# Higher K Groups & Cohomology — Topological K-Theory Learning Seminar, 19 Sep 2023

**Goal:** make K-theory into a generalized cohomology theory.

We'll work in the reduced form but can also do it unreduced.

**Axioms:**
- contravariant functors $\tilde{K}^i : \left\{ \begin{array}{l} \text{htpy classes of} \\ \text{all pairs of top.} \\ \text{spaces} \end{array} \right\} \to Ab$
- boundary maps $\tilde{K}^i(A) \to \tilde{K}^{i+1}(X, A)$
- homotopy equivalent spaces induce isomorphic $\tilde{K}^i(X)$
- $\exists$ LES $\cdots \to \tilde{K}^i(X,A) \to \tilde{K}^i(A) \to \tilde{K}^i(X) \to \tilde{K}^{i+1}(X,A) \to \cdots$
- (suspension) $\tilde{K}^i(X) \cong \tilde{K}^{i+1}(\Sigma X)$
- (additivity) $X = \bigvee_i X_i$, then $\tilde{K}^n(X) \cong \prod_i \tilde{K}^n(X_i)$

Main goal is to construct $\tilde{K}^n(X)$ and the LES. The other properties will follow.

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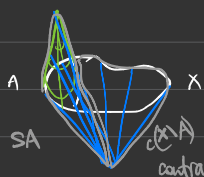
**Recall** $X$ compact Hausdorff. $A \subset X$ closed.

$\tilde{K}(X)$ reduced complex K-group

$f: X \to Y$ induces $f^*: \tilde{K}(Y) \to \tilde{K}(X)$ via pullback

Start with $A \hookrightarrow X \to X/A$. How to get a LES from this?

$$\begin{tikzcd}[column sep=small] A \arrow[r, hook] & X \arrow[r, hook] & X \cup CA \arrow[r, hook] \arrow[d] & (X \cup CA) \cup CX \arrow[r, hook] \arrow[d] & ((X \cup CA) \cup CX) \cup C(X \cup CA) \arrow[d] \\ & & X/A & SA & SX \end{tikzcd}$$




Each vertical map is a htpy equivalence arising from quotienting by a contractible space.

This induces isomorphisms on $\tilde{K}$.

Note the sequence of spaces itself is engineered to look exact: spaces included will be made homotopically trivial two terms later.

Get a sequence of $\tilde{K}$ groups:

$$\cdots \to \tilde{K}(SX) \to \tilde{K}(SA) \to \tilde{K}(X/A) \xrightarrow{q^*} \tilde{K}(X) \xrightarrow{i^*} \tilde{K}(A)$$

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**Claim.** The above sequence is exact.

**Pf idea:** ① $\mathrm{im}\, q^* = \ker i^*$ (Hatcher Prop 2.9)

$\mathrm{im}\, q^* \subset \ker i^*$ since $qi = \{pt\}$

$\ker i^* \subset \mathrm{im}\, q^*$: use trivialization of bundle over $A$ to construct bundle over a nbhd of $A/A \subset X/A$ & extend to all of $X/A$

② rest of sequence to the left
- composition of inclusions sends each space to a contractible space
- since all maps (on the sequence of spaces) are inclusions, only the above space is killed.

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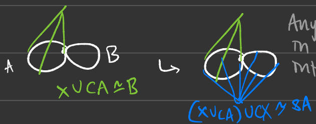
**e.g.** $X = A \vee B \Rightarrow X/A = B$

$$\cdots \to \tilde{K}(SX) \to \tilde{K}(SA) \to \tilde{K}(B) \to \tilde{K}(X) \to \tilde{K}(A)$$




Anything in im(B) is contractible in SA. So sequence breaks into split SES:

$\tilde{K}(X) \cong \tilde{K}(A) \oplus \tilde{K}(B)$

this was an axiom ✓

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**Interlude: Bott Periodicity** (recall from Isaac)

