & \end{array}$$

i.e. F induces a map $F_x : E_x \to E'_x$ for all $x \in X$, and F_x is linear for all x.

2 Clutching Functions

One way we might like to specify a morphism of vector bundles is by specifying one locally on open sets. The following result establishes the "sheafiness" of vector bundles and their morphisms.

Theorem 2.1. *Let (π, E, X) and (π', E', X) be vector bundles over X. Take the following data:*

i) An open cover $\{U_\alpha\}_{\alpha \in A}$ of X

ii) A collection of morphisms $F_\alpha : E\big|_{U_\alpha} \to E'\big|_{U_\alpha}$ such that for each $\alpha, \beta \in A$ $F_\alpha\big|_{U_\alpha \cap U_\beta} = F_\beta\big|_{U_\alpha \cap U_\beta}$

There exists a unique morphism of vector bundles $F : E \to E'$ *such that* $F\big|_{U_\alpha} = F_\alpha$.

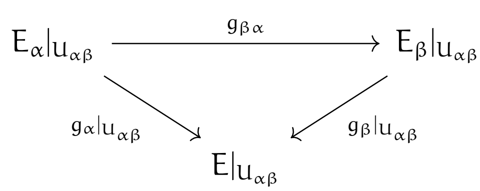
Proof. (Sketch) The map is uniquely defined, since its value must agree with the F_α on every fiber E_x. It suffices to show that such a map is continuous. Since it is continuous when restricting to the open covers $\pi^{-1}(U_\alpha), \pi'^{\prime -1}(U_\alpha)$ the map defined in the natural way is continuous. □

This way of producing morphisms is called clutching of morphisms, since we specify morphisms on an open cover and then "clutch" them together. We can produce a vector bundle in an analogous way by specifying a family of vector bundles over an open cover.

Theorem 2.2. *Let* $\{U_\alpha\}_{\alpha\in A}$ *be an open cover of* X *and* $(\pi_\alpha, E_\alpha, U_\alpha)$ *be a collection of vector bundles over the* U_α. *Define* $U_{\alpha\beta} = U_{alpha} \cap U_\beta$ *and* $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$. *Let* $g_{\beta\alpha} : E_\alpha|_{U_{\alpha\beta}} \to E_\beta|_{U_{\alpha\beta}}$ *be a family of isomorphisms which satisfy the cocycle condition:*

$$g_{\gamma\alpha}|_{U_{\alpha\beta\gamma}} = g_{\gamma\beta}|_{U_{\alpha\beta\gamma}} \circ g_{\beta\alpha}|_{U_{\alpha\beta\gamma}}.$$

There exists a unique bundle (π, E, X) *over* X *and isomorphisms* $g_i : E_\alpha \to E|_{U_\alpha}$ *making the following diagram commutes.*



Proof. (Sketch) The important point is that we can construct the total space of the bundle as a quotient of $\bigsqcup_{\alpha\in A} E_\alpha$ under the equivalence relation $e_\alpha \sim e_\beta$ if $g_{\beta\alpha}(e_\alpha) = e_\beta$. Because each E_α is equipped with a projection and the $g_{\beta\alpha}$'s are bundle maps, the map $\tilde{\pi} : \bigsqcup_\alpha E_\alpha \to X$ induces a map $\pi : (\bigsqcup_\alpha E_\alpha)/\sim \to X$ the bundle projection. The desired local isomorphisms are induced by the inclusion maps $E_\alpha \to \bigsqcup E_\alpha$. □

We can use this theorem to specify bundles over locally contractible spaces by specifying a contractible open cover and the data of "transition functions" on the intersections of this open cover. First, we must prove that any vector bundle over a contractible space is trivial which follows from the following lemma:

Lemma 2.3. *Let* $E \to Y$ *be a vector bundle over* Y *and* $f, g : X \to Y$ *homotopic maps. The pullback bundles* f^*E *and* g^*E *are isomorphic.*

Proof. cf. Dan Freed's K-theory notes. □

Now if U is contractible, we have $\mathrm{Id} : U \to U$ homotopic to $* : U \to U$ $u \mapsto *$. Given a vector bundle $E \to U$, we have $E \cong \mathrm{Id}^*E \cong *^*E = U \times E|_*$ meaning that E is trivial. This means that given an open cover of a space X by contractible neighborhoods, a given vector bundle restricts to trivial vector bundles on each neighborhood. Since vector bundles over the elements of the cover are trivial, we can describe the data of the transition functions via functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(k)$. Using the fact that the homotopic transition functions describe isomorphic bundles, we can conclude that the homotopy type of the $g_{\alpha\beta}$'s really determines the isomorphism type of E.

Example 3. Because S^n is covered by two hemispherical balls $B^\pm$, given $E \to S^n$ a vector bundle, we may trivialize E over the contractible cover $\{B^\pm\}$ and yield a transition function $B^+ \cap B^- \to GL_n(k)$. We can see that $B^+ \cap B^-$ deformation retracts to the equatorial copy of S^{n-1}. Since the contractible cover only has two elements, any function $B^+ \cap B^-$ yields a cocycle and determines a vector bundle. This means that vector bundles over S^n are classified by homotopy classes of maps $S^{n-1} \to GL_n(k)$.

Example 4. One way to build the tangent bundle of a smooth manifold is through the clutching construction. Let M^n be a smooth manifold with atlas given by $\varphi_\alpha : U_\alpha \to V_\alpha$. Over each U_α we have the trivial vector bundle $U_\alpha \times \mathbb{R}^n$. Since the charts in the atlas are compatible, we may form clutching functions over each chart by $g_{\alpha\beta}(x, v) := (x, d(\varphi_\beta \circ \varphi_\alpha)^{-1}(\varphi_\alpha))$. The $g_{\alpha\beta}$ satisfy the cocycle condition and hence give a well defined vector bundle $E \to M$. The proof that TM is a smooth manifold shows that TM is actually isomorphic to E.

