UT Austin **Spring 2024**

Corks

Abhishek Shivkumar

Introduction

Exotic pairs — pairs of homeomorphic but not diffeomorphic manifolds — in dimension four are controlled by certain contractible 4-manifolds equipped with diffeomorphisms of their boundary known as *corks*. More precisely, any exotic pair of four-dimensional manifolds are related by cutting out a contractible submanifold and gluing it back in with a twist. The proof of this result is dependent on the topological h-cobordism theorem in dimension five due to Freedman[1] together with the *failure* of its smooth counterpart. Our goal in these notes is to provide a relatively self contained proof of this result.

[1] Freedman, *The topology of four-dimensional manifolds*.

The proof strategy we follow is due to Kirby,[2] who reformulates and consolidates the original proofs of Curtis, Freedman, Hsiang, Stong,[3] and, separately, Matveyev,[4] and Bižaca[5]. The purpose of these notes is mainly to clarify my own understanding of the ideas involved and to force myself to learn Inkscape. The proofs presented are not original, although mistakes therein may be.

[2] Kirby, *Akbulut's corks and h-cobordisms of smooth, simply connected 4-manifolds*.

[3] Curtis, Freedman, Hsiang, and Stong, *A decomposition theorem for h-cobordant smooth simply-connected compact 4-manifolds*.

[4] Matveyev, *A decomposition of smooth simply-connected h-cobordant 4-manifolds*.

[5] Although Kirby's paper and others in the literature attribute a part of this result to him, I can't find a published paper attributed to Bižaca on this topic.

We use $\underset{\text{top}}{\cong}$ to denote homeomorphisms and $\underset{\text{sm}}{\cong}$ to denote diffeomorphisms.

Freedmanomicon

Our starting point is the famous pair of results due to Freedman in the topological category:

Theorem 2.1: Topological $5D$ h-Cobordism Theorem[6]

A compact 1-connected smooth 5-dimensional h-cobordism $(W; M, M')$ (which is a product over the possibly empty boundary ∂M) is topologically a product, i.e, $W \underset{\text{top}}{\cong} M \times [0, 1]$.

[6] Freedman, *The topology of four-dimensional manifolds*, Theorem 1.3

Freedman adds that, moreover, the product structure on W is smooth over the complement of a flat 4-cell in M, i.e, any obstruction to W being *diffeomorphic* to a product can be localized to essentially a point.

Note that there is a version of the above theorem that omits the smoothness requirement due to Quinn[7] that relies on his (and Freedman's) theory of topological handle decompositions.

[7] Quinn, *Ends of maps. III: Dimensions 4 and 5*, Proposition 2.7.2.

Theorem 2.2: Topological 4D Poincaré "Conjecture"[8]

If M is a topological 4-manifold homotopy equivalent to S^4, then $M \underset{\text{top}}{\cong} S^4$.

[8] Freedman, *The topology of four-dimensional manifolds*, Theorem 1.6

For our discussion, we will need one more result due to Freedman, although the proof strategy we use below is taken from a paper of Gompf.[9]

[9] Gompf, *Infinite order corks via handle diagrams*.

Lemma 2.3: Freedman

Let $f : \partial X \xrightarrow{\sim} \partial X$ be a diffeomorphism with X a compact, contractible 4-manifold. Then f extends to a *homeomorphism* $F : X \xrightarrow{\sim} X$ of X (i.e $F|_{\partial X} = f$).

Proof : Let W denote the *twisted double* of X along f, that is, the closed 4-manifold obtained by gluing X to itself along ∂X via f. $\pi_1(W) = \pi_1(X) *_{\pi_1(\partial X)} \pi_1(X) = 1$ by van Kampen's theorem, and $H_i(W) = H_{i-1}(\partial X)$ by Mayer-Vietoris (except for $i = 0$), and the boundaries of compact contractible n-manifolds are homology $(n-1)$-spheres (see, e.g., Hatcher,[10] or apply Lefschetz duality together with the long exact sequence of a pair), so $H_*(W) = H_*(S^4)$. Then, using a formulation of Whitehead's theorem — precisely, that, $f : X \to Y$ between simply-connected CW complexes is a homotopy equivalence if $f_* : H_i(X) \to H_i(Y)$ is an isomorphism for all i (see, e.g., Hatcher[11]) — we know that W is homotopy equivalent to S^4.

[10] Hatcher, *Algebraic Topology*, Exercise 3.3.33.

[11] Ibid., Corollary 4.33.

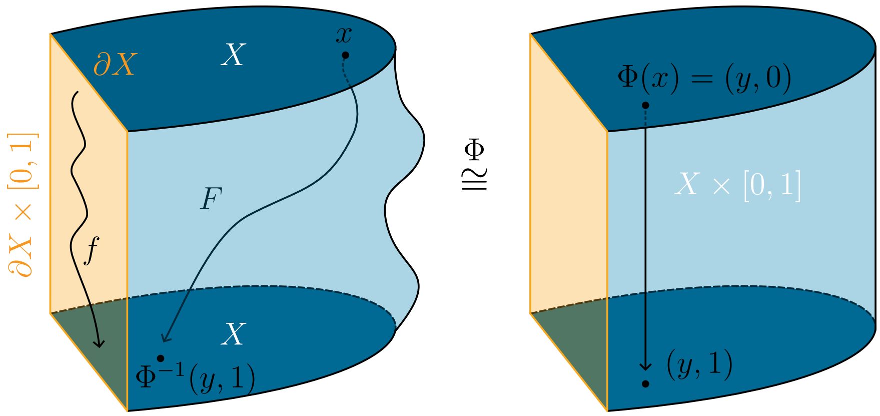
To produce a degree one map from W to S^4 in order to apply Whitehead's theorem, employ the standard construction of picking a small ball B in W and take the quotient map $W \to W/(W \setminus \text{int}B) = S^4$. This construction works in all dimensions.

Now, applying the topological Poincaré conjecture (see Corollary 2.2), we have that $W \underset{\text{top}}{\cong} S^4$ and therefore W bounds D^5. We can decompose W as

$$W = X \cup_{\partial X \times \{0\}} \partial X \times [0,1] \cup_{\partial X \times \{1\}} X$$

by construction and thereby view D^5 as a relative h-cobordism of pairs from $(X, \partial X)$ to itself (see Figure 1 below), which, restricted to $\partial X \times [0,1]$ is the mapping cylinder of the diffeomorphism f. By the rel-∂ version of the topological h-cobordism theorem (see Theorem 2.1), we have a homeomorphism $\Phi : D^5 \xrightarrow{\sim} X \times [0,1]$. The idea to produce F is to "project" $X \times \{0\}$ to $X \times \{1\}$ along the h-cobordism:

Figure 1: The h-cobordism induced by gluing along f, once "straightened out" by an application of the h-cobordism theorem, gives us the desired extension F.