**Recall** External product $K(X) \otimes K(Y) \to K(X \times Y)$

$\exists$ reduced version: $\tilde{K}(X) \otimes \tilde{K}(Y) \to \tilde{K}(\underbrace{X \wedge Y}_{X \times Y / X \vee Y})$

define by LES

$$\begin{array}{ccccccccc} \tilde{K}(S(X \times Y)) & \to & \tilde{K}(S(X \vee Y)) & \to & \tilde{K}(X \wedge Y) & \to & \tilde{K}(X \times Y) & \to & \tilde{K}(X \vee Y) \\ & & \wr\wr & & & \uparrow & & & \wr\wr \\ & & \tilde{K}(SX) \oplus \tilde{K}(SY) & & & \text{splits} & & & \tilde{K}(X) \oplus \tilde{K}(Y) \end{array}$$

quotient $SX \to \Sigma X \Rightarrow \tilde{K}(SX) \approx \tilde{K}(\Sigma X)$ since $\{x_0\} \times I$ contractible

$a * b = p_1^*(a) p_2^*(b)$ is zero in $K(X \vee Y)$ so lies in $\tilde{K}(X \times Y)$ & pulls back to unique elt of $\tilde{K}(X \wedge Y)$

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**Note** $S^n \wedge X = \Sigma^n X$

$S^n X \to S^n \wedge X$ induces $\tilde{K}(S^n X) \cong \tilde{K}(S^n \wedge X)$

So we can use the Fundamental Product Thm to state one of many formulations of Bott Periodicity:

**Thm.** $\tilde{K}(X) \xrightarrow{\cong} \tilde{K}(S^2 X)$, $\alpha \mapsto (H-1) * \alpha$

$\uparrow$ tautological line bundle over $S^2 \cong \mathbb{C}P^1$

**Pf.**

$$\begin{tikzcd} \tilde{K}(X) \arrow[r, "\cong"] \arrow[d, "\cong"', "\alpha \mapsto (H-1) \otimes \alpha"] & \tilde{K}(S^2 X) \\ \tilde{K}(S^2) \otimes \tilde{K}(X) \arrow[r, "\cong"] & \tilde{K}(S^2 \wedge X) \arrow[u, "\cong \text{ by above}"'] \end{tikzcd}$$

$\uparrow$ Fundamental Product Thm

can compute K of spheres: for more time in next week

**Cor.** $\tilde{K}(S^{2n+1}) = 0$, $\tilde{K}(S^{2n}) = \mathbb{Z}$ generated by

$(H-1) * \overset{n \text{ times}}{\cdots} * (H-1)$

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## Cohomology groups & LES

$$\cdots \to \tilde{K}(SX) \to \tilde{K}(SA) \to \tilde{K}(X/A) \xrightarrow{q^*} \tilde{K}(X) \xrightarrow{i^*} \tilde{K}(A)$$

Define $\tilde{K}^{-n}(X) = \tilde{K}(S^n X)$

$\tilde{K}^{-n}(X,A) = \tilde{K}(S^n(X/A))$

positive n: $\tilde{K}^{2i}(X) = \tilde{K}(X)$, $\tilde{K}^{2i+1}(X) = \tilde{K}(SX)$

So LES becomes

$$\cdots \to \tilde{K}^{-1}(X,A) \to \tilde{K}^{-1}(X) \to \tilde{K}^{-1}(A) \to \tilde{K}^{0}(X,A) \to \tilde{K}^{0}(X) \to \tilde{K}^{0}(A) \to \cdots$$

By periodicity, can write this as a 6 term cycle

So there's really only two groups $K^n$

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Can define $K^n(X)$ as $\tilde{K}^n(X_+)$, $X_+ = X \sqcup$ disjoint basepoint

## Ring structure:

- product $\tilde{K}^i(X) \otimes \tilde{K}^j(Y) \to \tilde{K}^{i+j}(X \wedge Y)$ from reduced external product
- Define $\tilde{K}^*(X) = \tilde{K}^0(X) \oplus \tilde{K}^1(X)$
  compose with map $\tilde{K}(X \wedge X) \to \tilde{K}(X)$ induced by diagonal
  extends ring str on $\tilde{K}^0(X)$.
- commutes up to sign: $\alpha \in \tilde{K}^i(X)$, $\beta \in \tilde{K}^j(X)$, $\alpha\beta = (-1)^{ij}\beta\alpha$