Ref: Spin Geo. Lawson - Michelsohn
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$$\chi(E)\cdot\mathcal{J}(E) = ch(\lambda_{-1}(E))$$

1. Chern Character defect

We have the following diagram

$$\begin{tikzcd} K(X) \arrow[r, "ch"] \arrow[d, "i_!"] & H^*(X,\mathbb{Q}) \arrow[d, "i_!"] \\ K(E) \arrow[r] & H^*_{cpt}(E,\mathbb{Q}) \end{tikzcd}$$

$i_!(u) = \Lambda_{-1}\cdot\pi^* u$

where ~~$\Lambda_{-1} = \pi^*\sum(-1)^k\Lambda^k_{\mathbb{C}}E$~~

~~with symbol~~

or $\Lambda_{-1} = [\pi^*\Lambda^{even}_{\mathbb{C}}E, \pi^*\Lambda^{odd}_{\mathbb{C}}E, \sigma]$

$\sigma = e\wedge - \iota_{e^*}$

on cohomology $\pi_* = (i_!)^{-1}$ ~~o fo tot but~~ where $\pi_!$ is integration along fibres.

Define the class $\mathcal{J}(E) = \pi_! ch(i_!(1))$ measures the failure of the above to commute. Since for $\xi\in K(X)$

$$\begin{aligned} \pi_!(ch(i_!(\xi))) &= \pi_!(ch(\Lambda_{-1}\cdot\pi^*\xi)) \\ &= \pi_!(ch(\Lambda_{-1})\cdot ch(\pi^*\xi)) \\ &= \pi_!(ch(\Lambda_{-1})\cdot\pi^* ch(\xi)) \\ &= \pi_!(ch(\Lambda_{-1}))\cdot ch(\xi) \\ &= \pi_!(ch(i_!(1)))\cdot ch(\xi). \end{aligned}$$

One can show that $\mathcal{J}(E)$ is natural and hence a characteristic class.
Since $\pi_! = (i_!)^{-1}$ $\quad i_!\mathcal{J}(E) = ch(\Lambda_{-1})$ and i^*

$$i^* i_!\mathcal{J}(E) = i^* ch(\Lambda_{-1}) = ch(i^*\Lambda_{-1}) = ch(\lambda_{-1}(E))$$

but $i^* i_! = \chi(E) = e(E)$ pretending E is a sum of lines

we have $\mathcal{J}(E) = \cancel{\pi\, ch(\lambda_{-1}(E))}\ \underline{ch(\Lambda^{even}E - \Lambda^{odd}E)} = \prod \frac{(1-e^{x_k})}{x_k}$

$$= (-1)^n Td_{\mathbb{C}}(\bar{E})^{-1}$$

2. towards the index thm.

We can define the analytical index of an elliptic operator $P:\Gamma(E)\to\Gamma(F)$ $\quad \mathrm{ind}\, P = \dim\ker P - \dim\operatorname{coker} P$. and we can actually define our index through K thy.

Consider an embedding $f: X \to \mathbb{R}^n$. We have the induced extension map

$$\cancel{f_!: K(X) \to K} \quad f_!: K(TX) \to K(T\mathbb{R}^n).$$

We have $q: K(T\mathbb{R}^n) \to \underset{\cong \mathbb{Z}}{K(pt)}$ by the Clutting map [or Thom iso (image $T\mathbb{R}^n = \mathbb{C}^n \to *$)]

define $\quad \text{top ind}(P) := q_! f_! \sigma(P)$

Thm: Atiyah Singer index thm.

$$\text{ind} P = \text{top ind} P.$$

One can prove this using axiomatics: the topological index clearly holds

P1: When $X = T^*X = pt$ $\text{ind}: K(pt) \to \mathbb{Z}$ is the identity

P2: If X, Y are cpt mfds and $f: X \hookrightarrow Y$ an embedding

$$\text{ind}(u) = \text{ind}(f_! u)$$

for all $u \in K_{cpt}(T^*X)$

these properties suffice as follows. Choose $f: X \hookrightarrow S^n$ an embedding, $j: pt \hookrightarrow S^n$ the inclusion of a pt.