By the construction of Φ as arising from an h-cobordism, it maps the top copy of X to $X \times \{0\}$ and the bottom copy to $X \times \{1\}$. Thus, to see where the point x in the top copy of X goes under the "flow" (scare quotes to avoid making this more precise) of the h-cobordism, we map it over to $(y, 0) = \Phi(x) \in X \times \{0\}$, whose image under the flow of the product h-cobordism is $(y, 1)$, and pull this back to the bottom copy of X as $F(x) := \Phi^{-1}(y, 1)$.

This is a homeomorphism (since Φ is a homeomorphism) which extends f by construction, so we are done. Note that we used the fact that f is a diffeomorphism (as opposed to a homeomorphism) in order to guarantee a smooth structure on W and therefore on the h-cobordism in order to apply Theorem 2.1. ■

The key point is that the diffeomorphism f is only guaranteed to extend as a homeomorphism over the larger manifold X, and it turns out that, in general, an extension to a self-diffeomorphism of X is not possible.

Corks on the Cob

Depending on who your friends are, corks as I've defined them may be called *loose* or *relative* corks.

Definition 3.1: Corks

A *cork* is a contractible 4-manifold C together with a self diffeomorphism (sometimes taken to be an involution) τ of its boundary ∂C which does not extend over C as a diffeomorphism.

Now, we are ready to state the main result discussed in this note:

Theorem 3.2: CFHSM[12]

Let M^5 be a smooth 5-dimensional h-cobordism between simply-connected closed 4-manifolds, M_0 and M_1. Then there exists a sub-h-cobordism $A^5 \subseteq M^5$ between $A_0 \subseteq M_0$ and $A_1 \subseteq M_1$ such

[12] Curtis, Freedman, Hsiang, and Stong, *A decomposition theorem for h-cobordant smooth simply-connected compact 4-manifolds*; Matveyev, *A decomposition of smooth simply-connected h-cobordant 4-manifolds*

that the A_i and A itself are compact contractible submanifolds and such that $M \setminus \text{int}A$ is diffeomorphic to a product, i.e, diffeomorphic to $(M_0 \setminus \text{int}A_0) \times [0,1]$.

The proof of this result follows by examining an arbitrary such h-cobordism and judiciously manipulating its handle structure, and gives as a corollary our main result:

Corollary 3.3: Cork Theorem

Let M_0 and M_1 be smooth, simply-connected, homeomorphic 4-manifolds which are not diffeomorphic, i.e, an exotic pair. Then there exists a cork $C \subseteq M_0$ such that M_1 is diffeomorphic to $(M_0 \setminus \text{int}\, C) \cup_\tau C$.

In other words, all exotic pairs are related by a *cork twist*, which is the operation of cutting out and gluing back in a cork along some automorphism of its boundary.

The proof of this corollary requires a few addenda and is found after the proof of the main theorem.

The proof of the main theorem will proceed as follows:

1. First, we pick a handle structure on the h-cobordism M and look at its middle slice $M_{\frac{1}{2}}$, where the 2-handle cocores and 3-handle cores meet to witness the fact their algebraic cancellation

2. Using these cores and cocores we build up a handle structure on $M_{\frac{1}{2}}$ that reflects the topology of M

3. We slide handles and potentially introduce new cancelling 2/3 handle pairs to $M_{\frac{1}{2}}$ in order to make the induced presentation of π_1 into the trivial presentation of the trivial group

4. Now, with our cleverly chosen handle structure on $M_{\frac{1}{2}}$ whose 2-handles are neatly sorted by the role they play, we are able to identify a contractible sub-h-cobordism of M which contains all of the handles of M, from which the result will follow

Proof : Pick a handle structure on M, induced by some Morse function $f : M \to [0,1]$, which, we can take to have no 0, 1, 4, or 5-handles.[13] If the 2 and 3-handles cancel geometrically, then M is a trivial h-cobordism, i.e, diffeomorphic to a cylinder $M_0 \times [0,1]$. In general, this is not true, however, since nontrivial smooth h-cobordisms exist in dimension four,[14] however, we do know that the 2 and 3-handles cancel *algebraically* by the fact that M is an h-cobordism (indeed, a non-cancelling (say) 2-handle in M would generate nontrivial $H_2(M, M_0)$ in the Morse complex). In particular, there are an equal number of 2 and 3-handles by this condition.

[13] Gompf and Stipsicz, *4-manifolds and Kirby calculus*, Proposition 9.2.3.
The 0-handles can be cancelled by 1-handles since M is connected, and, similarly, the 1-handles can be cancelled by 2-handles essentially due to the fact that M is simply-connected, though there are some details to work out (and the dimension being sufficiently large is important). Turning M upside down and repeating the argument eliminates the 4 and 5-handles. All this is done at the cost of new 2 and 3-handles.

[14] Donaldson, *Irrationality and the h-cobordism conjecture*.

Let $M_{\frac{1}{2}} := f^{-1}\left(\frac{1}{2}\right)$ denote the middle level of M, and assume that the 2-handles are attached below and the 3-handles above of $M_{\frac{1}{2}}$. Since we can build M without use of any 1-handles, $M_{\frac{1}{2}}$ is simply-connected. The goal is to produce a handle structure on $M_{\frac{1}{2}}$ that illuminates the structure of M, and the contractible manifold A that we will eventually produce will depend on the handle structure that we describe.

For those more familiar with the language of Morse theory, the cocore is the same as the ascending disc, and the core is the same as the descending disc.

We may arrange for the cocores of our 2-handles and the cores of our 3-handles to intersect $M_{\frac{1}{2}}$ in smoothly embedded 2-spheres (which we may take to be the belt spheres and attaching spheres respectively of the 2 and 3-handles). Call the 2-handle spheres $S_{0,i}$ and the 3-handle spheres $S_{1,i}$ for $i \in \{1, \cdots, n\}$.

Begin by choosing a basepoint $*$ in $M_{\frac{1}{2}}$, and basepoints $*_{k,i}$ for each sphere $S_{k,i}$. Running disjoint arcs from $*$ to each $*_{k,i}$ we obtain a tree in $M_{\frac{1}{2}}$, a tubular neighborhood of which is a copy of D^4, which we will take as a 0-handle.

