K-Theory Learning Seminar **Fall 2023**

Week 11: The Chern Character, the Thom Isomorphism(s) et al.

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Review

This talk is mostly just setting up Jacob for next week when he'll explain the K-theoretic formulation and proof of the Atiyah-Singer index theorem.

Since this is part of a series of talks, we will briefly review what Justin talked about last time, and how it relates to what Jacob will talk about next time.

Definition 1.1: Fredholm Operators

A *Fredholm operator* is a bounded operator between Banach spaces with finite dimensional kernel and cokernel. The *index* of such an operator T is

$$\text{index}(T) = \dim\ker T - \dim\operatorname{coker} T$$

The intuition is that Fredholm operators T are the operators for which solutions to the equation $Tx = y$ can be defined by a finite amount of data. The index is basically the infinite-dimensional stand in for the rank-nullity theorem. The ΨDOs (which are basically any operators that look like a differential operator after a Fourier and inverse Fourier transform) were our principal examples of Fredholm operators. Crucially, the index is homotopy invariant.

The relevance of the Fredholm operators (to our topological K-theory seminar) begins with the following theorem:

Theorem 1.2: Atiyah-Jänich

Given X compact Hausdorff, $\mathcal{F}$ the set of Fredholm operators from some infinite dimensional separable Hilbert space H to itself, then the index map

$$\text{index} : [X, \mathcal{F}] \to K^0(X)$$

is an isomorphism.

The fact that the choice of Hilbert space does not matter stems from the fact that all separable (containing a countable dense subset) infinite dimensional Hilbert spaces are isometrically isomorphic.

Thus, the space of Fredholm operators is a classifying space for K-theory, similar to the $K(G, n)$ for cohomology.

K-Theory with Compact Support

In this section, we follow Landweber[1] pretty closely.

[1] G. D. Landweber. K-theory and elliptic operators. *arXiv preprint math/0504555*, 2005

Recall that, nine weeks ago, we insisted on a compact base space when defining our K-groups. The technical reason for doing so at the time was the following lemma:

Lemma 2.1

For every finite rank vector bundle E over X compact Hausdorff, there exists another bundle E' such that $E \oplus E' \cong \varepsilon^m$.

Recall that ε^m denotes the rank m trivial vector bundle, and that Urysohn's lemma states that a topological space is normal iff disjoint closed subsets can be separated by continuous functions.

The proof of this lemma used, in an essential way, the compactness of X, in particular, applying Urysohn's lemma (since compact Hausdorff spaces are normal), to constructively build such an "inverse" vector bundle E'. We also used compactness explicitly for the existence of a finite trivializing cover. In fact, this lemma is *false* in the non-compact case:

Counterexample 2.2

The canonical bundle over $\mathbb{RP}^\infty$ is not a summand of any trivial vector bundle.

See Hatcher[2] for details.

[2] A. Hatcher. *Vector Bundles and K-Theory*. 2017. URL `https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf`

The above lemma was essential in defining the reduced K group $\tilde{K}(X)$, and without it, it is not possible to represent the formal inverses we recklessly added to our monoid by honest vector bundles. One potential resolution is to define the K-theory of a non-compact space in terms of its one-point compactification:

Definition 2.3: $K^0(X)$

For X Hausdorff, locally compact, set

$$K^0(X) := \tilde{K}^0(X_+)$$

where X_+ is the one-point compactification of X. If we restrict to proper maps, this is a functor from the category of locally compact Hausdorff spaces (and proper maps) to the category of commutative rings.

On its face, this is somewhat *ad-hoc*. To motivate this, note that in defining the K-group, we are modding out by relations of the form $[B] = [A] + [C]$ for every short exact sequence

$$0 \to A \to B \to C \to 0$$

Now, if we lose compactness of the base, we no longer know that every vector bundle is a summand of a trivial vector bundle $\rightsquigarrow$ short exact sequences no longer suffice. It is then maybe reasonable to consider instead chain complexes of vector bundles

$$0 \to E^0 \xrightarrow{\alpha_0} E^1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} E^n \to 0$$

where $\alpha_k \alpha_{k-1} = 0$. Two complexes $E^\bullet$ and $F^\bullet$ are *homotopic* if there exists a complex $G^\bullet$ over $X \times [0,1]$ such that $G^\bullet|_{X\times\{0\}} = E^\bullet$ and $G^\bullet|_{X\times\{1\}} = F^\bullet$. The *support* of a complex is the set of points $x \in X$ where the complex fails to be exact. Then, we claim:

Theorem 2.4

Let $C(X)$ denote the set of homotopy classes (via compactly supported homotopies) of complexes over X with compact support, and $C_\emptyset(X) \subseteq C(X)$ the set of complexes with empty support (i.e, exact sequences), then

$$K(X) \cong C(X)/C_\emptyset(X)$$

A reference for this formulation of K-theory is the appendix of Segal's thesis[3]. This results holds in the compact and non-compact case (i.e, it agrees with our "definition" above). One can show that we don't need arbitrary length complexes, in fact, complexes of the form $0 \to E \to F \to 0$ of vector bundles which are isomorphic outside of a compact set suffice.

