K-Theory Talk

At this point we have seen examples of vector bundles and introduced $K(X)$ and $\tilde{K}(X)$. I'd like to briefly review the definition of both of these groups and discuss functoriality.

For reference, I'm mostly sticking to Hatcher 2.1.

Setup: X our base space, usually compact, and I have to remind myself constantly that $p: E \to X$ need not have fibers of fixed dimension (only on connected components). • All vector bundles complex

Remember • $\mathcal{E}^n$ is dim n trivial bundle

- E_1, E_2 are <u>stably isomorphic</u> if $\exists n$ such that $E_1 \oplus \mathcal{E}^n \cong E_2 \oplus \mathcal{E}^n$; $E_1 \approx_s E_2$
- E_1, E_2 are <u>stably equivalent</u> if $\exists n, m$ such that $E_1 \oplus \mathcal{E}^n \cong E_2 \oplus \mathcal{E}^m$; $E_1 \sim E_2$
- $M(X) \text{ "=" }$ equiv classes $\approx_s$, $\tilde{M}(X) =$ equiv classes $\sim$.

Abhishek ended w/ discussion of the following proposition: (Prop 2.1 Hatcher)

Prop: If X compact Hausdorff, then $\tilde{M}(X)$ is a group under $\oplus$. Write $\tilde{K}(X) = \tilde{M}(X)$

Proof Comments: As we saw, only need to show existence of inverses.

- When E has constant dimension fibers, Prop 1.4 of Hatcher says exactly that $\exists E'$ such that $E \oplus E' \cong \mathcal{E}^n$. Construct E' locally through clever use of Urysohn's lemma.
- When E not constant dim on fibers, then make E' a v.b. trivial of appropriate rank on each
$$X_i = \{x \in X \mid \dim E_x = i\}.$$
then $E \oplus E'$ is constant dimensional.

Note: #1 Even after delving into the proof of Proposition 1.4, the construction of inverses remains opaque, and hence $\tilde{K}(X)$ remains opaque.

#2: The compactness assumption is necessary; for instance, the canonical line bundle over $\mathbb{RP}^\infty$ has no inverse in $\tilde{K}(X)$.

What about $K(X)$?

First, note that $E \approx_s E' \not\Rightarrow E \cong E'$.

Example: Take the tangent bundle τ over S^2. Hairy ball theorem $\Rightarrow$ τ non-trivial. However, if we consider the normal bundle 1 of S^2 embedded in $\mathbb{R}^3$, then

$$\tau \oplus 1 \cong \varepsilon^2 \oplus \varepsilon^1,$$

and 1 is trivial $\cong \varepsilon^1$. Hence $\tau \approx_s \varepsilon^2$ but $\tau \not\cong \varepsilon^2$.

However, I don't know if $E_1 \approx_s E_2 \Rightarrow E_1 \cong E_2$ in the complex case.

$M(X)$ is not a group under $\oplus$; only invertible element is ε^0: $E \oplus E' \approx_s \varepsilon^0 \Rightarrow E \oplus E' \oplus \varepsilon^n \cong \varepsilon^n \Rightarrow$ E and E' are zero dimensional.

Redefine $K(X)$ as the groupification of $\approx_s$ equiv classes by adding in formal inverses with

$$E_1 - E_1' = E_2 - E_2' \iff E_1 \oplus E_2' \approx_s E_2 \oplus E_1'.$$

every element in $K(X)$ is represented by a difference $E - \varepsilon^n$; given $E - E'$, can find E'' such that $E' \oplus E'' \approx \varepsilon^n \Rightarrow E - E' = E \oplus E'' - (E' \oplus E'') = (E \oplus E'') - \varepsilon^n$.

$K(X) \to \tilde{K}(X)$

Have a map $K(X) \to \tilde{K}(X)$ given by $E - \varepsilon^n$ to the $\sim$-class of E.

