Enumerative Geometry via the Chow ring

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I. INTRODUCTION

For our purposes, a scheme X is separated of finite type over a field k. Unless stated otherwise, k is algebraically closed and of characteristic 0. A variety V is simply an integral scheme. A subscheme is a closed subscheme unless stated otherwise.

The general outline by which we will proceed is to first discuss Bézout's Theorem in a few of its various forms, as these results serve as important guiding principles for intersection theory. We then define the Chow ring, which will allow us to abstract away a lot of the difficulty of conceptualizing scheme theoretic intersection. After developing a series of essential lemmas about the behavior of the Chow ring and its underlying group, we will conclude with some applications to enumerative geometry.

II. THREE COURSES OF BÉZOUT

Theorem 1 (Baby Bézout's Theorem) *Let X and Y be plane projective curves over a field k with no common factor. Then*

$$|X \cap Y| = \deg X \deg Y$$

Proof: Write the equations for X and Y as

$$X : a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$$

and

$$Y : b_0 z^m + b_1 z^{m-1} + \cdots + b_m = 0$$

where the a_i and b_i are polynomials in x and y, with z the homogenizing factor. Form the Sylvester matrix S of the two polynomials; for example, if $n = 3$, $m = 2$, the matrix would be

$$S = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 \end{pmatrix}$$

with the obvious generalization to other n, m thereof.

The determinant of the Sylvester matrix is zero iff the two curves have a common root (e.g they intersect); to see this, let P_d be the d-dimensional vector space over k of polynomials of degree at most $d-1$, and define $\varphi : P_m \times P_n \to P_{m+n}$ given by

$$\varphi(P, Q) = XP + QY$$

where, by abuse of notation, X and Y refer to the homogeneous equations above. Using the basis of $1, \cdots, z^k$ for all vector spaces involved, it is easy to see that S is the transpose of the matrix of φ, and a nonzero element of $\ker \varphi$ indicates that X and Y have a common zero. To see this, note that $XP = -QY$, hence any zero of X is a zero of QY. If X has no roots in common with Y, then X must divide Q, but Q is of degree at most $n-1$ (and nonzero), so this cannot hold.

Then, $\det S$ is a nonzero homogeneous polynomial of degree mn in x and y. To see this, we use the alternating sum definition of the determinant, and we need only show that

$$\prod_{i=1}^{m+n} S_{i,\sigma(i)}$$

has degree mn whenever it is nonzero. Note that a_k has degree $n-k$ and b_k has degree $m-k$ in x and y, so that

$$\deg S_{i,j} = \begin{cases} j - i & 1 \le i \le m \\ j - (i - m) & m+1 \le i \le m+n \end{cases}$$

Therefore, the degree of the product when nonzero is given by

$$\sum_{i=1}^{m+n} \sigma(i) - \sum_{i=1}^{n} i - \sum_{i=1}^{m} i = \frac{(m+n+1)(m+n)}{2} - \frac{n^2+n}{2} - \frac{m^2+m}{2} = mn$$

Finally, by the Fundamental Theorem of Algebra $\det S$ splits into linear factors, which follows by pulling out a factor of y^{mn} from $\det S$ and factoring the resulting polynomial in $\frac{x}{y}$ into linear terms, then pushing the factors of y back in. Therefore, since there are mn linear terms $ax + by$ in the factorization of $\det S$, each one gives one projective solution (together with $z = 1$) for a point in the intersection, therefore, there are mn points of intersection (counted with multiplicity). ■

Much of the intersection theory we will explore is modeled on this result; in particular, it is remarkable that the number mn does not depend on the curves, but this is in part because we have concocted projective space to avoid the "unlikely" scenario of parallel lines, and work over an algebraically closed field to avoid missing intersections. In that vein, we will develop a notion of rational equivalence, and a setting where intersections of subvarieties depend only on the rational equivalence class of the given subvarieties. Moreover, the fact that the number of intersections mn is the product of degrees suggests some sort of ring structure on subvarieties, which we construct in the next section. With that technology, we will be able to prove the following generalization of the above:

Theorem 2 (Teenage Bézout's Theorem) *If $X_1, \cdots, X_k \subset \mathbb{P}^n$ are subvarieties of codimensions $c_1, \cdots, c_k$ such that $\sum_i c_i \leq n$, and the X_i intersect transversely, then*

$$\deg(X_1 \cap \cdots \cap X_k) = \prod_i \deg(X_i)$$

This result essentially extends the domain of Bézout's Theorem from curves to subvarieties generally. From here, there is another level of abstraction we can consider: to deal with subvarieties of arbitrary smooth varieties as opposed to $\mathbb{P}^n$, and to loosen the requirement of generic transversality by introducing intersection multiplicities, which we already have some familiarity with in elementary settings; for example, a tangent to a conic, though not a transverse intersection, can be handled by our machinery as an intersection of multiplicity 2.

Theorem 3 (Adult Bézout's Theorem) *Let $A, B \subseteq X$ be subvarieties of a smooth variety X, and suppose that $\operatorname{codim} C = \operatorname{codim} A + \operatorname{codim} B$ for each irreducible component C of the intersection $A \cap B$, then we can associate to each irreducible component C a positive integer $m_C(A, B)$ in such a way that*

$$[A][B] = \sum_i m_{C_i}(A, B) \cdot [C_i]$$

with the additional constraint that $m_C(A, B) = 1$ iff A and B intersect transversely at a general point of C.