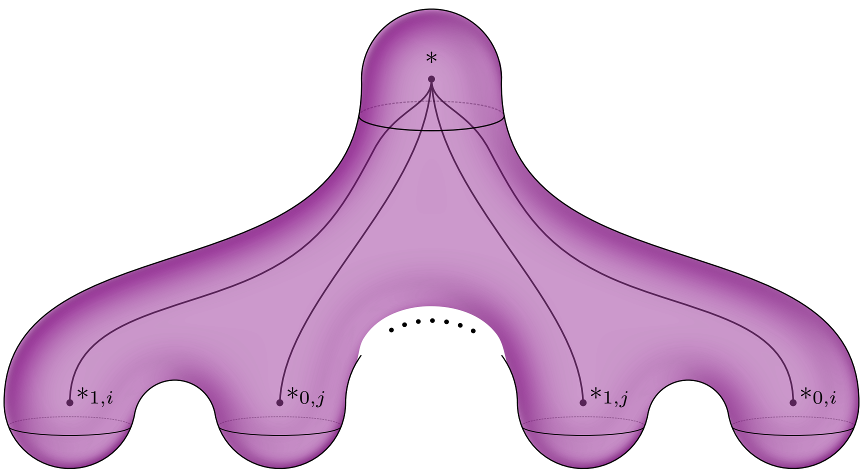


Figure 2: The neighborhood of a tree is a ball.

Next, we run disjoint arcs from $*_{k,i}$ along $S_{k,i}$ to each point of intersection with the $S_{1-k,j}$; note that i is not necessarily equal to j, since there may be algebraically cancelling but nonetheless essential intersections with other spheres. These arcs connect at the points of intersection and therefore can be paired up to form arcs that start and end on our 0-handle, and tubular neighborhoods of these arcs will be 1-handles for our handle decomposition.

Lastly, each $S_{k,i}$ minus a neighborhood of the tree of arcs along its surface forms a copy of D^2, which, when thickened up appropriately, becomes a 0-framed 2-handle labeled $H_{k,i}$ attached to the 0 and 1-handles. The 0-framing is due to the fact that the $S_{k,i}$ come with a trivial normal bundle $S^2 \times D^2$ inside of $M_{\frac{1}{2}}$ (and there is only one framing of this bundle corresponding to $\pi_2(\mathrm{SO}(2)) = 1$) thought of as the attaching region for the 3-handles and viewing the 2-handles as upside down 3-handles. Puncturing $S^2 \times D^2$ to $D^2 \times D^2$ then clearly results in a 0-framed 2-handle.

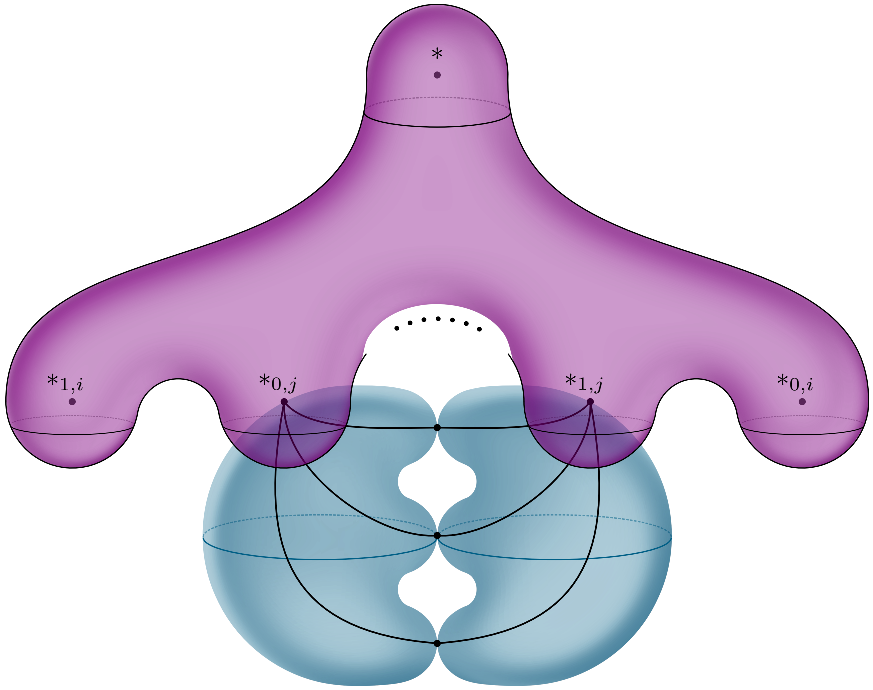


Figure 3: Arcs along the $S_{k,i}$ running from the $*_{k,i}$ to the points of intersection. Note that, for visual simplicity, we have excluded from this cartoon intersections between, say, $S_{0,j}$ and $S_{1,i}$ for $i \neq j$. We know that these intersections will algebraically cancel out, but we cannot in general rule them out as geometric data.

We can draw a Kirby diagram for the handles we have so far. The 2-handle attaching circles are all (0-framed) unknots by construction. We can show this by exhibiting discs they bound along the boundary of the 1-handles; for a given sphere, the disk is given by considering each arc in a given tree individually, taking a small two-dimensional neighborhood of it along the sphere (which sits in the three-dimensional boundary), and joining the resulting discs along their common base.

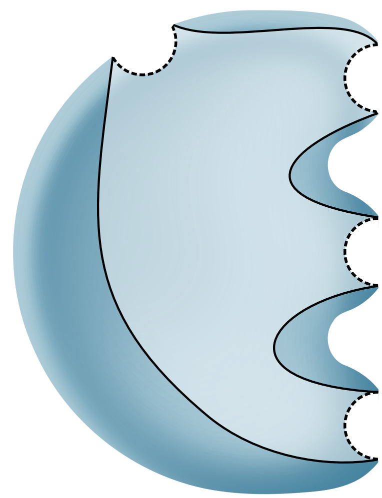


Figure 4: A sphere minus a tree of arcs gives a disc.

At each geometric intersection point of $S_{0,i}$ with $S_{1,i}$, the attaching circles for $H_{0,i}$ and $H_{1,i}$ clasp one another, with the sign of the clasp depending on the sign of the crossing (once the attaching circles are appropriately oriented). To see this, note that, unlike in Figure 3, where we are limited to a three-dimensional drawing, the spheres intersect *transversely* and the neighborhood of such a transverse intersection (e.g. the neighborhood of two D^2s intersecting transversely in D^4) is known to have boundary a Hopf link (see, e.g., Behrens et al.[15]). Since $S_{k,i}$ does not intersect $S_{k,j}$ for $i \neq j$, the $H_{0,i}$ and $H_{1,i}$ are each attached along n-component unlinks.

[15] Behrens, Kalmár, Kim, Powell, and Ray, *The disc embedding theorem*, Figure 1.5.

