Lie Groups **Fall 2024**

Overview

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Lie Groups

Lecture 1: August 26th

This is the third Fall in a row I'm attempting to take notes for a Tim Perutz class, which I always give up on because Tim posts his own notes and they are both more correct and more complete than mine. Let's see if I'm capable of change or if I'm doomed to repeat the same patterns forever.

To motivate our discussion, we want to think about groups which parametrize the continuous symmetries of geometric objects (for starters, vector spaces), e.g., $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SO}(n)$, and $\mathrm{SU}(n)$. We will also consider groups of affine transformations, e.g., the isometries of $\mathbb{R}^n$ of the form $x \mapsto AX + L$ (i.e. the group $\mathbb{R}^n \rtimes \mathrm{SO}(n)$), the torus group $\mathbb{R}^n/\Gamma$ for Γ a lattice, etc. These are to be contrasted with discrete groups such as all finite groups and e.g. $\mathrm{SL}_n(\mathbb{Z})$. We want the correct framework to think about groups which act continuously in a way that exploits this extra data.

The first such idea is that of a *topological group*:

Definition 1.1.1: Topological Groups

A *topological group* G is a group equipped with a topology such that the multiplication ($m : G \times G \to G$) and inversion ($i : G \to G$) maps are both continuous.

This is a perfectly fine definition, but it turns out that it will give rise to a quite varied category of objects. For example, all of $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{Q}_p)$, $\mathrm{GL}_n(\mathbb{F}_p)$ (with the Zariski topology) are topological groups, although only the first has the structure of a smooth manifold, and the second is totally disconnected. It turns out (and this is nontrivial) that the relatively mild restrictions of requiring G to be path-connected and a topological manifold will give rise to the "right" notion that we are interested in studying anyway.

It is useful in the definition to assume a smooth structure on G (for the purposes of doing calculus) even though one can prove from the above assumptions that such a smooth structure exists.

Definition 1.1.2: Lie Groups

A *Lie group* G is a group together with a smooth manifold structure on G such that m and i as above are smooth maps. This defines a category, whose arrows are *Lie group homomorphisms*, which are just smooth group homomorphisms.

Recall that C^∞ means infinitely differentiable and C^ω means that all functions have convergent local Taylor series expansions; functions of the form $f(x) = e^{-\frac{1}{x^2}}$ are what separate these classes, as all of the derivatives of f at 0 are 0 so f is infinitely differentiable but is not equal to its Taylor expansion on any neighborhood of $x = 0$.

There are two sensible meanings of smooth in this context, C^∞ and C^ω (real analytic). By default we will assume C^∞ in this course, as we will largely be thinking about finite dimensional Lie groups in this course, and there is no essential difference between the C^∞ and C^ω theories in finite dimension.

Example 1.1.3

Let

$$\mathrm{SU}(2) = \{U \in SL_2(\mathbb{C}) : UU^\dagger = I = U^\dagger U\} = \left\{ U = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \|\alpha\|^2 + \|\beta\|^2 = 1 \right\} = S^3$$

where † denotes the conjugate transpose. The evident homeomorphism with S^3 gives SU(2) the structure of a smooth manifold.

Lie Algebras

The most important fact about Lie groups G is that the vector space $\mathfrak{g} = T_eG$ has the structure of a Lie algebra over $\mathbb{R}$.

Definition 1.2.1: Lie Algebras

A *Lie algebra* $\mathfrak{g}$ is a vector space with a bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

- $[x, x] = 0$ (skew symmetry)
- $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ (Jacobi identity)

$[-,-]$ should be thought of as an infinitesimal form of the commutator $(g, h) = g^{-1}h^{-1}gh$. For matrix Lie groups $G \subseteq \mathrm{GL}_n(\mathbb{R})$, the Lie bracket reduces to $[A, B] = AB - BA$. The notion of Lie algebras is so important for this subject because much of the complexity of the manifold is contained in this vector space (which is *a priori* a simpler object to study). In fact, up to a small but well-understood deficit, $\mathfrak{g}$ knows everything about G (modulo, e.g., a Lie group (such as $\mathrm{SO}(3) = \mathbb{RP}^3$) and its universal cover ($\mathrm{SU}(2) = S^3$) have the same Lie algebra). Moreover, if we expand the multiplication map m in local coordinates centered on $e \in G$ we have

$$m(x, y) = x + y + \frac{1}{2}[x, y] + \cdots$$

Complex Lie Groups

Lecture 2: August 28th

Definition 1.3.1: Holomorphic

Let $U \subseteq \mathbb{C}^n$, $V \subseteq \mathbb{C}^m$ and $F : U \to V$. F is *holomorphic* if it is complex differentiable i.e. $F(z+h) = F(z) + (D_zF)h + \epsilon_z(h)$ where