The key step of proving this result will be in establishing *Serre's Intersection Formula*, which states that

$$m_C(A, B) = \sum_{i=0}^{\dim X} (-1)^i \operatorname{length}_{\mathcal{O}_{A\cap B,Z}}(\operatorname{Tor}_i^{\mathcal{O}_{X,Z}}(\mathcal{O}_{A,Z}, \mathcal{O}_{B,Z}))$$

This result will remain outside the scope of our discussion, but serves as a good guiding result for further study.

III. THE CHOW RING

The development of the Chow Ring is somewhat similar to the development of the divisor class group.

Definition 4 (Group of Cycles) *Let X be a scheme, then $Z(X)$ is the group of cycles on X, the free abelian group generated by the reduced irreducible subschemes (e.g subvarieties, or, equivalently, integral subschemes) of X.*

Clearly, $Z(X)$ is graded by dimension, e.g

$$Z(X) = \bigoplus_k Z_k(X)$$

where elements of $Z_k(X)$ are called k-cycles. Notice that $Z_{n-1}(X)$ is the group of divisors, from which we will inherit much of our terminology; for example, a cycle $\sum_i n_i Y_i$ is effective if all the coefficients n_i are nonnegative. For the following proof, recall the result that a scheme X is irreducible iff it has a unique generic point.

Lemma 5 *Let X be a scheme. Then $Z(X) = Z(X_{red})$.*

Proof: This follows from the fact that there is a bijective correspondence between integral subschemes $Y \subseteq X$ and points $x \in X$, given by taking the (unique) generic point of such a subscheme, with the reverse map taking a point to its closure. Therefore, since X and X_{red} have the same underlying topological space, the equality is immediate. ■

To regard arbitrary (not necessarily reduced or irreducible) subschemes $Y \subseteq X$ as elements of $Z(X)$, we introduce the notion of the effective cycle: let $Y_1, \cdots, Y_r$ be the irreducible components of Y_{red} (of which there are finitely many since schemes of finite type over a Noetherian ring are Noetherian), then we define

$$\langle Y \rangle = \sum_{i=1}^r l_i Y_i$$

where l_i are the lengths of the local rings $\mathcal{O}_{Y,x_i}$ where x_i is the unique generic point of Y_i. The l_i are called the multiplicity of Y along Y_i.

As in the study of the divisor class group, we only care about algebraic cycles up to a certain kind of equivalence. To that end, define $\operatorname{Rat}_i(X)$ to be the subgroup of $Z_i(X)$ consisting of all elements of the form

$$(f) = \sum_Y \operatorname{ord}_Y(f)\langle Y \rangle$$

where f is a rational function on some $(i+1)$-dimensional integral subscheme W of X, and the sum runs over all codimension 1 irreducible subschemes Y, and $\operatorname{ord}_Y(f)$ is the order of vanishing of f along Y.

Definition 6 (Chow Group) *Let X be a scheme, and let $\operatorname{CH}_k(X) = Z_i(X)/\operatorname{Rat}_i(X)$. Then the Chow Group of X is given by*

$$\operatorname{CH}(X) := \bigoplus_k \operatorname{CH}_k(X)$$

Note that we sometimes grade the Chow group by codimension, denoted $\operatorname{CH}^k(X) = \operatorname{CH}_{n-k}(X)$ with X n-dimensional.

An equivalent formulation is to define $\operatorname{Rat}(X)$ as the subgroup of $\mathbb{P}^1_k \times X$ generated by differences of the form

$\langle \Phi \cap (\{t_0\} \times X)\rangle - \langle \Phi \cap (\{t_1\} \times X)\rangle$ for $t_0, t_1 \in \mathbb{P}^1_k$ and subvarieties Φ of $\mathbb{P}^1_k \times X$ not contained in any fiber $\{t\} \times X$. The idea is that the fiber above $\{t_0\} \times X$ is one subvariety, and the fiber above $\{t_1\} \times X$ another one, and as t passes between t_0 and t_1, one subvariety "smoothly" (in an informal sense) deforms into the other. This has a nice visual analogy with cobordism, where two manifolds of a given dimension are cobordant if they together form the boundary for a manifold of one higher dimension; e.g, a pair of pants is a cobordism between a circle and a pair of circles, and the smooth deformation is obtained by taking cross sections from the waist down to the legs.

This data is also equivalent to defining $\mathrm{CH}(X)$ via the following right-exact sequence:

$$Z(\mathbb{P}^1 \times X) \to Z(X) \to \mathrm{CH}(X) \to 0$$

where the left hand map takes a variety $\Phi \subseteq \mathbb{P}^1 \times X$ to 0 if it is contained in a fiber, and to $\langle \Phi \cap (\{t_0\} \times X)\rangle - \langle \Phi \cap (\{t_1\} \times X)\rangle$ otherwise. This formulation will be useful later to prove some key lemmas.