By construction, the clasp region of each attaching circle is attached along a 1-handle, hence the position of the dotted circles in the diagram. As drawn, we may decide that the $H_{0,i}$ do not pass over any 1-handles, and have the $H_{1,i}$ pass over and back resulting in a trivial relator $x_i x_i^{-1}$ in the fundamental group of the manifold we have built so far.

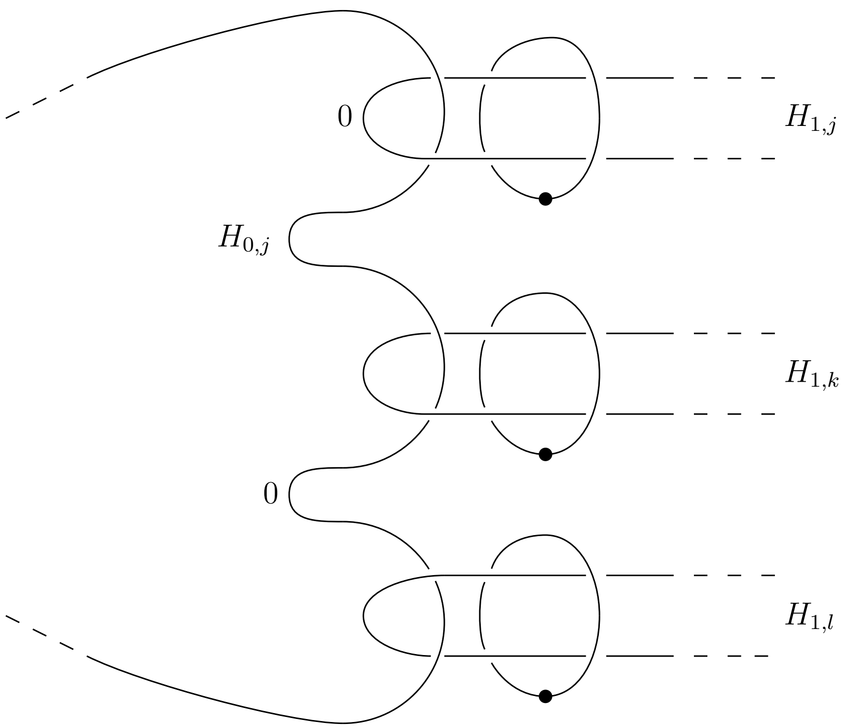


Figure 5: Figure adapted from Kirby's notes. Note that it is not necessary for $j = k = l$; while the only algebraic intersection is between $S_{0,j}$ and $S_{1,j}$, $S_{0,j}$ could have a cancelling pair of intersections with $S_{1,k}$. Note also that $S_{1,i}$ and $S_{1,j}$ have empty *geometric* intersection since we can assume that the 3-handles have disjoint attaching spheres (with the same true for the $S_{0,i}$ *mutatis mutandis*).

The point of the handles we have already chosen is to reflect in $M_{\frac{1}{2}}$ the structure of M; more specifically, to preserve the $H_{k,i}$ for a later step in the construction, where they will be thickened up to be algebraically cancelled by the 2 and 3-handles of the h-cobordism in order to form our desired contractible sub-h-cobordism that contains all of the handles of M. The 0 and 1-handles are necessary in order to attach the $H_{k,i}$.

The next step of the proof requires us to extend what we have constructed to a complete handlebody structure on $M_{\frac{1}{2}}$. This entails adding more 1 and 2-handles along with some 3-handles. Our now complete handle decomposition of $M_{\frac{1}{2}}$ has 2-handles algebraically cancelling our 1-handles so that we have $\pi_1(M_{\frac{1}{2}}) = 1$, as well as new 2-handles to potentially generate $H_2(M_{\frac{1}{2}})$. Any 3-handles must of course be algebraically cancelled as well.

For any smooth manifold M^n we can start with any random collection of handles and extend this to a handlebody structure on M^n by cancelling all the existing handles and then starting from scratch. In our case, our initial handlebody structure on $M_{\frac{1}{2}}$ has 1-handles that will need to be cancelled since we know that it is simply connected.

Since the $H_{k,i}$ give trivial relators in π_1, it follows that our new 2-handles must *homtopically* kill the 1-handles, old and new. Note that homotopic cancellation (meaning in π_1) is stronger than what we have thus far referred to as algebraic (really homological) cancellation. The difference is non-commutativity; for example, a 2-handle corresponding to the relator $x_1x_2x_1x_2^{-1}x_1^{-1}$ does not homotopically cancel the 1-handle corresponding to x_1, but it does algebraically cancel it in H_1.

Our handle decomposition gives us a possibly complicated presentation of the trivial group (where 1-handles correspond to generators and 2-handles to relators); our goal is to turn this presentation into the trivial one (where the relators are of the form x_i for a 1-handle generator x_i). The purpose of doing so is to distinguish the three different types of 2-handle: those which generate $H_2(M_{\frac{1}{2}})$, those that cancel the 1-handles, and those that cancel the 3-handles. In practice, these handles do not automatically fall into such neat disjoint classes, but we can try to arrange for them to do so.

More specifically, in building our contractible manifold A, which will include some of the 1 and 2-handles of $M_{\frac{1}{2}}$ appropriately thickened, we will need to be able to identify the 2-handles that cancel the 1-handles, and with an arbitrary presentation of the trivial group given by our handle decomposition, this would be difficult. If we just took all the 2-handles which run over a 1-handle, we may end up with some which give rise to trivial relators such as the $H_{1,i}$, and which, therefore, are nontrivial in second homology and obstruct the contractibility of A.

To the end of simplifying the interactions of our 1 and 2-handles, we will use the following result:

Lemma 3.4: π_1 Presentations = Handle Presentations[16]

Given a 4-dimensional relative handlebody X^4 with connected boundary and $\pi_1(X) = 1$, for every 1-handle h of this decomposition, one can introduce a cancelling 2/3-handle pair such that the 2-handle cancels h *homotopically*. Moreover, one can arrange the attaching circles for the new 2-handles to represent the canonical basis for the free group $\pi_1(X_1)$ induced by the 1-handles (where X_1 denotes the union of 0 and 1-handles in X).

[16] Gompf and Stipsicz, *4-manifolds and Kirby calculus*, Lemma 9.2.17

Note that the introduction of cancelling 2/3-handle pairs is how we avoid difficulties related to the *Andrews-Curtis conjecture* in combinatorial group theory, which, here, amounts to the claim that we can arrange for our 1 and 2-handles to (algebraically) cancel using only handle slides (which, roughly, correspond to the *Andrews-Curtis moves* on a presentation of π_1). The Andrews-Curtis conjecture is widely believed to be false.