[3] G. Segal. *Equivariant K-theory*. PhD thesis, University of Oxford, 1966

Our application for this formulation of K-theory is that the total spaces of our vector bundles (which are necessarily non-compact) are no longer second class citizens, and we can consider their K-groups as well. Given complex vector bundles E, F over X, and a (real or complex, but both will be important) vector bundle $\pi : V \to X$, we obtain vector bundles π^*E and π^*F over V. We say that a homomorphism $\alpha : \pi^*E \to \pi^*F$ is *homogeneous* of degree $m > 0$ if for all $v \in V$, $\lambda > 0$,

$$\alpha_{\lambda v} = \lambda^m \alpha_v : (\pi^*E)_v = E_{\pi(v)} \to (\pi^*F)_v = F_{\pi(v)}$$

Picking a metric on V, α is clearly determined completely by its restriction to the sphere bundle $S(V)$.

Suppose X is compact, and $E^\bullet$ is a complex over V

$$0 \to \pi^*E^0 \xrightarrow{\alpha_0} \pi^*E^1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} \pi^*E^n \to 0$$

where $\alpha^2 = 0$ and α is homogeneous of degree m. If $E^\bullet$ is exact on $S(V)$, then its support is clearly the zero section of $V \to X$, which in particular, is isomorphic to X and is therefore compact, so $E^\bullet$ is compact and represents an element of $K(V)$. One can show that $K(V)$ can be defined solely in terms of such homogeneous complexes for each $m > 0$. See Landweber[4] for a detailed proof.

[4] G. D. Landweber. K-theory and elliptic operators. *arXiv preprint math/0504555*, 2005

The Thom Isomorphism(s)

K-Theory

Let $\pi : V \to X$ be a complex rank n vector bundle over a compact space X. Then we have the following degree one homogeneous complex $\wedge^* V$ of vector bundles over V (not X!) called the *exterior complex*:

$$0 \to \pi^*(\wedge^0 V) \xrightarrow{\alpha} \pi^*(\wedge^1 V) \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} \pi^*(\wedge^n V) \to 0$$

where $\alpha : (v, w) \mapsto (v, v \wedge w)$ for all $v \in V$, $w \in (\pi^*(\wedge^k V))_v = \wedge^k V_{\pi(v)}$ where π^* pulls back vector bundles over X to vector bundles over the total space V.

Since $\alpha^2 = 0$, this complex defines an element $\lambda_V \in K(V) = \tilde{K}(V^+)$. As we will discuss below, the one-point compactification of a vector bundle over a compact base goes by another name: the *Thom space* of the bundle, which for a trivial bundle corresponds to iterated reduced suspension of the base.

Theorem 3.1: Thom Isomorphism Theorem

If V is a complex vector bundle over a compact space X, and $K(V)$ is a $K(X)$-module via the map $\pi^* : K(X) \to K(V)$ then the map $\varphi : K(X) \to K(V)$ given by multiplication by λ_V is an isomorphism.

By multiplication by λ_V, we are referring to the tensor product of chain complexes since V is necessarily non-compact.

Example 3.2

For X a point, $V = \mathbb{C}^n$, we have the Thom isomorphism

$$\varphi : K(\bullet) \xrightarrow{\sim} K(\mathbb{C}^n) \cong \tilde{K}(S^{2n})$$

so $\tilde{K}(S^{2n}) \cong \mathbb{Z} \implies K(S^{2n}) = \mathbb{Z}^2$.

The K-theoretic Thom isomorphism theorem is very powerful, and lends credence to this formulation of K-theory; for example, it gives an extremely quick proof of Bott periodicity.

Cohomology

The classical Thom isomorphism theorem is given in terms of ordinary cohomology, and is essentially a generalization of the fact $H^i(B) \cong H^{i+1}(\Sigma B)$, i.e, that reduced suspension just increases the indexing of cohomology by one.