- Well defined: $E - \varepsilon^n = E' - \varepsilon^m$ in $K(X)$ $\Rightarrow E \sim E'$.
- Surjective w/ kernel all elements of form $\varepsilon^m - \varepsilon^n$ $\left(E - \varepsilon^n \mapsto [\varepsilon^0] \Rightarrow E \approx_s \varepsilon^m\right)$
- $\{\varepsilon^m - \varepsilon^n\} \cong \mathbb{Z}$ as a group. (Justifies the notation $1 \cong \varepsilon^1$)
- For any x_0, $K(x_0) \cong \mathbb{Z}$. Restriction to x_0 gives map $K(X) \to K(x_0)$ and is an isomorphism on $\{\varepsilon^m - \varepsilon^n\}$.
- Hence $0 \to \{\varepsilon^m - \varepsilon^n\} \to K(X) \to \tilde{K}(X) \to 0$ splits, $K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}$ w/ isomorphism $\mathbb{Z} \cong \{\varepsilon^m - \varepsilon^n\}$ depending on x_0.

Call $\tilde{K}(X)$ reduced.

Ring Structure

Elements of $K(X)$ can be multiplied with $\otimes$:

$$(E_1 - E_1')(E_2 - E') = E_1 \otimes E_2 - E_1' \otimes E_2 + E_1 \otimes E_2' + E_1' \otimes E_2'.$$

This makes $K(X)$ a commutative ring w/ identity ε^1, the trivial line bundle.

The ring structure extends to $\tilde{K}(X)$ only if we fix a basepoint x_0:

$$K(X) \to K(x_0) \text{ is a ring hom w/ kernel } \cong \tilde{K}(X).$$

Thus $\tilde{K}(X)$ is a proper ideal of $K(X)$, can be regarded as a ring.

Functoriality

Not too much to say here. Given $f: X \to Y$ we have a map $f^*: K(Y) \to K(X)$ given by $E_1 - E' \mapsto f^*(E) - f^*(E')$. This is actually a ring homomorphism since pullback commutes w/ $\oplus$ and $\otimes$.

For $f^*: \tilde{K}(Y) \to \tilde{K}(X)$ to be a ring homomorphism, we must work in the category of pointed topological spaces.

External Product and Fundamental Product Theorem

Central to the discussion in the next few weeks is the notion of the external product;

$$\mu: K(X) \otimes_{\mathbb{Z}} K(Y) \longrightarrow K(X \times Y); \quad \mu(a \otimes b) = p_1^*(a) p_2^*(b)$$

where p_1, p_2 are the projections of $X \times Y$ onto X and Y. This is a ring homomorphism.

Taking Y to be S^2 gives a map

$$\mu: K(X) \otimes K(S^2) \to K(X \times S^2),$$

which is in fact an isomorphism. This is known as the Fundamental Product theorem.

Fundamental Product Theorem

It is better to write this in terms of the canonical line bundle H for $S^2 \cong \mathbb{CP}^1$. First let's do an example:

Example: $(H \otimes H) \oplus 1 \cong H \oplus H$, where 1 = 1-dim trivial line bundle.

Review of clutching functions for specifically S^2:

- write $S^2 = D_+ \cup D_-$ as union of upper and lower hemispheres.
- given $f: D_+ \cap D_- = S^1 \to GL_2(\mathbb{C})$, can construct bundle E_f by

$$E_f = \frac{(D_+ \times \mathbb{C}^2) \sqcup (D_- \times \mathbb{C}^2)}{(x,v) \sim (x, f(x)v)} \quad \text{for } (x,v) \in \partial D_- \times \mathbb{C}^2,\ (x, f(x)v) \in \partial D_+ \times \mathbb{C}^2.$$

- Then have projection $p: E_f \to S^2$
- It is theorem that $[S^{k-1}, GL_n(\mathbb{C})] \longrightarrow \mathrm{Vect}^n_{\mathbb{C}}(S^k)$ defined $f \mapsto E_f$ is an isomorphism.