Note that, as above with $Z(X)$, we have that $\mathrm{CH}(X) = \mathrm{CH}(X_{\mathrm{red}})$, since $\mathrm{Rat}_i(X) = \mathrm{Rat}_i(X_{\mathrm{red}})$ for each i by the same argument as for $Z(X)$. We are then ready to produce a ring from this group, to define the product on $\mathrm{CH}(X)$; to do so, we need to define what it means for an intersection to be transverse.

Definition 7 (Transverse Intersection) *Let X be a scheme, A, B subvarieties. A and B intersect transversely at a point p if A, B, and X are all smooth at p, and the tangent spaces at A and B at p taken together span the tangent space of X at p, e.g*

$$T_pA + T_pB = T_pX$$

Equivalently,

$$\mathrm{codim}(T_pA \cap T_pB) = \mathrm{codim}\, T_pA + \mathrm{codim}\, T_pB$$

A and B are generically transverse if they meet transversely at a generic point of each component of their intersection.

Similar to how the introduction of projective space forces all pairs of distinct lines to intersect at a unique point, in order to rule out counterexamples in affine space such as parallel lines, which are in some sense "unlikely," the transverse condition prevents the consideration of curves intersecting at a point where they are tangent to one another (although we will deal with these cases by essentially moving the curves until they intersect transversely).

Definition 8 (Intersection Products) *If X is a smooth variety over a field, then there is a unique product structure on $\mathrm{CH}(X)$ such that for any two subvarieties A, B of X which are generically transverse, then*

$$[A][B] = [A \cap B]$$

which gives $\mathrm{CH}(X)$ the structure of an associative, commutative graded ring.

This operation makes $\mathrm{CH}(X)$ into a ring; there is the slight issue that this definition is really more of a theorem, as there is quite a bit to check. First, note that we need the product to be well-defined for *all* subvarieties, not just those which are generically transverse to one another. Furthermore, we need to show that the product makes sense with respect to rational equivalence. Unfortunately, the proofs of these results are too involved to present concisely here, but the main result is as follows:

Theorem 9 *Let X be a smooth variety.*

1. *For every $\alpha, \beta \in \mathrm{CH}(X)$, there exist generically transverse cycles $A, B \in Z(X)$ such that $[A] = \alpha$, $[B] = \beta$.*
2. *The class $[A \cap B]$ is independent of the choice of such cycles A and B.*

The moral of the first part of this famous result is that a failure to be generically transverse is somehow "rare," e.g there are, in some natural sense, far more ways to arrange a circle and a line which intersect in two distinct points than a circle plus a tangent line. Given one of the unfavorable situations, we can always perturb it to a favorable one. The second part is the "uniqueness" counterpart to the "existence" of the first part. Indeed, if $[A \cap B]$ depended on the choice of cycles, we would need a weaker notion of rational equivalence to get any use out of the Chow ring, and we may not be able to ignore the unfavorable situations any longer in this "finer" setting. Assuming this result, we are ready to carry out some elementary calculations:

Definition 10 *Let X be a scheme, then $[X] \in \mathrm{CH}(X)$ is the fundamental class of X.*

Note that $[X]$ is never rationally equivalent to zero, since if X is n-dimensional, such a rational equivalence would be given by a divisor on an $n+1$-dimensional subscheme of X.

Proposition 11 $\mathrm{CH}(\mathbb{A}^n) \cong \mathbb{Z} \cdot [\mathbb{A}^n] = \mathbb{Z}$

Proof: Let Y be a proper subvariety of $\mathbb{A}^n$; we will show that Y is rationally equivalent to zero. To that end, choose coordinates $z_1, \cdots, z_n$ on $\mathbb{A}^n$ so that the origin does not lie in Y. Define

$$W^\circ = \{(t, tz) \subset (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^n : z \in Y\}$$

Clearly, then, we have

$$W^\circ = V(\{f(z/t) : f(z) \text{ vanishes on } Y\})$$

The fiber of W° over a point $t \in \mathbb{A}^1 \setminus \{0\}$ is tY. Let $W \subset \mathbb{P}^1 \times \mathbb{A}^1$ be the closure of W°, which is irreducible since W° (which is the image of $(\mathbb{A}^1 \setminus \{0\}) \times Y$) is.

Since the origin does not lie in Y, there is some polynomial $g(z)$ vanishing on Y which has a nonzero constant term c. Then $G(t,z) := g(z/t)$ on $(\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^n$ extends to a regular function on $\mathbb{P}^1 \times \mathbb{A}^n$ with constant value c above $\infty \times \mathbb{A}^n$; therefore, the fiber of W over $t = \infty$ is empty (by the latter characterization of W°) whereas the fiber over $t = 1$ is just Y, which establishes that $[Y] = 0$. ■

A direct approach, as above, to calculate the Chow ring is hopelessly difficult in a more general situation. To remedy that, we will now develop some results that allow us to build the Chow ring up piece by piece, towards the ultimate goal of proving an elementary result on what are known as quasi-affine stratifications.