The ability to introduce cancelling 2/3-handle pairs enlarges our moveset on π_1 to the set of *Tietze transformations*, and it is known that any two finite presentations of a given group are related by a finite sequence of Tietze transformations.

Using this result, we can introduce new 2-handles that homotopically cancel the 1-handles and therefore kill π_1. Any other 2-handles which run over a 1-handle can be slid off of them along the new, cancelling 2-handles, so we may assume all other 2-handles are disjoint from the 1-handles.

Now, we are nearly done. Let $B_{\frac{1}{2}}$ be the manifold formed by the 0-handle, all of the 1-handles, and the 2-handles which kill the 1-handles. $B_{\frac{1}{2}}$ is contractible since all of its algebraic invariants necessarily vanish (see the statement of Whitehead's theorem in the proof of Lemma 2.3). Define $A_{\frac{1}{2}}$ as $B_{\frac{1}{2}}$ together with the $H_{k,i}$ ($A_{\frac{1}{2}}$ is not contractible), but none of the other 2-handles we added to $M_{\frac{1}{2}}$, and let A be $A_{\frac{1}{2}}$ thickened up by $[\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$ within the h-cobordism M, together with the 3-handles added below to the $S_{0,i}$ and above to the $S_{1,i}$. A is contractible since these 2 and 3-handles cancel the $H_{k,i}$ *geometrically* by construction — that is, the belt spheres of the $H_{k,i}$ and the attaching spheres of the 3-handles intersect transversely in a single point (this point is $(0, 0)$ in the 2-handle $D^2 \times D^3$) — and the A_i are contractible since A itself is an h-cobordism (as we will argue below). Having isolated all of the 2 and 3-handles of our h-cobordism M to a contractible sub-h-cobordism A, we can conclude that $M \setminus \mathrm{int}A$ is a product.

Surgery on an embedded sphere $S^k \hookrightarrow M^n$ with trivial normal bundle first cuts out the sphere, then glues in $D^{k+1} \times S^{n-k-1}$ along a tubular neighborhood of S^k in such a way that the factors are switched; explicitly, if we have $f : S^k \times D^{n-k} \hookrightarrow M^n$, then the gluing map identifies $f(\theta, r\varphi)$ with $(r\theta, \varphi)$ with $0 \leq r \leq 1$, and the surgered manifold is $(M \setminus f(S^k \times \{0\})) \cup D^{k+1} \times S^{n-k-1}$.

Note that attaching a five-dimensional 3-handle has the effect of surgering the attaching sphere in the four-dimensional boundary, i.e, replacing $S^2 \times D^2$ with $D^3 \times S^1$. Note also that attaching a 0-framed 2-handle to a 4-manifold yields $S^2 \times D^2$ and attaching a 1-handle yields $S^1 \times D^3$, so

Attaching a k-handle along an embedded S^{k-1} in the boundary of some M^n gives a cobordism from M^n to the result of surgery on this S^{k-1}, and one can show that two manifolds are cobordant iff one can produce one from the other via sequence of surgeries.

attaching a five-dimensional 3-handle to the $S^2 \times D^2$ formed by the slice disk and core disk for the 0-framed 2-handle turns the attaching circle into the dotted circle for a 1-handle. Thus, the M_i have a handle decomposition given by the constructed handle decomposition for $M_{\frac{1}{2}}$ with the $H_{k,i}$ dotted instead of 0-framed.

To see that A itself is an h-cobordism, we need to see that it is an H-cobordism — a homology cobordism, i.e, the inclusion of the boundary components induces an isomorphism on homology — and that the inclusion preserves π_1 (this suffices by Whitehead's theorem as above). For the former claim, which is equivalent via the long exact sequence for relative homology to the claim that $H_*(A, A_i) = 0$. Since $H_*(A) = H_*(\bullet)$, this in turn amounts to the claim that $H_*(A_0) = H_*(\bullet)$, again by the long exact sequence. Note that $A_{\frac{1}{2}}$ is by construction $B_{\frac{1}{2}}$ (which is contractible) together with the $H_{k,i}$ and A_0 is $A_{\frac{1}{2}}$ with the attaching circles for $H_{0,i}$ now dotted. The $H_{k,i}$ generate $H_2(A_{\frac{1}{2}})$ (with all of its other homology groups trivial) but by construction the attaching circles for the $H_{0,i}$ have linking number 1 with the attaching circles for the $H_{1,i}$, so the new 1-handles are homologically cancelled by existing 2-handles, and A_0 has the homology of a point, as claimed (with A_1 handled symmetrically).

Curtis et. al. show that A is an h-cobordism by finding immersed Whitney discs for the extra $\pm$ cancelling intersections between $S_{0,i}$ and $S_{1,i}$, and these Whitney discs in turn give rise to immersed geometric duals for the $S_{k,i}$. They then use the fact that surgery on spheres with immersed geometric duals does not change π_1.

This all relies on results of Casson through techniques that are unrelated to our discussion, so we choose instead to perturb the initial construction, using an argument originally supplied by Matveyev to prove that $A_0 \times I \underset{\text{sm}}{\cong} B^5$.

To see that $\pi_1(A_0) = \pi_1(A_{\frac{1}{2}}) = \pi_1(A) = 1$ (with A_1 handled symmetrically as above), we need to return to the start and modify our construction slightly. Since A_0 is the same as $A_{\frac{1}{2}}$ with dotted circles in place of the $H_{0,i}$ we have n new generators $x_1, \cdots, x_n$ for $\pi_1(A_0)$ and n new relators $r_1, \cdots, r_n$ given by the $H_{1,i}$. We want *homotopic* cancellation of these, stronger than the homological cancellation that we used above. As noted above, the archetypal obstruction to upgrading our homological cancellation to homotopic cancellation is a 2-handle relator of the form $x_1x_2x_1x_2^{-1}x_1^{-1}$ which does not cancel out to x_1 (though it does in homology).

To remedy this, we return to our original construction and more carefully consider the tree of arcs which gave rise to the $H_{k,i}$; to wit, we first run arcs from $S_{1,i}$ to all points of intersection with $S_{0,1}$, then $S_{0,2}$, etc. up to $S_{0,n}$, and it follows that $r_i = w_1w_2\cdots w_n$ where w_j is a word in x_j and x_j^{-1} with exponent sum equal to 0 for $j \neq i$ and 1 for $j = i$. Thus, we can arrange for $\langle x_1, \cdots, x_n | r_1, \cdots, r_n\rangle$ to be the trivial presentation of the trivial group, and we have the desired homotopic cancellation of 1 and 2-handles. Thus, $\pi_1(A_0) = 1$, and, consequently, A is an h-cobordism, and the A_i are therefore contractible.