Theorem 3.3: Thom Isomorphism

Let $\xi : E \to B$ be a rank n real vector bundle. There is a unique cohomology class $u \in H^n(E, E_0) = H^n_c(E)$ (called the *Thom class*) whose restriction to $H^n(F, F_0)$ is nonzero for every fiber F (where E_0 is the complement of the zero section and $F_0 = F \cap E_0$) for a fiber F) giving an orientation. Moreover, the correspondence $y \mapsto y \smile u$ maps $H^j(E)$ isomorphically onto $H^{j+n}(E, E_0)$ for every integer j.

This can also be interpreted more geometrically in terms of the *Thom space* of the bundle ξ, where the above isomorphism becomes

$$H^i(B) \xrightarrow{\sim} \tilde{H}^{n+i}(T(\xi))$$

To construct the Thom space, choose a Riemannian metric on E (hence paracompactness of the base), and, using the metric, let $D(\xi)$ be the unit ball bundle (whose fibers are the unit balls in the fibers of E) and $S(\xi)$ the unit sphere bundle (defined similarly). Then $T(\xi) = D(\xi)/S(\xi)$ is the Thom space of ξ. This amounts to taking the one-point compactification of each fiber and identifying the "points at infinity." For a compact base, this is the same as taking the one-point compactification of the total space.

So, in particular, the Thom space of a trivial rank one bundle is the reduced suspension of the base, and, inductively, the Thom space of a trivial rank n bundle is the iterated reduced suspension, with the above isomorphism reducing to the suspension isomorphism:

$$H^i(B) \xrightarrow{\sim} \tilde{H}^{n+i}(\Sigma^n B)$$

For CW complexes, the reduced suspension is homotopy equivalent to the ordinary suspension.

For the base compact and oriented, the Thom isomoprhism is equivalent to Poincaré-Lefschetz.

Chern Classes/Characters

Characteristic Classes

Throughout this seminar, we have made passing references to the theory of characteristic classes. Here we will do our best to give a crash course in their construction and significance.

Definition 4.1

Given a (real or complex) vector bundle $E \to X$, a *characteristic class* for E is a cohomology class $c(E) \in H^*(X)$ which is *natural* in the sense that for any map $f : Y \to X$, $c(f^*E) = f^*c(E)$.

The two most basic families of characteristic classes are Stiefel-Whitney and Chern classes, for real and complex vector bundles respectively. Both can be understood as *obstructions* to the problem of finding $n-i+1$ linearly independent sections of the vector bundle in question, in a way which we can make precise.

This is the easiest way to state what Chern classes are, and it makes the applications to obstruction theory immediate, unfortunately, at the cost that it's not very useful for calculations.

Let $E \to X$ be a complex rank n vector bundle over X, and let $e_1, \cdots, e_{n-i+1}$ be $n-i+1$ generic sections of E (which we can choose to be pairwise transverse and transverse to the zero section). If these sections are linearly dependent anywhere, then their locus of degeneracy should be of complex codimension i (this dimension count is somewhat nontrivial), and therefore the Poincaré dual to this locus should be an element of $H^{2i}(X;\mathbb{Z})$. We call this element $c_i(E)$, and one can show that this is well-defined. If there exist $n-i+1$ linearly independent sections of E, then, clearly, $c_i(E)=0$, and therefore $c_{i+1}, c_{i+2}, \cdots = 0$.

The *total Chern class* $c(E) \in H^*(X;\mathbb{Z})$ is just the sum

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots$$

(where $c_0(E) = 1$)

The essential properties of the Chern classes are the following:

$c_0(E) = 1$, additivity, naturality, and one additional normalization criterion for the tautological bundle over $\mathbb{CP}^k$ actually suffice to uniquely *define* the Chern classes, though this is nontrivial to show.

- $c_0(E) = 1$ for all E
- Naturality: If $f : Y \to X$ is continuous, $c(f^*E) = f^*c(E)$ as required by our definition of a characteristic class
- Additivity: For E, F vector bundles over X,

  $$c(E \oplus F) = c(E) \smile c(F)i \implies c_k(E \oplus F) = \sum_{i=0}^{i} c_i(E) \smile c_{k-i}(F)$$

  This property can be rephrased in terms of short exact sequences (trivially, since all short exact sequences are split) as follows: for any SES of complex vector bundles

  $$0 \to E' \to E \to E'' \to 0$$

  $c(E) = c(E') \smile c(E'')$. This is useful for an axiomatic approach.