First, note that for any closed subscheme $Y \subseteq X$, cycles on Y are automatically cycles on X, and similarly cycles of $\mathbb{P}_1 \times Y$ are cycles of $\mathbb{P}^1 \times X$, so we have a natural map of Chow *groups* $\mathrm{CH}(Y) \to \mathrm{CH}(X)$. Similarly, let $U = Y^c$, and since the intersection of a subvariety of X with U is a possibly empty subvariety of U, there is a natural restriction morphism $Z(X) \to Z(U)$. As above, this also holds for $Z(\mathbb{P}^1 \times X) \to Z(\mathbb{P}^1 \times U)$, so we have a natural restriction morphism of Chow groups $\mathrm{CH}(X) \to \mathrm{CH}(U)$ given by intersecting with U.

Lemma 12 (Excision) *Let X be a scheme, Y a closed subscheme of X, $U = Y^c$ its complement, then there is a right exact sequence of groups*

$$\mathrm{CH}_k(Y) \to \mathrm{CH}_k(X) \to \mathrm{CH}_k(U) \to 0$$

for all k. Moreover, if X is smooth, then the map $\mathrm{CH}(X) \to \mathrm{CH}(U)$ is a ring homomorphism.

Proof: We have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & Z_k(Y \times \mathbb{P}^1) \arrow[r] \arrow[d] & Z_k(X \times \mathbb{P}^1) \arrow[r] \arrow[d] & Z_k(U \times \mathbb{P}^1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_k(Y) \arrow[r] \arrow[d] & Z_k(X) \arrow[r] \arrow[d] & Z_k(U) \arrow[r] \arrow[d] & 0 \\
 & \mathrm{CH}_k(Y) \arrow[r] \arrow[d] & \mathrm{CH}_k(X) \arrow[r] \arrow[d] & \mathrm{CH}_k(U) \arrow[r, dashed] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

That the columns are exact is immediate, by our discussion of rational equivalence above. Since cycles on Y are cycles on X, $Z_k(Y) \to Z_k(X)$ is clearly an inclusion. The map $Z_k(X) \to Z_k(U)$ acts by $[A] \mapsto [A \cap U]$ which is clearly surjective since subvarieties of U are subvarieties of X, and exactness in the middle is satisfied since U is the complement of Y. Therefore, the middle row is exact, and by the same reasoning, the top row as well. It remains only to show that $\mathrm{CH}_k(X) \to \mathrm{CH}_k(U)$ is a surjection; to see this, note that we can pull elements in $\mathrm{CH}_k(U)$ back to $Z_k(U)$ via surjectivity, then pull that back to $Z_k(X)$ again by surjectivity, and map that down to $\mathrm{CH}_k(X)$. That $\mathrm{CH}(X) \to \mathrm{CH}(U)$ is a ring homomorphism if X is smooth simply reflects the fact that the intersection product is only well-defined on smooth schemes. ■

As an immediate corollary, for any nonempty open set $U \subseteq \mathbb{A}^n$, $\mathrm{CH}(U) = \mathbb{Z}$, since $\mathrm{CH}(\mathbb{A}^n) \to \mathrm{CH}(U)$ given by $[Z] \mapsto [Z \cap U]$ is a surjection of rings, so $\mathrm{CH}(U)$ is a quotient of $\mathbb{Z}$ generated by $[U]$, but the fundamental class generates a copy of $\mathbb{Z}$, so $\mathrm{CH}(U)$ is $\mathbb{Z}$ itself.

Lemma 13 (Mayer-Vietoris) *Let X be a scheme, X_1, X_2 closed subschemes of X, then there is a right exact sequence of groups*

$$\mathrm{CH}(X_1 \cap X_2) \to \mathrm{CH}(X_1) \oplus \mathrm{CH}(X_2) \to \mathrm{CH}(X_1 \cup X_2) \to 0$$

Proof: Let $Y = X_1 \cap X_2$, and we may assume that $X = X_1 \cup X_2$ by shrinking X if necessary. Then we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & Z(Y \times \mathbb{P}^1) \arrow[r] \arrow[d] & Z(X_1 \times \mathbb{P}^1) \oplus Z(X_2 \times \mathbb{P}^1) \arrow[r] \arrow[d] & Z(X \times \mathbb{P}^1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z(Y) \arrow[r] \arrow[d] & Z(X_1) \oplus Z(X_2) \arrow[r] \arrow[d] & Z(X) \arrow[r] \arrow[d] & 0 \\
 & \mathrm{CH}(Y) \arrow[r] \arrow[d] & \mathrm{CH}(X_1) \oplus \mathrm{CH}(X_2) \arrow[r] \arrow[d] & \mathrm{CH}(X) \arrow[r, dashed] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The columns are again exact by construction. The rows need some explaining; the map $Z(Y) \to Z(X_1) \oplus Z(X_2)$ acts by $[A] \mapsto ([A], -[A])$, and the map $Z(X_1) \oplus Z(X_2) \to Z(X)$ is addition. From this, exactness follows for the middle row, and similarly for the top row. By this reasoning, the bottom row is clearly exact in the middle, and to show surjectivity on the right, we can diagram chase as before. Elements of $\mathrm{CH}(X)$ retract to elements of $Z(X)$, which retract to elements of $Z(X_1) \oplus Z(X_2)$, which we can there map down to $\mathrm{CH}(X_1) \oplus \mathrm{CH}(X_2)$. ■

Finally, we are ready to define stratifications. As we stated above, in general, it is difficult to calculate the the Chow ring of a scheme except in some special circumstances; one of these is when our scheme has an open cover satisfying certain properties.

