K-Theory and Differential Operators

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The sources I cite below are:

- (Atiyah Global Theory): Atiyah, "Global theory of elliptic operators"
- (Gilkey): Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theorem*
- (Lang): Lang, *Real and Functional Analysis*
- (Mukherjee) Mukherjee, *Atiyah–Singer Index Theorem: An Introduction*
- (Yale): `https://gauss.math.yale.edu/~mr2245/func2018Data/fredholm.pdf`

1 Pseudo-differential Operators

The Fourier transform is a fundamental tool in the study of partial differential equations. There are a variety of reasons, including the classic orthogonality, but one of its fundamental properties is how it simplifies more complicated operations. Specifically, it turns derivatives into polynomials and convolutions into products:

$$\mathcal{F}(\partial_i f) = \xi_i \mathcal{F}(f) \qquad\qquad \mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g).$$

In general, the point of using Fourier transforms is to turn your problem into a simpler one in the "phase space", solve it there, and then pass back to the original context using the inverse Fourier transform.

One particular way using such manipulations to solve PDEs is the parametrix method. Suppose we have a PDE of the form $Pf = g$. Using the differentiation relation above, this turns into multiplication by a polynomial in the Fourier world: $p\hat{f} = \hat{g}$. Assuming p does not vanish, we then have $\hat{f} = \hat{g}/p$, and we then find f by taking the inverse Fourier transform. If we have an operator Q such that $\widehat{Qg} = \hat{g}/p$ (called a *multiplier operator*), it would provide the solution to our problem.

This train of thought leads us to the concept of pseudo-differential operators, which are particularly nice multiplier operators. To start, we define a (linear) **differential operator** of order d to be an operator of the form

$$P = p(x, D) = \sum_{|\alpha| \le d} a_\alpha(x) D_x^\alpha,$$

where the coefficients a_α are smooth functions, and

$$D_x^\alpha = (-i)^{|\alpha|} \partial_x^\alpha.$$

(The $-i$ factor is to account for the factors that come from the Fourier transform.) We define the symbol to be the corresponding polynomial in ξ

$$\sigma(P) = p(x, \xi) = \sum_{|\alpha| \le d} a_\alpha(x) \xi^\alpha.$$

For a Schwartz function $f \in \mathcal{S}$,

$$\begin{aligned} Pf(x) &= \sum_{|\alpha| \le d} a_\alpha(x)(D_x^\alpha f(x)) \\ &= \sum_{|\alpha| \le d} a_\alpha(x) \int e^{ix\cdot\xi} \xi^\alpha \hat{f}(x)\, d\xi \\ &= \int e^{ix\cdot\xi} p(x, \xi) \hat{f}(\xi)\, d\xi \\ &= \int e^{i(x-y)\cdot\xi} p(x, \xi) f(y)\, dy\, d\xi. \end{aligned}$$

Accordingly, we define a **pseudo-differential operator** of order $d \in \mathbb{R}$ to be an operator of the form

$$Pf(x) = \int e^{(x-y)\cdot\xi} p(x, \xi) f(y)\, dy\, d\xi$$

where $p(x, \xi)$ is a function satisfying

(a) $p(x, \xi)$ is C_c^∞ in x and C^∞ in ξ

(b) For all $\alpha, \beta \in \mathbb{N}$,

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{d-|\beta|}.$$

This defines a linear operator $\mathcal{S} \to \mathcal{S}$. Moreover, for all s, it extends to a map $H^s \to H^{s-d}$. This generalizes differential operators; an order d differential operator is an order d ΨDO.

Of these, the nicest operators are those whose symbols lie in the class $S^{-\infty} = \bigcap_d S^d$, the so-called **infinitely smoothing** operators. They are called this because they map each Sobolev space H^s to C_c^∞. In a sense, they are negligible with regards to the overall theory of ΨDOs, and thus many results in the theory are stated mod $S^{-\infty}$.

The formalism of pseudo-differential operators allows us to study the parametrix method. We say that a pseudo-differential operator $P(x,D) \in \Psi^m$ is **elliptic** if its symbol $p(x,\xi)$ satisfies

$$|p(x,\xi)^{-1}| \le C\langle\xi\rangle^{-m}$$

for all $|\xi| \ge r$. This is equivalent to the existence of a pseudo-differential operator Q of order $-m$ such that $PQ, QP \equiv I \pmod{\Psi^{-\infty}}$. In other words, P is "almost" invertible. Such a Q is a **parametrix** for P.

One can say a great deal about elliptic ΨDOs. They generalize differential operators whose so-called **leading symbol**, the highest order terms of their symbol, does not vanish for $\xi \neq 0$. The classic example is the Laplacian, whose leading symbol is $\xi_1^2 + \cdots + \xi_n^2$. Easily the most important fact about elliptic operators is **elliptic regularity**: if P is an elliptic pseudo-differential operator, then

$$\operatorname{singsupp} Pu = \operatorname{singsupp} u$$

for every tempered distribution u. In particular, if we have the equation $Pf = g$, where g is a smooth function, then any weak solution is actually a smooth solution. This is an important part of the solution of such PDEs, as it allows us to reason in the much looser world of distributions with the assurance that the result we get will be a classical solution.