Remark. The proceeding construction of the tangent bundle is a mathematical rephrasing of the oft heard physics phrase "a vector is something that transforms like a vector". The clutching functions tell us that if we are locally working on two open sets U_α and U_β, then in order to go from an expression over U_α (say a section $\sigma : U_\alpha \to E_\alpha$) to an expression over $U_{\alpha\beta}$ expressed with respect to E_β we need only apply $g_{\alpha\beta}$. Unravelling the defintions of $g_{\alpha\beta}$ for the tangent bundle, one finds that in the overlapp of two charts, with coordinates x^i and y^j one has $\frac{\partial}{\partial y^j} = \frac{\partial x^i}{\partial y^j}\frac{\partial}{\partial x^i}$, the familiar expression from differential geometry.

Applying a group homomorphism $h : GL(V) \to GL(V')$ allows us to again produce new vector bundles from old ones. Over the trivializing neighborhoods, composing $h \circ g_{\alpha\beta}$ gives a new cocycle and hence a new vector bundle. For example, the map $GL_n(V) \to GL_n(V^*)$ by the adjoint, yields the dual bundle to the original cocycle (the bundle of linear functions $E_x \to k$).

Remark. One can do many natural operations on vector bundles. Given a functor $\varphi : \mathrm{Vect}_k \to \mathrm{Vect}_k$ such that the map $\varphi_{U,V} : \mathrm{Hom}(U, V) \to \mathrm{Hom}(\varphi(U), \varphi(V))$ is continuous, we can produce a family of functors $\varphi_X : \mathrm{Vect}_k(X) \to \mathrm{Vect}_k(X)$ which applies φ fiberwise in X. This leads to the notions of Whitney sum and tensor product of vector bundles. The tensor product $\otimes : \mathrm{Vect}_k(X) \times \mathrm{Vect}_k(X) \to \mathrm{Vect}_k(X)$ makes $\mathrm{Vect}_k(X)$ into a monoidal category, this structure along with direct sums will be crucial to defining the K-groups.

3 Grassmannians

One approach which allows one to classify vector bundles through homotopy is via Grassmannians.

Let $\mathrm{Proj}_n(V)$ denote the space of projections $P : V \to V$ with $\mathrm{rank}(P) = n$. One can verify that a continuous map $g : X \to \mathrm{Proj}_n(V)$ yields a projection P on the trivial bundle $X \times V$, and its image defines a vector bundle $E_f \to X$. We can see that for $g : Y \to X$ we have $E_{f\circ g} \cong f^*E_g$. If $X = Y = \mathrm{Proj}_n(V)$ and $g = 1$ the bundle $E_n(V) = E_g$ is the *tautological bundle* over $\mathrm{Proj}_n(V)$. Concretely, this is the bundle over $\mathrm{Proj}_n(V)$ with fiber given by $E_n(V)_P := \mathrm{Im}(P)$.

The isomorphism class of E_f only depends the homotopy type of f and hence $f \mapsto E_f$ gives a correspondence

$$[X, \mathrm{Proj}_n(V)] \to \pi_0(\mathrm{Vect}_k(X))$$

Block matrices give us a maps $\iota_{N,N'} : \mathrm{Proj}_n(K^N) \to \mathrm{Proj}_n(K^{N'})$ and taking a direct limit gives a map

$$\lim[X, \mathrm{Proj}_n(K^N)] \to \pi_0(\mathrm{Vect}_k(X))$$

Theorem 3.1. *For "nice enough" spaces the map* $\lim[X, \mathrm{Proj}_n(K^N)] \to \pi_0(\mathrm{Vect}_k(X))$ *is bijective.*

In preceding discussion, we used the correspondence between projections and their image. This correspondence is surjective, but not bijective. To any n dimensional subspace $W \subset V$ there are at least as many projections $V \to V$ with image V as there are metrics on V (taking the orthogonal compliment and the relevant projection). If we equip V with a metric, then there is a natural bijection between orthogonal projections and n-dimensional hyperspaces in V. Denote the space $\mathrm{Gr}_n(V) := \{p \in \mathrm{Proj}_n(V) | p \text{ is orthogonal}\}$ with the subspace topology, this is called the Grassmannian of n-planes in V.

Theorem 3.2. *The space of projections* $\mathrm{Proj}_n(V)$ *deformation retracts onto* $\mathrm{Gr}_n(V)$.

Proof. Use the square root to define the homotopy $F(p,t) = \sqrt{1 + tJ^*J}p\sqrt{1+tJ^*J}^{-1}$, where $J = 2p - 1$ for $p \in \mathrm{Proj}_n(V)$. □

Define the limit $BO(n) := \lim \mathrm{Gr}_n(\mathbb{R}^N)$ and $BU(n) := \lim \mathrm{Gr}_n(\mathbb{C}^N)$. The deformation retract means that the previous isomorphisms give

$$[X, BO(n)] \cong \pi_0(\mathrm{Vect}_{\mathbb{R}}(X)), \quad [X, BU(n)] \cong \pi_0(\mathrm{Vect}_{\mathbb{C}}(X))$$

The spaces $BO(n)$ and $BU(n)$ a classifying spaces since they classify euclidean and hermitian bundles (i.e. they represent the functor $X \mapsto \pi_0(\mathrm{Vect}_k(X))$.)