Definition 14 (Stratification) *A scheme X is stratified by a finite collection of irreducible, locally closed subschemes U_i if X is the disjoint union of the U_i, and the closure of any U_i is the union of such U_j. The latter condition is equivalent to the following: if $\overline{U_i}$ intersects U_j nontrivially, then $\overline{U_i}$ contains U_j. The sets U_i are the strata or open strata of the stratification, while the sets $Y_i = \overline{U_i}$ are the closed strata. A stratification of X is affine if each U_i is isomorphic to some $\mathbb{A}^k$, and quasi-affine if each U_i is isomorphic to an open subset of some $\mathbb{A}^k$.*

Proposition 15 *If a scheme X has a quasi-affine stratification, then $\mathrm{CH}(X)$ is generated by the classes of the closed strata.*

Proof: We induct on the number of strata. If there is one stratum, then the result follows from our observation above that $\mathrm{CH}(U) = \mathbb{Z}$ for a nonempty open subset of affine space. In the general case, let U_0 be a stratum which is minimal with respect to inclusion. Since $\overline{U_0}$ is the union of U_i, by minimality, we must conclude that U_0 is closed; therefore, $Y = U_0^c$ is stratified by the strata other than U_0. By the inductive hypothesis, $\mathrm{CH}(Y)$ is generated by the classes of the closures of these strata, and by the $n = 1$ case, $\mathrm{CH}(U_0)$ is generated by $[U_0]$. By the excision right exact sequence, we have

$$\mathrm{CH}(U_0) = \mathbb{Z} \to \mathrm{CH}(X) \to \mathrm{CH}(Y) \to 0$$

Since $\mathrm{CH}(Y)$ is generated by the closed strata other than U_0 by induction, and these closed strata are naturally elements of X, it therefore follows that $\mathrm{CH}(X)$ is generated by the classes of the closed strata. ■

We are now ready to determine the the Chow ring of $\mathbb{P}^n$, an essential calculation to any problem in enumerative geometry.

Theorem 16 $\mathrm{CH}(\mathbb{P}^n) = \mathbb{Z}[\zeta]/(\zeta^{n+1})$ *where ζ is the rational equivalence class of a hyperplane. Moreover, the class of a variety of codimension k and degree d is $d\zeta^k$.*

Proof: Let

$$\{p\} \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^n$$

be a complete flag of subspaces, and let $U_i = \mathbb{P}^i \setminus \mathbb{P}^{i-1} \cong \mathbb{A}^i$. The U_i give an affine stratification of $\mathbb{P}^n$, since $\overline{U_i} = \mathbb{P}^i$ and therefore contains all U_k for $k \leq i$. By Proposition 15, $\mathrm{CH}_k(\mathbb{P}^n)$ is generated by the class of $\mathbb{P}^k$, e.g any k-plane.

Moreover, $\mathrm{CH}_0(\mathbb{P}^n) = \mathbb{Z}$ since there is a surjection $\mathrm{CH}_0(\mathbb{P}^n) \to \mathbb{Z}$ taking a linear combination of points to the sum of their coefficients (reminiscent of the degree morphism for divisors on curves), and because we can move all the points to one point by rational equivalence (given points p, q, we can construct a rational function vanishing at p with a pole at q, so that the element $[p] + [q] \in \mathrm{CH}_0(\mathbb{P}^n)$ is equal to $[p] + [q] + [p] - [q] = 2[p]$).

This in fact implies that $\mathrm{CH}_k(\mathbb{P}^n) = \mathbb{Z}$ for all k; to see this, note that a a general k-plane L intersects a general $(n-k)$-plane M transversely at one point. This follows from the fact that a copy of $\mathbb{P}^k$ in $\mathbb{P}^{n+1}$ is given by demanding that $n-k$ of the coordinates are equal to 0, and a copy of $\mathbb{P}^{n-k}$ by demanding that k of the coordinates be 0. For the $\mathbb{P}^k$ to intersect transversely with the $\mathbb{P}^{n-k}$, these indices must be disjoint, leaving a 1 in one coordinate and 0s in all other coordinates, e.g, a single point. Therefore, multiplication by $[M]$ induces a surjection $\mathrm{CH}_k(\mathbb{P}^n) \to \mathrm{CH}_0(\mathbb{P}^n) = \mathbb{Z}$, so $\mathrm{CH}_k(\mathbb{P}^n) = \mathbb{Z}$ for all k, since we know that $\mathrm{CH}_k(\mathbb{P}^n)$ is generated by $[L]$ and therefore both contains and is contained in $\mathbb{Z}$.

A k-plane in $\mathbb{P}^n$ is the transverse intersection of $n-k$ hyperplanes by the coordinate counting argument above, so any such plane L has class $[L] = \zeta^{n-k}$ where ζ is the class of a hyperplane, from which the result follows. Note that the class of a variety X of *codimension* k and degree d is $d\zeta^k$; to see this, note that X intersects a general k-plane transversely in d points, so $\deg([X]\zeta^{n-k}) = d$, with $\deg(\zeta^n) = 1$, so that $[X] = d\zeta^k$. ■

Immediately, we have a proof of Theorem 2 as an easy corollary; if $\deg X_i = d_i$, then $[X_i] = d_i\zeta^{c_i}$, so that

$$[X_1 \cap \cdots \cap X_k] = \left(\prod_i d_i\right)\zeta^{\sum_i c_i}$$

That the sum of the codimensions does not exceed n, the dimension of the ambient space, ensures that the ζ term does not vanish. The result on degrees follows immediately from this calculation.