For the future, we want to be able to extend the notion of pseudo-differential operators to compact manifolds and their vector bundles. Let M be a compact n-dimensional Riemannian manifold without boundary. Let P be an operator $C^\infty(M) \to C^\infty(M)$. We say that P is a pseudo-differential operator of order d

on M if for every open chart U of M and $\phi, \psi \in C_c^\infty(U)$, the operators $\phi P \psi$ is a pseudo-differential operator of order d when pulled back to Euclidean space through the chart.

From there, we can define the notion of infinitely smoothing operators in the same way, and prove a similar symbol calculus as in the Euclidean case. We say that P is elliptic if $\phi P \psi$ is elliptic in the previous sense when $\phi(x), \psi(x) \neq 0$. We can also define the Sobolev spaces H^s on M via a partition of unity, and from this view P once again maps $H^s \to H^{s-d}$ continuously. (Gilkey 29)

If V is a vector bundle over X, we can use a partition of unity subordinate to an open cover by local trivializations to define the Sobolev space $H^s(V)$. An operator $P : C^\infty(V) \to C^\infty(W)$ is a ΨDO of order d if $\phi P \psi$ is given by a matrix of dth order ΨDOs for $\phi, \psi \in C_c^\infty(U)$ for any coordinate chart U over which V and W are trivial. Once again, we can extend all previous notions locally.

The last observation we make about elliptic ΨDOs (over $\mathbb{R}^n$, a compact manifold, a vector bundle) is that they are of a special class known as Fredholm operators. Such operators have a similar parametrix characterization, though modulo "compact operators" rather than infinitely smoothing ones. (This is a more general case, as infinitely smoothing operators are compact.) We should examine the theory of Fredholm operators, which will lead us to what the index of the Atiyah–Singer index theorem is.

2 Basic Properties of Fredholm Operators

- Fredholm operators are bounded operators between Banach spaces with finite dimensional kernels and cokernels. (Some sources include that the range is closed; this is redundant.) This naturally leads to the definition of the index.

- Intuitively, Fredholm operators are those where can describe the solutions to the inhomogeneous equation $Tx = y$ with only finitely many data. More exactly, the solution space is a finite dimensional affine space at some particular solution x_0, and this equation y is solvable for some y if and only if $\varphi_i(y) = 0$, where the φ_i are the (finitely many) dual basis vectors of the cokernel. (Intuition for Fredholm operators)

- The index is essentially the substitute for rank-nullity, and is notable for its stability. Note that neither the dimension of the kernel nor the dimension of the cokernel is usually stable: consider the operator $T - \lambda I$ as λ passes through an eigenvalue. But in finite dimensions the index is always $\dim V - \dim W$, and a weaker form of stability still holds. As we will see in the Fredholm alternative, this allows us to draw conclusions if we know something about the kernel or the cokernel.

- If $T : E \to E$ is a compact operator on a Banach space, then $I - T$ is Fredholm. (Lang Theorem 2.1)

- If E, F are Banach spaces, then $\mathcal{F}(E, F)$ is open in $\mathcal{B}(E, F)$, and the index is continuous. In particular it is constant on connected components (and thus path components since topological vector spaces are locally path connected). (Lang Theorem 2.3)

- Fredholm operators are those whose images in the Calkin algebra are invertible. In other words, they are invertible modulo compact operators. This inverse can be chosen to have finite dimensional cokernel. (Lang Theorem 2.5, Mukherjee Theorem 2.2.6). This is the *parametrix* or *pseudoinverse*.

- The Fredholm operators on a Banach space form a monoid. (Lang Corollary 2.6)

- $\mathcal{F}(E, F)$ is closed under compact perturbations, and index is also constant in this regard. (Lang Corollary 2.6, 2.7) Proof of the first part is that all the perturbations can have the same parametrix.

- Index is additive on compositions (even when not on the same Banach space). In particular, it is a monoid homomorphism $\mathcal{F}(E) \to \mathbb{Z}$. (Lang Theorem 2.8) Mukherjee gives a neat proof using the snake lemma and Euler characteristic. (Mukherjee Corollary 2.1.7)

- The stability of index can be used to show that if $T - \lambda I$ is injective for some $\lambda \neq 0$, then $T - \lambda I$ is invertible. This is because it has a path through $\mathcal{F}(E)$ to λI, which has index 0. (Lang Theorem 3.1) This allows us to conclude the Fredholm alternative: either $(T - \lambda I)x = y$ has a unique solution for each y, or $(T - \lambda I)x = 0$ has non-trivial solutions.

