K-Theory Learning Seminar **Fall 2023**

Week 2: More Background and K^0

Abhishek Shivkumar

Operations and Examples

To recap, last time we defined the notion of vector bundles and some operations on them. The three operations to keep in mind are:

Of course, one has to check that all of these satisfy the local triviality condition. As an aside, the direct sum can be realized as a special case of the pullback operation via the following construction: let E_1, E_2 be vector bundles over X, $E_1 \oplus E_2$ is the pullback of the vector bundle $E_1 \times E_2 \to X \times X$ by the diagonal map $\Delta : X \to X \times X$.

- Direct sum of vector bundles over the same base, where the fiber of the direct sum is the direct sum of the fibers (this is sometimes called the Whitney sum)
- Tensor product of vector bundles over the same base, where the fiber is the tensor product of the fibers
- Pullbacks: given a vector bundle $E \xrightarrow{p} Y$ and a smooth map $f : X \to Y$ we have the pullback bundle f^*E which sits in the following diagram:

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

  As a set, $f^*E = \{(x, e) \in X \times E : f(x) = p(e)\} \subseteq X \times E$, from which there are obvious projection maps making the above square commute. The fibers of f^*E are given by $(f^*E)_x = E_{f(y)}$

Of course, there are other operations we care about on vector spaces that will end up porting over to vector bundles (dualizing, complexifying, etc.). In Milnor and Stasheff[1], there is a proof sketched that any functor $T : C^n \to C$ where C is the category (in fact, a groupoid) of finite dimensional vector spaces and isomorphisms among them that is "continuous" in the sense that its action on morphisms depends continuously on the morphisms, gives rise to an operation on vector bundles. This gives us (in addition to tensor products and direct sums) Homs, exterior powers, duals, etc. Basically, the philosophy is that any reasonable operation on vector spaces should port to vector bundles.

[1] J. Milnor and J. Stasheff. *Characteristic Classes*. Annals of Mathematics Studies. Princeton University Press, 1974. ISBN 9780691081229

For completeness, we will list some basic examples of vector bundles:

Example 1.1: Tangent Bundles

Given a smooth manifold M^n, the *tangent bundle* TM is the union of all the tangent spaces of M, which has a natural topology and is also a smooth manifold. If TM is trivial, i.e, splits as $TM \cong M \times \mathbb{R}^n$, then M is called *parallelizable*. For example, S^1 is parallelizable, but S^2 is not.

In fact, TS^2 does not admit even one non-vanishing section (i.e, a non-vanishing vector field); this is the hairy ball theorem which one can see e.g via Poincaré-Hopf (the sum of the degrees of the zeroes of a vector field on an orientable smooth manifold equal its Euler characteristic, and $\chi(S^2) = 2$).

In fact, all Lie groups are parallelizable; the idea is that one can pick a basis (or frame) on the tangent space at the identity, and then slide it around by the (derivative of the) left action of the group on itself.

Example 1.2: Tautological Bundles

Consider $\mathbb{RP}^n$ or $\mathbb{CP}^n$ which parameterize lines in $\mathbb{R}^{n+1}$ and $\mathbb{C}^{n+1}$ respectively; the idea of the *tautological bundle* on a projective space is that the line (real or complex) above a point is the line corresponding to that point. The total space is $\{(l, x) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : x \in l\}$ equipped with the obvious projection maps (and similarly for $\mathbb{C}$).

Similarly, one can define tautological bundles over Grassmanians $\mathrm{Gr}_k(\mathbb{R}^n)$ and $\mathrm{Gr}_k(\mathbb{C}^n)$, with total space

$$\{(H, x) \in \mathrm{Gr}_k(F^n) \times F^n : x \in H\}$$

for $F = \mathbb{R}$ or $\mathbb{C}$ and again equipped with the obvious projection maps giving the vector bundle structure.

Some sources call the tautological bundle the canonical bundle, which is an unfortunate clash of notation that should be avoided if possible, as the canonical bundle is more commonly given to mean the determinant (top exterior power) of the cotangent bundle, particularly in algebraic geometry; canonical is maybe justified in this setting due to the role the canonical bundle plays in the adjunction formula.

Example 1.3: Subbundles

There is a natural notion of a subbundle of a vector bundle, whose definition we will not belabor here. As an example, take the trivial bundle $S^1 \times \mathbb{R}^2$, which has as a subbundle the Möbius strip (bundle), which one can see by picking a line through the origin, and then allowing it to rotate through a full circle as the base point varies over S^1 (this is exactly the construction of the tautological line bundle on $\mathbb{RP}^1 = S^1$).