Theorem 17

$$\mathrm{CH}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}) = \bigotimes_{k=1}^{r} \mathrm{CH}(\mathbb{P}^n_k)$$

Together with the previous result, this implies that

$$\mathrm{CH}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}) \cong \mathbb{Z}[t_1, \cdots, t_r]/(t_1^{n_1+1}, \cdots, t_r^{n_r+1})$$

where t_k is the class of the pullback (via projection) of the hyperplane class on $\mathbb{P}^{n_k}$. Moreover, the class of the hypersurface defined by a homogeneous form of degree $(a_1, \cdots, a_r)$ is $\sum_{k=1}^r a_k t_k$.

Proof: For notational clarity, we will restrict to the case where $r = 2$, though the underlying ideas are the same. We need to prove that

$$\mathrm{CH}(\mathbb{P}^m \times \mathbb{P}^n) \cong \mathrm{CH}(\mathbb{P}^m) \otimes \mathrm{CH}(\mathbb{P}^n)$$

and we proceed as above by constructing an affine stratification on $\mathbb{P}^m \times \mathbb{P}^n$. We have complete flags

$$\Lambda_0 \subset \cdots \subset \Lambda_m = \mathbb{P}^m \quad \Gamma_0 \subset \cdots \subset \Gamma_n = \mathbb{P}^n$$

and take the closed strata to be

$$\Xi_{a,b} = \Lambda_{m-a} \times \Gamma_{n-b}$$

with corresponding open strata

$$\tilde{\Xi}_{a,b} := \Xi_{a,b} \setminus (\Xi_{a-1,b} \cup \Xi_{a,b-1})$$

By Proposition 15, we can conclude that $\mathrm{CH}(\mathbb{P}^m \times \mathbb{P}^n)$ is generated by the classes $[\Xi_{a,b}] \in \mathrm{CH}^{a+b}(\mathbb{P}^m \times \mathbb{P}^n)$. Since $\Xi_{a,b}$ is the transverse intersection of the pullbacks (via projection) of a hyperplanes in $\mathbb{P}^m$ and b hyperplanes in $\mathbb{P}^n$ (since each hyperplane increases the codimension by 1), we have $[\Xi_{a,b}] = \alpha^a\beta^b$ where α and β are, respectively, the pullbacks via projection of the hyperplane classes in $\mathbb{P}^m$ and $\mathbb{P}^n$. Therefore, it is easy to see that $\alpha^{m+1} = \beta^{n+1} = 0$, since their projections vanish. This shows that there is a surjective morphism from

$$\mathbb{Z}[\alpha, \beta]/(\alpha^{m+1}, \beta^{n+1}) = \mathbb{Z}[\alpha]/(\alpha^{m+1}) \otimes \mathbb{Z}[\beta]/(\beta^{n+1})$$

to $\mathrm{CH}(\mathbb{P}^m \times \mathbb{P}^n)$. To see that it is a bijection, note that $\Xi_{m,n}$ is a single point, so its degree is 1, and consider the pairing

$$\mathrm{CH}^{a+b}(\mathbb{P}^m \times \mathbb{P}^n) \times \mathrm{CH}^{m+n-a-b}(\mathbb{P}^m \times \mathbb{P}^n) \to \mathbb{Z}$$

given by $([X], [Y]) \mapsto \deg([X][Y])$ for $[X]$, $[Y]$ of appropriate dimension. Under this map, $(\alpha^a\beta^b, \alpha^r\beta^s)$ is sent to 1 if $a + r = m$ and $b + s = n$, since the intersection in this case is transverse and consists of a single point, and to 0 otherwise, since the intersection would be empty. This shows that monomials of degree (a, b) are linearly independent with a, b in the appropriate range, which proves the result.

If $F \in k[x, y]$ is homogeneous of degree (d, e), then because $F(x, y)/x_0^d y_0^e$ is a rational function on $\mathbb{P}^m \times \mathbb{P}^n$ (and therefore its vanishing locus is rationally equivalent to zero), the class defined by $F = 0$ is $d\alpha + e\beta$. ■

We then have some results on the functoriality of Chow groups, which we will use later in some calculations but will not prove in the interest of brevity.