> $D_z F$ is a $\mathbb{C}$-linear map $\mathbb{C}^n \to \mathbb{C}^m$ and $\epsilon_z(h) = o(|h|)$. One can show that F is holomorphic iff it is C^∞ and $D_z F$ is $\mathbb{C}$-linear.

Using this, one can define a complex manifold analogously to a smooth manifold, but with charts valued in open sets of $\mathbb{C}^n$ and with holomorphic transition functions. It is then evident how to define a complex Lie group by analogy, where the underlying space is a complex manifold and the multiplication and inversion maps are holomorphic.

This "analogous" definition can be made precise using the categorical definition of a group object, where real/complex Lie groups are simply the group objects in the category of smooth manifolds/complex manifolds respectively.

The standard examples of complex Lie groups are $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$ etc. Another important example is a complex torus (i.e. an elliptic curve): given a lattice $\Lambda \subseteq \mathbb{C}$, $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ with $e_1, e_2 \in \mathbb{C}$ linearly independent over $\mathbb{R}$, $\mathbb{C}/\Lambda$ is topologically a torus and has the structure of an abelian group.

$\mathbb{C}/\Lambda$ is isomorphic to a real Lie group (as a real Lie group) $S^1 \times S^1$; if e_1, e_2 generate Λ, then we have an explicit map given by

$$\mathbb{C}/\Lambda \ni [xe_1 + ye_2] \mapsto (x, y) \in S^1 \times S^1$$

However, $\mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda'$ as complex manifolds iff Λ and Λ' are *homothetic*, i.e, $\Lambda' = c\Lambda$ for some $c \in \mathbb{C}^\times$, so different tori as described are not even isomorphic as complex manifolds, let alone as complex Lie groups, so forgetting the complex Lie group structure can in general destroy nontrivial data.

There is an analogous story for algebraic groups, i.e. group objects in the category of algebraic varieties over a field k, although there is some more complexity there. Tim says something about these in his notes but we will not discuss them in class.

Compact Lie groups and Classification

Any countable discrete group (e.g. any finite group) is trivially a Lie group, so it is unwise to try to classify all Lie groups (as this would include a classification of all finite groups, which is notably quite hard). To avoid such pathologies, we can begin by assuming connectedness, though this also removes some important examples such as $\mathrm{O}(n)$ and $\mathrm{GL}_n(\mathbb{R})$. For the purposes of this overview, we will assume that our Lie groups are simply connected and compact; later in the course we will try to broaden the scope of our story to study the non-simply-connected case (which is not too hard) and perhaps the noncompact case (which is harder).

We define the following groups:

- $A_n = \mathrm{SU}(n+1)$
- $B_n = \mathrm{Spin}(2n+1)$ where $\mathrm{Spin}(k)$ is the universal (double) cover of $\mathrm{SO}(k)$
- $C_n = \mathrm{Sp}(n)$, the compact symplectic group, or the group of norm-

preserving $\mathbb{H}$-linear automorphisms of $\mathbb{H}^n$ (where $\mathbb{H}$ denotes the quaternions)

- $D_n = \mathrm{Spin}(2n)$

Then, we have the following:

Theorem 1.4.1: Classification of compact Lie Groups

Let G be a compact, simply connected Lie group, then $G \cong G_1 \times \cdots \times G_k$ where the factors G_i are isomorphic to one of the following:

- A_n for $n \geq 1$
- B_n for $n \geq 2$
- C_n for $n \geq 3$
- D_n for $n \geq 4$
- one of E_6, E_7, E_8, F_4, or G_2 (the exceptional compact Lie groups)

Moreover, the factorization is unique up to reordering.

The omitted indices are largely to guarantee uniqueness e.g. $B_1 = \mathrm{Spin}(3) = \mathrm{SU}(2) = A_1$.

One of the main themes of this course will be proving this result. There is a very similar classification for a class of complex Lie groups, the *semisimple* complex Lie groups. The Lie algebra of G is crucial to proving this classification, as $\mathfrak{g}$ determines G once we have set the above adjectives.