Example 1.4: Normal Bundles

Given an embedding $M \hookrightarrow N$, we can consider its *normal bundle* inside N, which one should think of interchangeably with a tubular neighborhood of the embedded copy of M (technically we need a Riemannian structure on N to take orthogonal complements). So, for example, if you know all of the possible vector bundles over M, then you know all of the possible tubular neighborhoods it can have in another manifold; I will give an example of the utility of this below.

Sections

Another basic definition in the world of vector bundles is that of a section. In particular, given a vector bundle $E \xrightarrow{p} B$, a section is simply a map $B \xrightarrow{s} E$ such that $p \circ s = \mathrm{id}_B$, i.e, $s(b)$ is a vector lying in the fiber above b for all $b \in B$.

Section is kind of an opaque word choice for this concept; it comes from shortening the more old-fashioned cross-section of a vector bundle.

Example 2.1: Zero Section

Any vector bundle $E \to B$ has a canonical section, the *zero section*, which picks out the origin in each fiber.

Example 2.2: Vector Fields

Any vector field on a manifold is the same as a section of the tangent bundle.

An easy result that one can show is that a vector bundle $E \xrightarrow{p} B$ is trivial iff it has n linearly independent sections (i.e sections whose values are linearly independent vectors above each point $b \in B$).

This is straightforward if one knows the following fact: a continuous map $h : E_1 \to E_2$ between vector bundles over the same base space B is an isomorphism if it takes each fiber $p_1^{-1}(b)$ to $p_2^{-1}(b)$ by a linear isomorphism.

As an aside, much of what we studied last year in the characteristic classes learning seminar can be phrased as finding obstructions to finding some number of linearly independent sections on a vector bundle (along with variations of this type of problem).

Clutching

We also touched on clutching last time, which is the technique of building vector bundles by looking at explicit open covers of a given manifold and considering the restrictions on the transition maps. Before we can discuss the utility of clutching, we need some basic results:

Lemma 3.1

The restrictions of a vector bundle $E \to X \times I$ over $X \times \{0\}$ and $X \times \{1\}$ are isomorphic if X is paracompact.

Corollary 3.2

Let $\mathrm{Vect}^n(X)$ denote the set of isomorphism classes of n-dimensional real vector bundles over a space X. A homotopy equivalence $f : A \to B$ induces a bijection $f^* : \mathrm{Vect}^n(B) \to \mathrm{Vect}^n(A)$ for all n (the same result holds for complex vector bundles, and, in fact, for general fiber bundles). In particular, every vector bundle over a contractible (paracompact) base is trivial.

Proofs of the above results can be found in Hatcher[2]. The point is now that in studying vector bundles over a topological space, we only care about the homotopy type of the base space.

[2] A. Hatcher. Vector bundles and k-theory, 2017. URL https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf

Returning to clutching, the only case I've ever personally used in practice is to study vector bundles (or, more generally, fiber bundles) on S^n, with its standard covering by stereographic projections. The data of a k-vector bundle over S^n, then is a map from the intersection of the two charts (which is homotopy equivalent to the equatorial S^{n-1}) to $\mathrm{GL}_k(\mathbb{R})$ (or $\mathbb{C}$ for complex vector bundles), and, by the results above, we only care about the homotopy type of this map. Thus, an element of $\pi_{n-1}(\mathrm{GL}_k(\mathbb{R}))$ gives a real k-vector bundle over S^n. Conversely, given an arbitrary real k-vector bundle over S^n, its transition map in the standard open cover must be an invertible $k \times k$ matrix, so we have the following:

Theorem 3.3

The map
$$\Phi : \pi_{n-1}(\mathrm{GL}_k(\mathbb{C})) \to \mathrm{Vect}^k_{\mathbb{C}}(S^n)$$
(which sends a clutching map to its corresponding vector bundle) is a bijection (of *sets*; we have not defined a group structure on the target to make this a homomorphism). Similarly, in the real case, the map
$$\Phi : \pi_{n-1}(\mathrm{GL}^+_k(\mathbb{R})) \to \mathrm{Vect}^k_{+}(S^n)$$
where $\mathrm{GL}^+_n(\mathbb{R})$ denotes the positive determinant component, and $\mathrm{Vect}^k_+(S^n)$ denotes the groupoid of oriented vector bundles and orientation preserving isomorphisms among them. Moreover, by Gram-Schmidt, $\mathrm{GL}^+_k(\mathbb{R})$ deformation retracts onto $\mathrm{SO}(k)$ and $\mathrm{GL}_k(\mathbb{C})$ onto $\mathrm{U}(k)$, so we may replace $\mathrm{GL}_k(\mathbb{C})$ above with $\mathrm{U}(k)$ and similarly in the real case (this will often simplify calculations conceptually).