Definition 18 (Pushforward for cycles) *Let $f : X \to Y$ be a proper map of schemes, $A \subseteq Y$ a subvariety.*

1. *If $f(A)$ has strictly lower dimension than A, then we set $f_*\langle A\rangle = 0$*
2. *If $\dim f(A) = \dim A$ and $f|_A$ has degree n, then we set $f_*\langle A\rangle = n \cdot \langle f(A)\rangle$*

Theorem 19 (Pullback for cycles) *Let $f : Y \to X$ be a map of smooth quasi-projective varieties. There is a unique map of groups $f^* : \mathrm{CH}^c(X) \to \mathrm{CH}^c(Y)$ such that whenever $A \subseteq X$ is a subvariety generically transverse to f, that is $f^{-1}(A)$ is generically reduced and $\mathrm{codim}_Y(f^{-1}(A)) = \mathrm{codim}_X(A)$, then*

$$f^*([A]) = [f^{-1}(A)]$$

Note that this equality also holds when the hypothesis of generic transversality is weakened to $\mathrm{codim}_Y(f^{-1}(A)) = \mathrm{codim}_X(A)$ and A is Cohen-Macaulay. The map f^* is a ring homomorphism making CH *into a contravariant functor from the category of smooth projective varieties into the category of graded rings.*

With the pullback well-defined, we have another result in the vein of our right-exact sequence lemmas above:

Proposition 20 *Let X be a scheme, and $\pi : E \to X$ a locally free sheaf of rank r on X. Then for all $k \geq 0$, the pull back homomorphism $\pi^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+r}(E)$ is surjective.*

Proof: Note that π^* maps $\mathrm{CH}_k(X)$ to $\mathrm{CH}_{k+r}(E)$ since a cycle will gain a factor of $\mathbb{A}^r$ under preimage. We proceed via induction on $\dim X$; let U be an open affine on which E is trivial, e.g, such that $E|_U \cong U \times \mathbb{A}^r$, and set $Y = U^c$. By excision, we have the following diagram:

$$\begin{tikzcd}
\mathrm{CH}_k(Y) \arrow[r] \arrow[d] & \mathrm{CH}_k(X) \arrow[r] \arrow[d, "\pi^*"'] & \mathrm{CH}_k(U) \arrow[r] \arrow[d] & 0 \\
\mathrm{CH}_{k+r}(E|_Y) \arrow[r] & \mathrm{CH}_{k+r}(E) \arrow[r] & \mathrm{CH}_{k+r}(U \times \mathbb{A}^r) \arrow[r] & 0
\end{tikzcd}$$

By the four lemma, if we can show that $\mathrm{CH}_k(Y) \to \mathrm{CH}_{k+r}(E|_Y)$ and $\mathrm{CH}_k(U) \to \mathrm{CH}_K(U \times \mathbb{A}^r)$ are surjections, it follows that π^* is a surjection, and the former map is a surjection by the inductive hypothesis. Therefore, we need only show that the rightmost arrow factors, e.g, we have reduced the proof the assertion to the case of $X = \operatorname{Spec} R$ affine, and $E = X \times \mathbb{A}^r$ is the trivial bundle (by choice of U above). Moreover, we can factor π as a series of projections:

$$\pi : E = X \times \mathbb{A}^r \to X \times \mathbb{A}^{r-1} \to \cdots \to X \times \mathbb{A}^1 \to X$$

Therefore, we may assume that $r = 1$ by treating each locally free sheaf in the chain as a rank 1 free sheaf on the scheme below, and we have that

$$E = X \times \mathbb{A}^1 = \operatorname{Spec} R \otimes_k \operatorname{Spec} k[t] = \operatorname{Spec} R[t]$$

We are then tasked with showing that $\pi^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+1}(X \times \mathbb{A}^1)$ is surjective; to that end, let $V \subset X \times \mathbb{A}^1$ be a $(k+1)$-dimensional subvariety, and let $W = \overline{\pi(V)}$. If $\dim W = k$, then $V = W \times \mathbb{A}^1$ since π is a projection, so $[V] = \pi^*[W]$. If $\dim W = k+1$ (this is the only other option) we must show that $[V]$ is in the image of the induced pullback map $\mathrm{CH}_k(W) \to \mathrm{CH}_{k+1}(W \times \mathbb{A}^1)$. We can therefore assume that $W = X$, let $I(V)$ be the vanishing ideal of V, and consider the related ideal

$$I(V) \otimes_R K \subset K[t]$$

where K is the fraction field of R. This ideal is not the unit ideal, as otherwise $V = W \times \mathbb{A}^1$ which we handled above. Since $K[t]$ is a PID, $I(V) \otimes_R K[t] = (\varphi)$ for some $\varphi \in K[t]$. Then the divisor of φ (taken as a function on $X \times \mathbb{A}^1$) is $[V]$ by construction, up to terms of the form $\pi^*[W_i]$ for $W_i \subset X$ corresponding to tensoring with K, from which the result follows. ■

Note that this result gives an easy alternative derivation of the Chow ring of affine space by showing that all subvarieties are rationally equivalent to zero. First, we have that $\mathrm{CH}_0(\mathbb{A}^n) = 0$ for all n; to see this, for any $x \in \mathbb{A}^n$, pick a line $L \cong \mathbb{A}^1 \subseteq \mathbb{A}^n$ through x and a function on L vanishing (only) on x. Then, by the above result, $\mathrm{CH}_0(\mathbb{A}^{n-k}) \to \mathrm{CH}_k(\mathbb{A}^n)$ is surjective for all $k < n$, from which the result follows.