$\pi_1(A_0)$) has other generators and relators of course, corresponding to the 1 and 2-handles of $A_{\frac{1}{2}}$ that were unaffected by surgery on the $H_{0,i}$, but these should homotopically cancel each other as they did in $A_{\frac{1}{2}}$.

There is one slight cause for concern, which is that some of these 2-handles may have linked the $H_{0,i}$ (which are now 1-handles) so the corresponding relator is different in $\pi_1(A_0)$ than it was in $\pi_1(A_{\frac{1}{2}})$. I.e., a relator $r = x$ in $A_{\frac{1}{2}}$ might become $r = x_ix_jxx_k^{-1}$ in A_0. However the x_i are homotopically trivial (via r_i) by our construction in this proof so the other 1 and 2-handles in A_0 still homotopically cancel as desired.

Improvements

There are a few other things we can say about A that follow either by inspection or by slightly perturbing our construction (note that we made no claim of uniqueness above):

Addendum 3.5

A is diffeomorphic to B^5 (though, of course, not preserving the structure of the h-cobordism).

PROOF : We have that $A \underset{\text{sm}}{\cong} B_{\frac{1}{2}} \times I$ since the 3-handles of the h-cobordism cancel the $H_{k,i}$ geometrically as noted above, and, $B_{\frac{1}{2}} \times I$ is diffeomorphic to B^5 since the 1 and 2-handles of $B_{\frac{1}{2}}$ *homotopically* cancel one another (see Lemma 3.4), and with an extra dimension of room in $B_{\frac{1}{2}} \times I$, we can promote this homotopy to isotopy[17] and these handles are in *geometric* cancellation. ■

[17] Gompf and Stipsicz, *4-manifolds and Kirby calculus*, Example 4.1.3.

Addendum 3.6

$A_0 \times I$ and $A_1 \times I$ are diffeomorphic to B^5.

The idea of non-diffeomorphic manifolds becoming diffeomorphic after $\times I$ is not exclusively a high(er)-dimensional phenomenon; note, for example, that the punctured torus and the pair of pants become diffeomorphic after $\times I$ (or more generally, the punctured genus g surface and the disk with $2g$ punctures).

PROOF : By construction, we know that that the 1-handles of A_0 are homotopically cancelled by its 2-handles, so since $r_i = w_1 w_2 \cdots w_n$ is homotopic to x_i, and homotopy implies isotopy in the now four-dimensional boundary, the 1 and 2-handles cancel and $A_0 \times I$ has an empty handle decomposition. ■

Note that, to Curtis et. al., $A_i \times I \underset{\text{sm}}{\cong} B^5$ is not automatic but requires special consideration in the construction of $A_{\frac{1}{2}}$ since they achieve simple-connectedness of $\pi_1(A_i)$ in a different manner.

Addendum 3.7

We can construct A so that A_0 and A_1 are diffeomorphic by a map that is an involution when restricted to the boundary.

PROOF : Our ultimate goal is to replace A with $A \cup A^{-1}$ as in Figure 6, and the diffeomorphism between the ends of of $A \cup A^{-1}$ will be given by the obvious involution interchanging A_0 and A_1. We can achieve this by enlarging A, as in the following figure (where we set $N := M \setminus \mathrm{int}A$, $N_0 := M_0 \setminus \mathrm{int}A_0$ etc.):

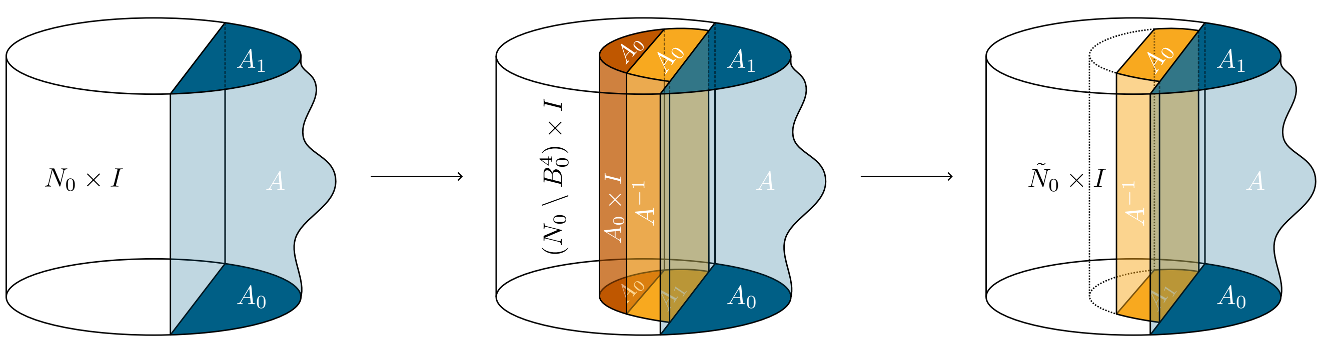


Figure 6: Enlarging A to obtain an involution on the boundary.

First, we find a 4-ball $B_0^4 \subset M_0$ whose boundary meets ∂A_0 in a 3-ball, and which is otherwise contained in N_0. Since N is a product cobordism, there is a $B^5 = B^4 \times I$ above B_0^4.

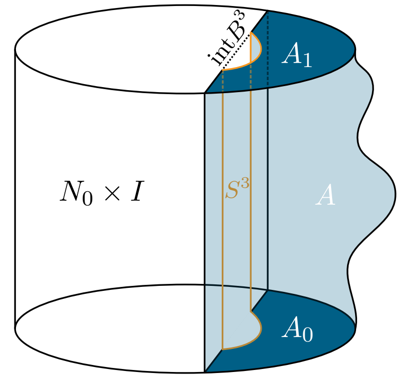


Figure 7: The effect of deleting an open 4-ball from ∂A (specifically, one which meets $\partial A_0 = \partial A_1$ in an open three-ball).

By Addendum 3.5, $A \underset{\text{sm}}{\cong} B^5$, hence $\partial A = A_0 \cup_\partial A_1 \underset{\text{sm}}{\cong} S^4$, so removing an open 4-ball from ∂A will give us a copy of B^4. Specifically, we want to remove an open 4-ball that meets $\partial A_0 = \partial A_1$ in an open three-ball, and identify the remaining (closed) four-ball with $B_0^4 = (A_0 \cup_\partial A_1)^\circ$ (see Figure 7).

By Addendum 3.6, $A_0 \times I$ can also be identified with B^5, so $A_0 \cup_\partial A_0 \underset{\text{sm}}{\cong} S^4$, and we can similarly puncture this S^4 to obtain a four-ball labeled $B_1^4 = (A_0 \cup_\partial A_0)^\circ$.