$\cancel{\text{ind}(f_! \circ j_!}$ $\text{ind}\, j_! = \text{ind} \Rightarrow \text{ind} \circ j_!^{-1} = \text{ind}$

we have $\quad \text{ind}(u) = \text{ind}(f_! u) = \text{ind}(j_!^{-1} f_! u)$

and since $q \circ j_!^{-1} \cong j_!^{-1} = q_! \Rightarrow \text{ind}\, u = q_! f_!(u) = \text{top ind}$. $\square$

it is clear that the analytical index satisfies 1 since an elt of $K(pt)$ is represented by $[\mathbb{C}^r] - [\mathbb{C}^s]$ and ~~the operator~~ any operator

$$0 \to \mathbb{C}^r \to \mathbb{C}^s \to 0 \text{ has index } r - s \text{ by rk nullity.}$$

the ~~excision~~ second property is much more difficult to establish.

An excision theorem tells us that it suffices to consider $X \hookrightarrow V$ as the zero section of a VB over X.

~~$0 \to V \to \pi^* TX \to$~~

$0 \to V \to TP \to \pi^* TX \to 0$

One can define $R(G)$ the space of virtual reps of G (the space of f.d. reps beyond cr. domain under $\oplus$ & $\otimes$ so also a space ring)

note $K(*) = R(1)$. $\rightsquigarrow$ equivariant K theory

for now we need only that there is an ind map for

$\mathrm{ind}_{O_n} : K_{O(n)}(X) \to R(O(n))$ "the G index"

Generalizes ind. $R(O(n)) \to K(X)$ by

$P \mapsto P \times_{\rho} \mathbb{R}^k$. also there is a mult map

$K(T^*X) \otimes K_G(T^*S^n) \to K_G(T^*Z)$ where Z is the frame bundle

($O(n)$ principal bundle) of a vector bundle $V \to X$.

these satisfy $\mathrm{ind}(u \cdot v) = \mathrm{ind}(u \cdot \mathrm{ind}_{O(n)}(v))$. (generalized product formula)

then $\mathrm{ind}(u \cdot i_! 1) = \mathrm{ind}(u \cdot \mathrm{ind}_{O_n}(i_! 1))$

but $\mathrm{ind}_{O(n)} i_! 1 = 1$ so $\mathrm{ind}(u \cdot i_! 1) = \mathrm{ind}(u)$

and $f_! u = \mathrm{ext}\, u \cdot i_! 1$ $\square$

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thm: $\mathrm{ind}\, P = (-1)^n \{ ch(\sigma(P)) \cdot \hat{A}(X)^2 \} [TX]$.

Consider $q : T\mathbb{R}^n \cong \mathbb{C}^n \to *$ and $i : * \to \mathbb{C}^n$ the inclusion.

for fixed $u \in K(X) K(\mathbb{C}^n)$ we apply defect for $u = i_! \xi$.

$\pi_! ch((i_! \xi)) = \pi_! ch(u) = ch\, u\, [T\mathbb{R}^n]$

so $q_! u = ch\, u\, [T\mathbb{R}^n]$.

now consider $P : V \to X$ a Real v.b. $i : X \to V$ the zero. $p : TV \to TX$ the derivative.

$i : TX \to TV$ 0. we have $TV \cong \pi^* V \oplus \pi^* V$ with the complex structure for each $\ldots$

$\sigma \in K(TX)$ $\quad \cong V \otimes \mathbb{C}$

we have $p_! ch\, i_! \sigma = \mathcal{J}(V \otimes \mathbb{C})\, ch\, \sigma$

$\Rightarrow ch\, i_! \sigma\, [TV] = (\mathcal{J}(V \otimes \mathbb{C}))\, ch\, \sigma [TX]$

$(-1)^n(Td_{\mathbb{C}}(TX\otimes\mathbb{C}$

now ~~take~~ $f: X\to\mathbb{R}^n$ with normal bundle ν. Identifying ν with a nbd of X. and hence an embedding $T\nu\hookrightarrow T\mathbb{R}^n$. Given $\sigma\in K(TX)$, $i_!\sigma$ has cpt support and extends to $\mathbb{R}^n$ giving $f_!\sigma\in K_{cpt}(T\mathbb{R}^n)$.