  Suppose that $E = L_1 \oplus \cdots \oplus L_n$ splits as a direct sum of line bundles. Then, letting $x_i = c_1(L_i) \in H^2(X;\mathbb{Z})$, the above formula tells us that

  $$c(E) = \prod_{i=1}^{n} c_1(L_i) = \prod_{i=1}^{n}(1+x_i) = 1 + \sum_i x_i + \sum_{i<j} x_i x_j + \cdots$$

  whence it is clear that $c_k(E)$ is the k^{th} *elementary symmetric polynomial* of the x_i. It follows that any symmetric polynomial in the x_i can be expressed as a polynomial in the Chern classes.

  Note that additivity also implies that c is *stable*, that is, it is insensitive to summing with a trivial bundle.

As an example of a complex vector bundle that does not split as a sum of line bundles, take $T\mathbb{CP}^2$, which has total Chern class $1 + 3h + 3h^2$ (where h is Poincaré dual to the class of a hyperplane $\mathbb{CP}^1 \subseteq \mathbb{CP}^2$) and this polynomial does not factor.

- Splitting Principle: For any $E \to X$, there exists a space $F(E)$ (called the flag bundle of E) and a map $\pi : F(E) \to X$ such that $\pi^* : H^*(X) \to H^*(F(E))$ is injective and $\pi^*(E)$ splits as the direct sum of complex line bundles. This is then useful for calculating $c(E)$ since

$$\pi^* c(E) = c(\pi^* E) = c(L_1 \oplus \cdots \oplus L_n) = \prod_{i=1}^{n} c(L_i) = \prod_i (1 + x_i)$$

as above, where $x_i = c_1(L_i)$ are the *Chern roots* of E, living in some enlargement of the cohomology ring.

The Chern Character

Why does one write down the Chern character? One possible reason is that this is all secretly about symmetric functions, and $\sum_i e^{x_i}$ is a convenient way to package together all of the power-sum symmetric polynomials. I imagine that other power series whose expansions are symmetric polynomials in increasing degrees should have similar significance (we will see another one below).

Definition 4.2: The Chern Character

The *Chern character* of a rank n complex vector bundle $E \to X$ is given by

$$\mathrm{ch}(E) = \sum_i e^{x_i} = n + \sum_i x_i + \frac{1}{2!} \sum_i x_i^2 + \cdots =$$
$$n + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots$$

where, for example,

$$c_1^2(E) - 2c_2(E) = \left(\sum_i x_i\right)^2 - 2\sum_{i<j} x_i x_j = \sum_i x_i^2$$

so the terms in the expansion for the Chern character simply rewrite the power-sum symmetric polynomials in terms of the elementary symmetric polynomials.

It follows that

$$\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$$

and

$$\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \smile \mathrm{ch}(F)$$

Extending the Chern character to $K(X)$ (since Chern classes are stable), we may regard the Chern character as a ring homomorphism $\mathrm{ch} : K^0(X) \to H^{2*}(X; \mathbb{Q})$. Replacing X with ΣX and using the degree shift isomorphism between the cohomology of X and that of ΣX (and using $K^1(X) = K^0(\Sigma X)$ from Bott periodicity), we also have $\mathrm{ch} : K^1(X) \to H^{2*+1}(X; \mathbb{Q})$. Together, this can be regarded as a homomorphism

$$\mathrm{ch} : K^*(X) \to H^*(X; \mathbb{Q})$$

Theorem 4.3

For X a finite CW-complex, the Chern character

$$\mathrm{ch} : K^*(X) \otimes \mathbb{Q} \to H^*(X; \mathbb{Q})$$

is an isomorphism.

Sidestepping for now the moral significance of this result, we now have Thom isomorphisms in K^* and H^* and an isomorphism between them. Does the natural square one can draw commute?

$$\begin{array}{ccc} K(X) & \longrightarrow & K(E) \\ {\scriptstyle \mathrm{ch}}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^*(X;\mathbb{Q}) & \longrightarrow & H_c^*(E;\mathbb{Q}) \end{array}$$

Unfortunately, no. The difference between the two paths turns out to be essentially the following correction factor:

Note that the power series in the definition of the Todd class is manifestly symmetric in the variables, and, therefore, its degree expansion should consist of symmetric polynomials.

Definition 4.4: Todd Class

The *Todd class* of a complex vector bundle E is given by

$$\mathrm{td}(E) = \prod_{i=1}^{n} \frac{x_i}{1 - e^{-x_i}} 1 + \frac{1}{2} c_1 + \frac{1}{12}(c_2 + c_1^2) + \frac{c_1 c_2}{24} + \cdots$$

By definition, $\mathrm{td}(E \oplus F) = \mathrm{td}(E) \smile \mathrm{td}(F)$.

Jacob will tell us more about some/all of this next week.

References

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