Examples of compact operators:

- Operators with finite rank (finite dimensional image). In fact compact operators in some sense generalize finite rank operators.

- On a separable Hilbert space, we can define diagonal operators to be those defined by $Te_k = \lambda_k e_k$, where e_k is a Schauder basis. If $\lambda_k \to 0$ then such an operator is compact. (Yale)

- If $K : [a, b] \times [a, b] \to \mathbb{R}$ is continuous, then the integral operator

$$Tf(x) = \int_a^b K(x, y)f(y)\, dy$$

  is compact as an operator of $C([a, b])$ and $L^2([a, b])$.

Examples of Fredholm operators:

- The identity operator, and in general the shift operators.

- Operators of the form $K - \lambda I$, where K is compact. The fact that these are Fredholm is useful in studying eigenvalue problems of compact operators.

3 The Atiyah–Jänich Theorem

In general, assume that all spaces are complex infinite dimensional separable Hilbert spaces.

- If $T \in \mathcal{F}(H_1, H_2)$ and $V \subset H_1$ is a closed subspace of finite codimension with $V \cap \ker T = 0$, then $H_2/T(V)$ is isomorphic to a finite dimensional subspace of H_2. Also we can make a vector bundle $\bigcup_{S \in U} H_2/S(V)$ over U in $\mathcal{B}(H_1, H_2)$. (Mukherjee Lemma 2.3.1) The proof uses orthogonal complements so this might not extend to Banach spaces.

- The above lemma quickly proves that $\mathcal{F}(H_1, H_2)$ is closed in $\mathcal{B}(H_1, H_2)$. Granted, this can be proved for Banach spaces without this lemma.

- Index is a homotopy invariant of $\mathcal{F}(H_1, H_2)$. This proof also uses orthgonal complements. (Mukherjee Lemma 2.3.3)

- Let X be compact. If $T : X \to \mathcal{F}$ is a continuous family such that $\dim\ker T_x$ is constant, then $\bigcup_{x\in X} \ker T_x$ and $\bigcup_{x\in X} \operatorname{coker} T_x$ form vector bundles over X, and we might then define a generalization of the index as the "virtual bundle"

$$\operatorname{ind} T = [\ker T] - [\operatorname{coker} T] \in K(X).$$

  In general this property will not hold, but we can reduce to this case by composing with a suitable projection. (Atiyah Global Theory)

- Specifically, there exists an open set V such that $H/T(V) := \bigcup_{x\in X} H/T_x(V)$ forms a vector bundle over X (plus some other stuff). (Mukherjee Lemma 2.3.5) This is our version of the cokernel bundle, while the trivial bundle $H/V := X \times T/V$ acts as our kernel bundle. We then define

$$\operatorname{ind} T = [H/V] - [H/T(V)]$$

  (Mukherjee Definition 2.3.6) Homological reasoning shows that this is independent of V.

- The bundle index is a homotopy invariant (homotopies of maps $X \to \mathcal{F}$), so it defines a map $[X, \mathcal{F}] \to \mathbb{Z}$. The set $[X, \mathcal{F}]$ is a monoid via pointwise multiplication.

- The index map is functorial and is a semigroup homomorphism. (Mukherjee Proposition 2.3.8)

- If X is compact, then the index map is an isomorphism from $[X, \mathcal{F}]$ to $K(X)$. (Mukherjee Theorem 2.4.1) The proof relies on Kuiper's theorem that $\mathrm{GL}(H) = \mathcal{B}(H)^\times$ is contractible for a complex infinite dimensional separable Hilbert space. (This is not true in finite dimensions, similar to S^∞ versus the normal spheres.)

$$[X, \mathcal{B}^\times] \to [X, \mathcal{F}] \xrightarrow{\text{ind}} K(X) \to 0$$

  is exact, as this implies the middle is an isomorphism. (Mukherjee says $\mathcal{F}^\times$, and maybe Kuiper does this as well, but the more famous result of Kuiper is $\mathcal{B}^\times$, and this is what Atiyah uses. Note that $\mathcal{B}^\times \subset \mathcal{F}$ since both kernel and cokernel are trivial.)

- Kuiper's theorem is apparently also true for real and non-separable Hilbert spaces (Mukherjee Remark 2.5.8)

- The most basic case of this is that $[*, \mathcal{F}] \cong K(*) \cong \mathbb{Z}$, which recovers our original result about the index of Fredholm operators since $[*, \mathcal{F}]$ represents the path components.

4 Conclusion

We end by returning to the study of elliptic operators. Let $P : C^\infty(V) \to C^\infty(W)$ be an elliptic ΨDO of order d for vector bundles over a compact Riemannian manifold without boundary. Then, for all s, P is a Fredholm operator $H^s(V) \to H^{s-d}(W)$. The index of P does not depend on s. In principle, this can tell us a great deal about the solutions of elliptic operators provided we can calculate this index. This is the problem the Atiyah–Singer index theorem seeks to answer.