If we glue $A_0 \times I$ to A^{-1} appropriately, we can identify the resulting cobordism as $(B^4 \times I, B_0^4, B_1^4)$. Note that $\partial A_0 \underset{\text{sm}}{\cong} \partial A_1$; this is implicitly used above to decompose $\partial A = A_0 \cup_\partial A_1$, and follows from Theorem 2.1 together with the fact that there is no difference between $\underset{\text{top}}{\cong}$ and $\underset{\text{sm}}{\cong}$ for 3-manifolds. Using this fact, we can glue $A_0 \times I$ to A^{-1} along $(\partial A_0 \setminus \mathrm{int}B^3) \times I$. The bottom boundary of this cobordism is evidently $(A_0 \cup_\partial A_1)^\circ = B_0^4$ and the top is $(A_0 \cup_\partial A_0)^\circ = B_1^4$ as claimed.

Note that, despite appearances, it only makes sense to call the restriction to the boundary an involution since we already know that the boundaries are otherwise diffeomorphic, whereas this constructed "swapping" map is the only means we have of comparing the ends of A to each other. Saying anything about the order of a diffeomorphism between manifolds requires an alternative method of identifying these manifolds (something that serves the role of the identity map).

Then, as depicted in Figure 6, we may fold $A_0 \times I$ into the product side of the cobordism and the ends of the remaining contractible sub-h-cobordism $A \cup A^{-1}$ has ends which are diffeomorphic, and this diffeomorphism is an involution when restricted to the boundary. ■

With the addenda in hand, we are ready now to prove Corkollary 3.3:

Proof : M_0 and M_1 are evidently homotopy equivalent, so it follows by[18] that there exists a smooth h-cobordism M between them. Theorem 3.2 then supplies us with a contractible submanifold $C := A_0 \subseteq M_0$, which we claim is our cork.

[18] Wall, *On simply-connected 4-manifolds*, Theorem 2.

Our construction supplies us with two different (smooth) automorphisms of $\partial A_0 \underset{\text{sm}}{\cong} \partial A_1$. The first is the restriction to the boundary of the diffeomorphism $\Phi : M_0 \setminus \mathrm{int}A_0 \xrightarrow{\sim} M_1 \setminus \mathrm{int}A_1$ given by the product half of the h-cobordism, and the second is the restriction to the boundary of $\Psi : A_0 \xrightarrow{\sim} A_1$. Set $\varphi := \Phi|_\partial$ and $\psi := \Psi|_\partial$, and $\tau := \varphi^{-1} \circ \psi : \partial A_0 \to \partial A_0$. Then, we have the following sequence of diffeomorphisms:

$$M_1 = (M_1 \setminus \mathrm{int}A_1) \cup_{\mathrm{id}_{A_1}} A_1 \underset{\Phi^{-1}}{\cong} (M_0 \setminus \mathrm{int}A_0) \cup_{\varphi^{-1}} A_1 \underset{\Psi}{\cong} (M_0 \setminus \mathrm{int}A_0) \cup_\tau A_0$$

By Addendum 3.7, we can choose φ so that τ is an involution. If we can show that M_0 is not diffeomorphic to M_1, then we know that τ does not extend over A_i (this is frequently the hard part). ■

Further Results and Im~~prove~~ments

There are a few more improvements we can make, which we will recount but not prove. First, we can take $M \setminus A$ (where M is a general h-cobordism as above) to be simply connected[19] by even more carefully considering our handle decompositions when building A and carefully sliding 2-handles so that the corresponding relators are modified by products of commutators.

[19] Kirby, *Akbulut's corks and h-cobordisms of smooth, simply connected 4-manifolds*; Curtis, Freedman, Hsiang, and Stong, *A decomposition theorem for h-cobordant smooth simply-connected compact 4-manifolds*.

Another improvement that can be made is due to Akbulut and Matveyev,[20] who show that we can take the cork C to be a *compact Stein domain* (i.e with boundary), which is also sometimes called a *strictly pseudoconvex manifold*. Akbulut's personal definition of a cork requires the Stein condition, but it seems that most of the literature has moved away from this convention.

[20] Akbulut and Matveyev, *A Convex decomposition theorem for four-manifolds*.

C is *strictly pseudoconvex* if it admits a *strictly plurisubharmonic* Morse function whose maximum points coincide with ∂C. This term along with the notion of Stein domains and manifolds come from complex geometry, and it's never been clear to me why one should care about Stein corks.

Akbulut's Cork

The results we proved above are all classical, and proven with classical techniques, but in order to actually prove that a given contractible 4-manifold is a cork we need a way to obstruct diffeomorphism without obstructing homeomorphism, which typically means using gauge-theoretic invariants. An important early example, due to Akbulut, uses Donaldson invariants to distinguish the smooth structures on either side of A. Here we will briefly recount his construction apart from the application of Donaldson invariants.

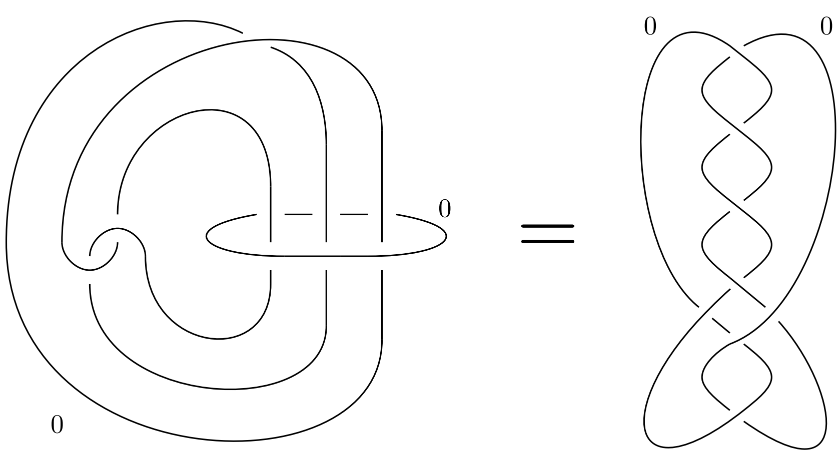


Figure 8: Figure adapted from Kirby's notes.

Akbulut's cork is one of the first explicit examples of a non-product h-cobordism, the construction of which is based on the above Kirby diagram. This diagram represents a manifold $A_{\frac{1}{2}}$ which is a homology $(S^2 \times S^2)^\circ$ since the algebraic linking number of the two components is one — even though, crucially, the disk bounded by one of the components of the link of unknots intersects the other component three times geometrically.