For the rest of this section we just list some facts that are important towards the classification result.

Definition 1.4.2: Tori

A *torus* in a Lie group G is a closed, connected, compact abelian subgroup.

Any compact Lie group G contains a *maximal torus* $T \cong (S^1)^n$ for some n, called the *rank* of G (this n is the subscript in the A_n, B_n etc.). A maximal torus T is unique up to conjugacy. Every element of G lies in some conjugate of T. In $\mathrm{SU}(n)$, the maximal torus consists of the diagonal matrices, and this is in general how we think about maximal tori.

To recognize T as a diagonal matrix in general, note that G acts linearly on $\mathfrak{g}$ via $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, which when restricted to T becomes $\mathrm{Ad}|_T : T \to \mathrm{GL}(\mathfrak{g})$, and we can realize eigenvalues of $\mathrm{Ad}|_T$ as homomorphisms $T \to S^1$. These homomorphisms are called the *roots* of G.

Unclear to me why this gives us a map from $T_e(T)$ to $\mathbb{C}$ and not $\mathbb{R}$. Tim says there's something subtle about this but doesn't elaborate further (for now).

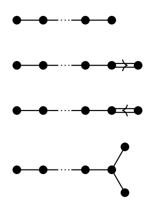
After taking the log of $T \to S^1 \ni e^{i\theta}$, the roots give a distinguished subset of $\mathrm{Hom}(T_e(T), \mathbb{C})$, and this subset has various properties that are abstracted in the notion of a *root system*. A step in the classification will be the classification of abstract root systems, which is combinatorial via *Dynkin* *diagrams*:

A_n :

B_n :

C_n :

D_n :



One then proves that the root system determines the complex Lie algebra $\mathfrak{g} \otimes \mathbb{C}$ and then the existence and uniqueness of compact real forms of complex Lie algebras.

Lie Groups **Fall 2024**

Basics of Lie Groups

Professor Tim Perutz Abhishek Shivkumar

Matrix Lie Groups

Lecture 3: August 30^{th}

The first and most basic example of a (matrix) Lie group is $\mathrm{GL}(V)$ which denotes the set of linear invertible maps $\alpha : V \to V$ where V is a finite dimensional real vector space. This is a manifold because it's open in $\mathrm{End}(V)$, and a Lie group because matrix multiplication/composition is smooth (since matrix multiplication is a polynomial in each entry) and inversion is smooth (because $A^{-1} = (\det A)^{-1}(\mathrm{adj}A)$ where $\mathrm{adj}A$ is the *adjugate* of A and its entries are polynomials in the entries of A). $\dim \mathrm{GL}(V) = n^2$ since it is open in $\mathbb{R}^{n^2} = \mathrm{End}(V)$, and $T_I \,\mathrm{GL}(V) = \mathrm{End}(V)$ since $I + \epsilon M$ for any matrix M will be invertible for ϵ sufficiently small.

We also have $\mathrm{GL}^+(V)$ which is the kernel of $\frac{\det}{\|\det\|} : \mathrm{GL}(V) \to \{\pm 1\}$ consisting of the positive determinant matrices, which is an open and closed normal subgroup of $\mathrm{GL}(V)$ (and therefore $\mathrm{GL}(V)$ is not connected). A linear isomorphism $\alpha : U \xrightarrow{\sim} V$ induces an isomorphism of Lie groups $\mathrm{GL}(U) \xrightarrow{\sim} \mathrm{GL}(V)$ given by $A \mapsto \alpha A \alpha^{-1}$. By picking a basis and identifying V with $\mathbb{R}^n$, we have $\mathrm{GL}(V) = \mathrm{GL}(\mathbb{R}^n) = \mathrm{GL}_n(\mathbb{R})$.

We can also run all of these constructions starting with a complex vector space E, with the one difference (in what we have said so far) that $\mathrm{GL}(E)$ is path-connected.

Lemma 2.1.1

$\mathrm{GL}_n(\mathbb{C})$ is path-connected.

$\mathrm{GL}^+(\mathbb{R}^n)$ is path-connected as well but this argument will not work in that case.