Returning to our example:

$\iota: S^1 \hookrightarrow GL_1(\mathbb{C}) = \mathbb{C}$

$$(H \otimes H) \oplus 1 \quad \longleftrightarrow \quad \underset{z}{\overset{S^1}{\rotatebox{90}{$\in$}}} \mapsto \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{C}) \qquad \left(= (\iota \cdot \iota) \oplus 1\right)$$

$$(H \oplus H) \quad \longleftrightarrow \quad z \mapsto \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \qquad \left(= \iota \oplus \iota\right)$$

Since $GL_2(\mathbb{C})$ path connected, there is a path α_t from $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$ to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the matrix interchanging the factors of $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$.

Then $\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \alpha_t \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \alpha_t$ is a homotopy from $\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$ to $\begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}$,

hence the clutching functions of $(H \otimes H) \oplus 1$ and $(H \oplus H)$ are homotopic $\Rightarrow$ these bundles are isomorphic.

In $K(S^2)$, this means $H^2 \oplus 1 = H \oplus H = 2H$, so $(H-1)^2 = 0$ in $K(S^2)$.

We therefore have a ring homomorphism

$$\mathbb{Z}[H]/(H-1)^2 \longrightarrow K(S^2),$$

and define μ as the composition

$$K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X) \otimes K(S^2) \to K(X \times S^2).$$

Now we can state the fundamental product.

Theorem (2.2 in Hatcher): The homomorphism $\mu : K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X \times S^2)$ is an isomorphism of rings whenever X is compact Hausdorff.

Cor: $K(S^2) \cong \mathbb{Z}[H]/(H-1)^2$ and $\tilde{K}(S^2) = (H-1)$.

Pf of Cor: Obtain first isomorphism by taking $X = pt$. We then have s.e.s. of $\mathbb{Z}$-modules

$$0 \to \tilde{K}(S^2) \to \mathbb{Z}[H]/(H-1)^2 \xrightarrow{\alpha} \mathbb{Z} \to 0$$

and hence $\tilde{K}(S^2) = \ker \alpha = (H-1) \cong \mathbb{Z}$ (as a group). $\square$.

This means multiplication on $\tilde{K}(S^2)$ is entirely trivial. Compare this to the cohomology of S^2, $\tilde{H}^*(S^2;\mathbb{Z})$ w/ cup product, with $H-1$ behaving as the generator of $H^2(S^2;\mathbb{Z})$.

Fundamental product theorem is core of Bott Periodicity. Once we have that, we'll get that $\tilde{K}(S^{2n+1}) = 0$ and $\tilde{K}(S^{2n}) = \mathbb{Z}$.

The setup in algebraic geometry

X a Noetherian scheme. Now vector bundles are locally free $\mathcal{O}_X$-modules: $\mathcal{F}$ every $x \in X$ has nbhd $U \ni x$ so $\mathcal{F}$ ∃ so that $\mathcal{F}|_U$ is isomorphic to a finite direct sum of copies of $\mathcal{O}_X|_U$.

- $\mathrm{Vect}(X)$ = isomorphism classes of locally free $\mathcal{O}_X$-modules.
- $\oplus$ still works, and then obtain $K(X)$ as before, we call

$$\left(K^{\circ}, \oplus\right) = \mathrm{Gr}\left(\mathrm{Vect}(X), \oplus\right).$$

There is also another construction, however. If we instead look at isomorphism classes of coherent sheaves $\mathrm{Coh}(X)$ ($\mathcal{O}_X$-modules which locally look like sheaves associated to finitely generated modules), we can get a similar construction by moding out by $[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}'']$ whenever these fit into a s.e.s.

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0.$$

This gives us a group $K_0(X)$, called the Grothendieck group. When X is smooth and Noetherian,

$$K_0(X) \cong K^0(X).$$

However $K_0(X)$ always has a ring structure, ~~which is~~ not particularly enlightening to me:

$$[\mathcal{E}]\cdot[\mathcal{E}'] = \sum (-1)^k \left[\mathrm{Tor}_k^{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}')\right]$$

Grothendieck Riemann Roch:

$$\mathrm{ch}: K_0(X) \otimes \mathbb{Q} \to A(X) \otimes \mathbb{Q}.$$