Similar results hold for fiber bundles where the fibers are not necessarily vector spaces (for example, when the fibers are spheres).

Example 3.4

Consider a smooth embedding $S^{n-2} \hookrightarrow S^n$, a knotted sphere (codimension two is not actually necessary for knottedness in the smooth category, though it is in the PL category; there exists a knotted S^3 inside S^6 and in general S^{4k-1}s inside S^{6k}). An open tubular neighborhood of our embedded S^{n-2} is diffeomorphic to the total space of a rank two vector bundle over S^{n-2}, which are classified by $\pi_{n-3}(\mathrm{SO}(2)) = \pi_{n-3}(S^1)$. Thus, we know that for $n \geq 3$, $n \neq 4$, the normal bundle is trivial and therefore the tubular neighborhood is diffeomorphic to $S^{n-2} \times \mathring{D}^2$. In fact, this is also true for $n = 4$, but requires some further arguments.

Details of the $n = 4$ case can be found in Zeeman[3], section 5.

[3] E. C. Zeeman. Twisting spun knots. *Trans. Am. Math. Soc.*, 115(0):471–495, 1965. URL `https://api.semanticscholar.org/CorpusID:16020709`

Classifying Vector Bundles

Now that we have defined something, we must be good mathematicians and try to classify all possible instances of that thing. For the purposes of our discussion, we will assume that our base spaces are all compact Hausdorff, and all vector bundles are complex unless stated otherwise, as the theory is a little cleaner in the complex case (though we won't see this until next week at least).

Note that by a complex vector bundle, we only mean that the fibers have a complex structure and that any maps are complex linear. This differs from a *holomorphic* vector bundle, which is a complex vector bundle over a complex manifold, such that the total space is also a complex manifold and the projection map is holomorphic. Alternatively, one can require that the transition maps are all biholomorphisms.

Note that $\mathrm{Vect}^n_{\mathbb{C}}(X)$ has a natural structure as an abelian monoid under direct sum; we want a group instead so we will make one:

Definition 4.1: Grothendieck Completion

Let M be an abelian monoid, then let $\mathrm{Gr}(M)$ be the *Grothendieck completion* of M, defined as

$$\mathrm{Gr}(M) = \mathbb{Z}\langle M\rangle/([m] + [n] - [m + n] : m, n \in M)$$

Grothendieck completion amounts to adjoining formal inverses to a monoid, and as a functor is the left adjoint to the forgetful functor.

We take $\mathbb{Z}$-linear combinations of elements of M (to get formal negatives of elements in M), and the relations we have adjoined simply ensure that the addition in M is reflected in $\mathrm{Gr}(M)$. Note that any element of $\mathrm{Gr}(M)$ can be written as $[m]-[n]$ for $m, n \in M$, with $[m]-[n]\oplus[a]-[b] = [m\oplus a]-[n\oplus b]$ where $\oplus$ is the monoid addition.

Example 4.2

$\mathrm{Gr}(\mathbb{N}) = \mathbb{Z}$.

Then, we can set $K^0(X) := \mathrm{Gr}(\mathrm{Vect}_{\mathbb{C}}(X))$. Note that we can write every element of $K^0(X)$ as $[E] - \varepsilon^n$ since (as we will show below), for any vector bundle E, there exists E' s.t $E \oplus E' \cong \varepsilon^n$ for some n, so, starting with a formal difference $[E] - [E']$, we can add a bundle E'' s.t $E' \oplus E'' \cong \varepsilon^n$, and therefore $[E] - [E'] \oplus [E''] - [E''] = [E \oplus E''] - [\varepsilon^n]$.

This leads us to consider the notion of *stable equivalence* of vector bundles, where two vector bundles E_1, E_2 both over X are *stably equivalent* if $E_1 \oplus \varepsilon^n = E_2 \oplus \varepsilon^m$, denoted $E_1 \cong_s E_2$, (where ε denotes the trivial bundle) for some m, n (and *stably isomorphic* if we may take $m = n$).

Example 4.3

TS^n is stably trivial (as a *real* vector bundle); consider S^n sitting inside $\mathbb{R}^{n+1}$, then the sum of the tangent bundles and the (trivial) normal bundle is a trivial bundle since the elements of this bundle are of the form $(x, v, tx) \in S^n \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ such that x is perpendicular to v, and there is a map $(x, v, tx) \mapsto (x, v + tx)$ which gives an isomorphism to the trivial bundle $S^n \times \mathbb{R}^{n+1}$.