Proof : If V is a complex vector space, $f : V \to \mathbb{C}$ holomorphic, set

$$S_k(f) = \{v \in V : D_v^{(j)} f = 0 \text{ for all } j = 0, \cdots, k \text{ and } D_v^{(k+1)} f \neq 0\}$$

This is a complex submanifold of V by the complex inverse function theorem. Then consider $\mathrm{End}(\mathbb{C}^n) \xrightarrow{\det} \mathbb{C}$, and note that

$$\mathrm{End}(\mathbb{C}^n) \setminus \mathrm{GL}_n(\mathbb{C}) = S_1(\det) \cup S_2(\det) \cup \cdots$$

a finite union of complex submanifolds of $\mathrm{End}(\mathbb{C}^n)$ (since det as a polynomial has degree n), each of positive codimension. Take $A \in \mathrm{GL}_n(\mathbb{C})$, and connect it to I by a path in $\mathrm{End}(\mathbb{C}^n)$ (which is just $\mathbb{C}^{n^2}$), and perturb the path to be transverse to all the S_k (rel. endpoints) via general transversality theory. Then since the *real* codimension of S_k is at least two since its complex codimension is at least one, the transverse intersection with a path is generically empty, so the path is in $\mathrm{GL}_n(\mathbb{C})$. ■

Embedded Lie Subgroups

Given G a Lie group, an *embedded Lie subgroup* is a subgroup $H \subseteq G$ that is also an embedded submanifold. Explicitly, for all $h \in H$ there exists a neighborhood $U \ni h$ in G, a diffeomorphism $\phi : (U, h) \to (U', 0)$ for U' open in $\mathbb{R}^d$, and a vector subspace $L \subseteq \mathbb{R}^d$ s.t $\phi(U \cap H) = U' \cap L$.

Recall that for M a manifold, $f : M \to N$ a smooth map, then $c \in N$ is a regular value if $f(x) = c$ implies that $D_x f : T_x M \to T_c N$ is surjective. Then, the implicit function theorem tells us that $f^{-1}(c)$ is a submanifold of M.

Definition 2.2.1: Special Linear Group

$\mathrm{SL}(V)$ is defined as $\ker(\det : \mathrm{GL}(V) \to \mathbb{R}^\times)$ and analogously for $\mathbb{C}$.

We want to see that $\mathrm{SL}(V)$ is an embedded Lie subgroup of $\mathrm{GL}(V)$, so it will be helpful to consider a slightly different definition; set $V \cong \mathbb{R}^n$ (i.e. pick a basis) and consider the exterior powers $\wedge^d V$ of dimension $\binom{n}{d}$, where $\alpha : U \to V$ induces $\wedge^d \alpha : \wedge^d U \to \wedge^d V$ given by

$$u_1 \wedge \cdots u_d \mapsto \alpha(v_1) \wedge \cdots \wedge \alpha(v_d)$$

Then we have $\det V := \wedge^n V$ and for any $\alpha : V \to V$ we have the induced map $\wedge^n \alpha(x) = \det \alpha x$ for $x \in \det V$ (since $\det V$ is one-dimensional, $\wedge^n \alpha = \det \alpha$ is a scalar). Thus, we may realize $\mathrm{SL}(V)$ as the subset of $\mathrm{GL}(V)$ consisting of α such that the action of α is trivial on $\det V$ (this is a convoluted way to say $\det \alpha = 1$). Then, to see that $\mathrm{SL}(V)$ is an embedded Lie subgroup of $\mathrm{GL}(V)$, we have the following steps:

Allegedly this argument is a model for how to show that something is a Lie subgroup in general.

1. I is a regular point of det, i.e, $D_I \det \neq 0$. To see this, note that

$$I + tX = \begin{pmatrix} 1 + tx_{11} & \cdots & tx_{1n} \\ \vdots & \ddots & \vdots \\ tx_{n1} & \cdots & 1 + tx_{nn} \end{pmatrix}$$

for an arbitrary perturbation X, so to prove that $D_I \det \neq 0$, we want to compute $\det(I + tX)$ up to $O(t^2)$. The main diagonal contributes the term

$$(1 + tx_{11}) \cdots (1 + tx_{nn}) = 1 + t\,\mathrm{Tr}(X) + O(t^2)$$

and any other (non-identity) permutation will contribute a term that has at least two off-diagonal elements as factors, and thus is $O(t^2)$, so

$$\det(I + tX) = 1 + t\,\mathrm{Tr}(X) + O(t^2)$$

Since there exist matrices with nonzero trace, $(D_I \det)(X) = \mathrm{Tr}(X)$ is not the zero map, so I is a regular point for det. Therefore, elements of $\mathrm{SL}(V)$ near I are also regular points for det.