This "excess linking" exhibits the first required characteristic of corks. The linking number one is required for us to extract a contractible from our setup, and the extra geometric linking is required for the resulting contractible to be nontrivial, i.e, not $\underset{\text{sm}}{\cong} B^4$.

Because this link has a symmetric presentation as above right, there is a natural involution $\tau : \partial A_{\frac{1}{2}} \to \partial A_{\frac{1}{2}}$ induced by the 180° rotation about the z-axis that interchanges the two components of the link. To extend this to a diffeomorphism $\overline{\tau} : A_{\frac{1}{2}} \to A_{\frac{1}{2}}$ that switches the 2-handles we need only extend the initial automorphism of S^3 over our 0-handle B^4; note that the rotation is clearly smoothly isotopic to the identity, so we may smoothly extend over B^4 by taking the isotopy from the identity $f_t : S^3 \times I \to S^3$ as the "outer shell" of the extension, and fill in the rest with the identity map. This diffeomorphism $\overline{\tau}$ should evoke Addendum 3.7.

Explicitly, if $f : S^3 \to S^3$ is our initial rotation (or any diffeomorphism isotopic to the identity), then our extension to B^4 is given by

$$F(r,\theta) = \begin{cases} (r, f_{2r-1}(\theta)) & \frac{1}{2} < r \leq 1 \\ (r,\theta) & 0 \leq r \leq \frac{1}{2} \end{cases}$$

where $f_0 = \mathrm{id}_{S^3}$ and $f_1 = f$.

Both components of the given link are unknots, so it is possible to replace either of the 0-framed unknots with a dotted circle, i.e, an orientable 1-handle, obtaining A_0 and A_1 depending on which component is dotted. This operation does not affect the boundary 3-manifold since 0-surgery on the unknot in S^3 results in $S^2 \times S^1$, and attaching an orientable 1-handle to B^4 gives $S^1 \times B^3$ whose boundary is $S^1 \times S^2$, so $\partial A_0 = \partial A_{\frac{1}{2}} = \partial A_1$. The main result of Akbulut using the Donaldson polynomials is that the identity map $\partial A_0 \xrightarrow{\sim} \partial A_1$ does not extend to a diffeomorphism between A_0 and A_1 (even though A_0 is clearly diffeomorphic to A_1).

The operation of turning a 0-framed unknot into a dotted circle can be performed by attaching a five-dimensional 3-handle as noted above, and we can therefore realize A_0 and A_1 as the ends of an h-cobordism. Explicitly, each unknotted component of our unlink determines a 2-sphere given by the union of its slice disk with its core disk (called S_0 and S_1 resp.), and we build our h-cobordism between A_0 and A_1 by first thickening $A_{\frac{1}{2}}$ and attaching a 3-handle to to S_1 and an upside down 3-handle to S_0. Akbulut then proves that this h-cobordism is nontrivial by showing that $\partial A_0 \xrightarrow{\sim} \partial A_1$ does not extend over A_0; this suffices because A is a product above ∂A_0 (the mapping cylinder of the diffeomorphism), and if $\partial A_0 \underset{\mathrm{sm}}{\cong} \partial A_1$ extended over A_0, then A would be a mapping cylinder for this extension.

One key insight from this construction is that the handles of A (for any such contractible sub-h-cobordism A as in Theorem 3.2) have the effect of dotting some of the 0-framed unknots in the middle layer. This is a general feature we can therefore expect of corks, for there to be a link of unknots where some components are 0-framed and others are dotted, and the cork diffeomorphism interchanges some of the dots and 0s. For involutory corks, we can perhaps expect the link we start with to even be symmetric as in Akbulut's example.

Complexity of Corks

To be continued.

References

Selman Akbulut and Rostislav Matveyev. *A Convex decomposition theorem for four-manifolds.* 2000. URL: https://arxiv.org/abs/math/0010166.

Stefan Behrens, Boldizsár Kalmár, Min Hoon Kim, Mark Powell, and Arunima Ray. *The disc embedding theorem.* Oxford University Press, 2021.

C. L. Curtis, M. H. Freedman, W. C. Hsiang, and R. Stong. *A decomposition theorem for h-cobordant smooth simply-connected compact 4-manifolds.* In: *Inventiones mathematicae* 123.2 (1996), pp. 343–348.

Simon K Donaldson. *Irrationality and the h-cobordism conjecture.* In: *Journal of Differential Geometry* 26.1 (1987), pp. 141–168.

M. H. Freedman. *The topology of four-dimensional manifolds.* In: *Journal of Differential Geometry* 17.3 (1982), pp. 357–453.

Robert E Gompf. *Infinite order corks via handle diagrams.* In: *Algebraic & Geometric Topology* 17.5 (2017), pp. 2863–2891.

Robert E Gompf and András Stipsicz. *4-manifolds and Kirby calculus.* 20. American Mathematical Soc., 1999.

Allen Hatcher. *Algebraic Topology.* Cambridge: Cambridge University Press, 2001.

Rob Kirby. *Akbulut's corks and h-cobordisms of smooth, simply connected 4-manifolds.* In: *Turkish Journal of Mathematics* 20.1 (1996), pp. 85–93.

Matthias Kreck. *h–cobordisms between 1–connected 4–manifolds.* In: *Geometry & Topology* 5.1 (Jan. 2001), pp. 1–6. URL: http://dx.doi.org/10.2140/gt.2001.5.1.

Rostislav Matveyev. *A decomposition of smooth simply-connected h-cobordant 4-manifolds.* In: *Journal of Differential Geometry* 44.3 (1996), pp. 571–582.

John Milnor. *A procedure for killing homotopy groups of differentiable manifolds.* In: *Proceedings of Symposia in Pure Mathematics* (1961), pp. 39–55. URL: https://cir.nii.ac.jp/crid/1360855570196982528.

– *Lectures on the h-cobordism theorem.* Vol. 2258. Princeton university press, 2015.

Frank Quinn. *Ends of maps. III: Dimensions 4 and 5.* In: *Journal of Differential Geometry* 17.3 (1982), pp. 503–521.

– *Isotopy of 4-manifolds.* In: *Journal of Differential Geometry* 24.3 (1986), pp. 343–372.

Benjamin Matthias Ruppik. *Some of Freedman's Results on Topological 4-Manifolds.* 2020. URL: https://ben300694.github.io/pdfs/2020/2020-11-09_Freedman_Exotic_4-mflds.pdf.

Stephen Smale. *On the structure of manifolds.* In: *American Journal of Mathematics* 84.3 (1962), pp. 387–399.

C Terence C Wall. *On simply-connected 4-manifolds.* In: *Journal of the London Mathematical Society* 1.1 (1964), pp. 141–149.