Note that $TS^2 = T\mathbb{CP}^1$ is not stably trivial as a *complex* vector bundle, since its Chern class (equivalently, here, its Euler class which should equal $\chi(S^2)$) is 2 and the Chern class is stable; thanks to Jacob for this example.

One reason for considering this equivalence relation is that, as we saw last year in the characteristic classes learning seminar, it can be extraordinarily difficult to classify vector bundles up to actual equivalence, and, moreover, the characteristic classes we came up with to differentiate vector bundles were generally insensitive to direct summing with a trivial bundle (the Stiefel-Whitney, Chern, and Pontryagin classes but *not* the Euler class).

Proposition 4.4

For X compact Hausdorff, the set of equivalence classes of vector bundles over X forms an abelian group.

This is true over $\mathbb{R}$ or $\mathbb{C}$.

PROOF : That direct summing is associative and commutative is straightforward. The identity element is (the class of) ε^0. For inverses to exist, we need that for all bundles E there exists a bundle E' such that $E \oplus E' \cong \varepsilon^m$ for some m, that is, every vector bundle is a subbundle of some trivial bundle. We assume X is connected for simplicity.

The idea is as follows: if we had an embedding $E \hookrightarrow X \times \mathbb{R}^m$ for some m, then projecting to $\mathbb{R}^m$ gives a map $E \to \mathbb{R}^m$ that is a linear injection on each fiber. We will reverse this logic, first building an injection that is linear on each fiber of E into $\mathbb{R}^m$, then showing that this gives an embedding of E into $X \times \mathbb{R}^m$.

Let $E \xrightarrow{p} X$; for each $x \in X$, there is an open neighborhood $U_x \ni x$ s.t E trivializes over U_x, and by Urysohn's lemma, there is a function $\varphi_x : B \to [0, 1]$ that is nonzero at x and has support contained in U_x. Since X is compact, we may pass to a finite subcover and relabel $\varphi_{x_i} = \varphi_i$, $U_{x_i} = U_i$. Define $g_i : E \to \mathbb{R}^n$ (where n is the dimension of the fibers of E; here is where we use that X is connected) by

$$g_i(v) = \varphi_i(p(v)) \cdot \pi_i(h_i(v))$$

where we have the following diagram:

$$\begin{tikzcd} p^{-1}(U_i) \arrow[r, "h_i"] & U_i \times \mathbb{R}^n \arrow[d, "\pi_i"] \\ & \mathbb{R}^n \end{tikzcd}$$

and h_i is the local trivialization, π_i the projection map. Then g_i is a linear injection on each fiber in the support of φ_i, and so we can tuple the g_i together to form $g : E \to \mathbb{R}^N$ which is a linear injection on each fiber in E. We then have the map $f := (p, g) : E \to X \times \mathbb{R}^N$, and it remains only to show that the image of f is a subbundle, and one can show that the image trivializes over the supports of the φ_i; for the details see Hatcher[4] Proposition 1.4. In fact, more is true: one can show that every short exact sequence

$$0 \to A \to B \to C \to 0$$

of vector bundles is split by picking a metric and taking orthogonal complements. ■

[4] A. Hatcher. Vector bundles and k-theory, 2017. URL https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf

Even though inverses don't really exist, we do have a cancellation property, i.e, if $E_1 \oplus E_2 \cong_s E_1 \oplus E_3$ then $E_2 \cong_s E_3$ since we can direct sum both sides with a vector bundle E_1' s.t $E_1 \oplus E_1' \cong \varepsilon^n$.

We denote the group of such equivalence classes $\tilde{K}^0(X)$. Evidently, there is a surjective homomorphism $K^0(X) \to \tilde{K}^0(X)$ sending $[E] - [\varepsilon^n]$ to the class of E with kernel generated by elements of the form $[\varepsilon^m] - [\varepsilon^n]$ hence $K^0(X) \cong \tilde{K}^0(X) \oplus \mathbb{Z}$. We call $K^0(X)$ and $\tilde{K}^0(X)$ the *unreduced* and *reduced complex K-group of X* respectively. Another way to arrive at the reduced K-group is

$$\tilde{K}^0(X) = \ker(l^* : K^0(X) \to K^0(\bullet))$$

where l is the inclusion of some basepoint to X, and $K^0(\bullet) = \mathbb{Z}$ since finite dimensional vector spaces are classified by their dimension.

References

A. Hatcher. Vector bundles and k-theory, 2017. URL https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf.

J. Milnor and J. Stasheff. *Characteristic Classes*. Annals of Mathematics Studies. Princeton University Press, 1974. ISBN 9780691081229.

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