$$Ch(i_!\sigma)[T\nu]=ch(f_!\sigma)(T\mathbb{R}^n]$$

by integration.

$$\Rightarrow \operatorname{ind}P=a_!f_!\sigma(P)$$
$$=Ch f_!\sigma(P)[T\mathbb{R}^n]$$
$$=Ch(i_!\sigma)[T\nu]$$
$$=\{\mathcal{J}(\nu\otimes\mathbb{C})\,Ch\,\sigma\}[TX]$$

Since $\mathcal{J}$ is multiplicative $\mathcal{J}((TX\oplus\nu)\otimes\mathbb{C})=\mathcal{J}((TX\otimes\mathbb{C})\oplus(\nu\otimes\mathbb{C}))$
$$=\mathcal{J}(TX\otimes\mathbb{C})\mathcal{J}(\nu\otimes\mathbb{C})$$
$$=1.$$

$$\mathcal{J}(\nu\otimes\mathbb{C})=\mathcal{J}(TX\otimes\mathbb{C})^{-1}$$

and $\operatorname{ind}P=$ ~~$(\mathcal{J}(\nu\otimes$~~
$$=\left((-1)^n Td_{\mathbb{C}}(TX\otimes\mathbb{C})\,Ch\,\sigma\right)[TX]$$
$$=(-1)^{\frac{n(n+1)}{2}}\left(\pi_! Ch\,\sigma\cdot Td_{\mathbb{C}}(TX\otimes\mathbb{C})\right)[X]$$

thm: (Chern Gauss Bonnet) if (X,g) is an oriented Riemannian $2n$ mfld

$$\int_X Ric\, dvol=\chi(X)$$

Pf: Consider the operator $d+d^*:\bigoplus\Omega^{2k}T^*X^{\otimes\mathbb{C}}\longrightarrow\bigoplus\Omega^{2k+1}T^*X^{\otimes\mathbb{C}}$

this is elliptic w/ symbol $e\wedge+\iota_{e^*}$ so $\sigma=\lambda_{-1}(T^*X)$

and
$$Ch\,\sigma=e(\underset{\otimes\mathbb{C}}{T^*X})\cdot Td_{\mathbb{C}}(T^*X\otimes\mathbb{C})$$

and $\operatorname{ind}P=(-1)^n e(T^*X\otimes\mathbb{C})[TX]=e(T^*X)[X]$

$\operatorname{ind}P=\dim\ker(d+d^*)-\dim\operatorname{Cok}(d+d^*)=\sum\dim H^{2k+1}-\sum\dim H^{2k}=\chi(X)$

by Chern Weil, $e(T^*X)[X]=\int_X Pf(R)$ $\square$

__Hirzebruch signature__ Consider now $d+d^*: \Omega^+\otimes\mathbb{C} \to \Omega^-\otimes\mathbb{C}$

where the $\Omega^\pm$ is the $\pm$ eval of $*$.

$$\operatorname{ind} P = \dim H^+ - \dim H^-$$

where the are the $\pm$ eigenspaces of $*$ acting on H^*.

The pairing $\alpha, \beta \mapsto \int \alpha\wedge\beta$ is a perfect pairing on H^*

and the $\pm$ eigenspaces correspond to the maximal positive definite/negative definite subspaces of

H^{2k}. $\Rightarrow$ $\operatorname{ind} P = \operatorname{Sig} X = \cancel{(-1)^{\frac{n(n+1)}{2}}(\pi_! \operatorname{ch}\sigma \cdot Td_{\mathbb{C}}(TX\otimes\mathbb{C}))[X]}$

$$(-1)^n(\operatorname{ch}\sigma \cdot Td_{\mathbb{C}}(TX\otimes\mathbb{C}))[TX]$$

$$= (-1)^n(-2)^n \hat{L}(X)\hat{A}(X)^2 \cdot Td_{\mathbb{C}}(TX\otimes\mathbb{C})[TX]$$

$$= (2)^n \hat{L}(X)[TX] = L(X) \quad \square$$