IV. COUNTING CURVES

With these results, we are already able to answer some nontrivial enumerative problems via parameter spaces. First note that the set of all curves of degree n in $\mathbb{P}^2$ can be parametrized by $\mathbb{P}^{\binom{n+2}{2}-1}$, since such a curve is a homogeneous polynomial in three variables, and there are $\binom{n+2}{2}$ solutions to $a + b + c = n$ (each solution corresponding to a term with a coefficient determined up to a constant overall scale). In particular, the parameter space for lines is $\mathbb{P}^2$, for conics, $\mathbb{P}^5$, for cubics, $\mathbb{P}^9$.

Our first example provides an archetypal use of pullbacks to calculate degrees, which will be central to the problems that follow.

Proposition 21 *Let $\Sigma_{m,n}$ be the Segre variety, defined by the embedding $\sigma_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{(m+1)(n+1)-1}$ given by*

$$([X_0, \cdots, X_m], [Y_0, \cdots, Y_n]) \mapsto [X_0Y_0, \cdots, X_iY_j, \cdots X_mY_n]$$

Then

$$\deg \Sigma_{m,n} = \deg(\alpha + \beta)^{m+n} = \binom{m+n}{n}$$

where α and β are the hyperplane classes in $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively.

Proof: By definition, the degree of $\Sigma_{m,n}$ is the number of points in which it meets the intersection of $m+n$ general hypersurfaces in $\mathbb{P}^{(m+1)(n+1)-1}$; since $\sigma_{m,n}$ is an embedding we may compute this number by pulling back these hypersurfaces to $\mathbb{P}^m \times \mathbb{P}^n$. Thus $\deg \Sigma_{m,n} = \deg(\alpha+\beta)^{m+n}$, since the class of a hyperplane in the codomain pulls back to $\alpha+\beta$, and because the intersection with $\Sigma_{m,n}$ is taken for granted in the domain since $\sigma_{m,n}$ is an embedding. In $\mathrm{CH}(\mathbb{P}^m \times \mathbb{P}^n) = \mathbb{Z}[\alpha, \beta]/(\alpha^{m+1}, \beta^{n+1})$, $(\alpha+\beta)^{m+n} = \binom{m+n}{n}\alpha^m\beta^n$ since every other term vanishes by the quotient, from which the result follows. ■

Proposition 22 *Let $F_0, F_1, F_2 \in k[x, y, z]$ be general homogeneous cubic polynomials. Up to scalars, 21 linear combinations $t_0F_0 + t_1F_1 + t_2F_2$ factor as a product of a linear and a quadratic polynomial.*

Proof: Let $\Gamma \subset \mathbb{P}^9$ be the closure of the locus of cubics consisting of a conic and a transverse line. Γ is the image of the map

$$\tau : \mathbb{P}^2 \times \mathbb{P}^5 \to \mathbb{P}^9$$

sending a line and a conic to their product. The coefficients of the product are bilinear in the coefficients of the factors, so the pullback of the hyperplane class $\zeta \in \mathrm{CH}^1(\mathbb{P}^9)$ is

$$\tau^*(\zeta) = \alpha + \beta$$

where α and β are the pullbacks (via projections) to $\mathbb{P}^2 \times \mathbb{P}^5$ of the hyperplane classes on $\mathbb{P}^2$ and $\mathbb{P}^5$ respectively. Since polynomials over a field factor uniquely, τ is birational onto its image (and therefore we are not overcounting); therefore,

$$\deg \Gamma = \deg(\tau^*(\zeta)^7) = \deg(\alpha + \beta)^7 = \binom{7}{2} = 21$$

The first and final equalities follow from the logic in Proposition 21.

The reason that $\deg \Gamma$ counts what we want it to here is that F_0, F_1, and F_2 are points in our parameter space $\mathbb{P}^9$, so $t_0F_0 + t_1F_1 + t_2F_2$ forms a copy of $\mathbb{P}^2$ in $\mathbb{P}^9$ parametrized by $[t_0, t_1, t_2]$, and $\dim \Gamma = 7$, so $\deg \Gamma$ is the number of intersections between this plane of curves and Γ by the definition of degree.

Note that, geometrically, we are counting the number of degenerate cubics which factor as a conic and a transverse line. Further degeneracies (e.g a conic with a tangent line, or three lines) are excluded from this count by assumption of generic transversality for the pullback to exist. ■

Proposition 23 *Let $F_0, F_1, F_2, F_3 \in k[x, y, z]$ be general homogeneous cubic polynomials. Up to scalars,* 15 *linear combinations $t_0F_0 + t_1F_1 + t_2F_2 + t_3F_3$ factor as a product of three distinct linear polynomials.*

Proof: Proceeding as above, let $\Sigma \subset \mathbb{P}^9$ be the closure of the locus of such cubics which factor as three lines meeting transversely; geometrically, this is the set of "triangles," a very degenerate class of cubics. By Bertini's theorem, and the arguments made above, this is the degree of Σ.

Σ is the image of the map

$$\mu : \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^9$$

that sends three lines to their product, e.g $([L_1], [L_2], [L_3]) \mapsto [L_1L_2L_3]$. This map is no longer birational, but generically six-to-one, so we can calculate

$$\deg \Sigma = \frac{1}{6} \deg((\alpha + \beta + \gamma)^6) = \frac{1}{6} \binom{6}{2, 2, 2} = 15$$

where α, β, and γ are the pullbacks via projection of hyperplane classes in successive factors of $\mathbb{P}^2$. ■