The geometric interpretation of this is that the first order change to the volume of a parallelopiped is the trace.

This shows that, near I, $\mathrm{SL}(V)$ is a submanifold of $\mathrm{GL}(V)$ with

$$T_I\,\mathrm{SL}(V) = \ker \mathrm{Tr} = \{X \in \mathrm{End}\,V : \mathrm{Tr}\,X = 0\}$$

More generally, if we have $f : X \to Y$ with $y \in Y$ a regular value of f, and $x \in M = f^{-1}(y)$,

$$T_x M = \ker df_x$$

2. Now, we can slide our regular neighborhood of $\mathrm{SL}(V)$ around using the group structure. Explicitly, for $A \in \mathrm{SL}(V)$ there is the left multiplication automorphism L_A given by $x \mapsto Ax$. L_A is a linear endomorphism of $\mathrm{End}(V)$ and a diffeomorphism of $\mathrm{GL}(V)$ that preserves $\mathrm{SL}(V)$. Since $\mathrm{SL}(V)$ is an embedded submanifold near I, applying L_A shows that $\mathrm{SL}(V)$ is an embedded submanifold near A with

$$T_A\,\mathrm{SL}(V) = D_I(L_A)T_I\,\mathrm{SL}(V) = \{X \in \mathrm{End}\,V : \mathrm{Tr}\big(A^{-1}X\big) = 0\}$$

Thus $\mathrm{SL}(V)$ is an embedded Lie subgroup of $\mathrm{GL}(V)$ of dimension $n^2 - 1$.

Flags

Let B denote the subgroup of *upper triangular* matrices in $\mathrm{GL}(V)$ (i.e, having nonzero entries on the diagonal and above), called the *Borel subgroup*. Let $U \subset B$ be the subgroup of *unipotent* matrices, whose diagonal entries are all 1. By counting entries, we have that

$$\dim B = \frac{1}{2}n(n+1) \qquad \dim U = \frac{1}{2}n(n-1)$$

We can describe B and U in a coordinate-free manner.

Definition 2.3.1: Flags

In a vector space V, a *flag* is a nested sequence $0 = F_0 \subset F_1 \subset \cdots \subset F_k = V$ of linear subspaces. In the case that $k = \dim V$, the F_i together form a *complete flag* (where $\dim F_i = i$).

If we have a complete flag $V = F_n \supset F_{n-1} \supset \cdots \supset F_1 \supset F_0 = 0$ note that we can build a basis for V by first picking a basis for F_1, then F_2/F_1, etc., adding one vector at a time. Set $B_F = \{\alpha \in \mathrm{GL}(V) : \alpha(F_i) = F_i\}$; in our chosen basis, B_F clearly corresponds to the set of upper triangular matrices, so one can also define the Borel subgroup in terms of preservation

I suppose the utility of this is probably that we can define Borel subgroups in other (perhaps non-matrix) Lie groups, but in this case the the auxiliary data of a complete flag is not too far from just picking a basis in which case we can just say “upper triangular matrices”.

of a complete flag. We also get interesting subgroups (called *parabolic* subgroups) by considering the analogous construction using incomplete flags, which correspond to subgroups of block upper triangular matrices in the GL_n picture.

Orthogonal and Unitary Groups

Lecture 4: September 4th

Let V be a finite dimensional vector space over $\mathbb{R}$, with a norm $\|-\|$, then

$$\mathrm{O}(V, \|-\|) = \{\alpha : V \to V \text{ linear} : \|\alpha(v)\| = \|v\|\}$$

is the *orthogonal group* with respect to this norm. We assume this norm comes from an inner product $\langle -, - \rangle$ so we can equivalently define

$$\mathrm{O}(V, \langle -, - \rangle) = \{\alpha : V \to \text{ linear} : \langle \alpha(u), \alpha(v) \rangle = \langle u, v \rangle\}$$

There's some discussion about norms not coming from inner products e.g. the ℓ^∞ norm which takes the maximum norm of entries in a vector. This norm generates the same topology as the ℓ^2 norm but does not satisfy the parallelogram law and therefore does not come from an inner product.

Of course, $(V, \langle -, - \rangle)$ is isomorphic to $(\mathbb{R}^n, \cdot)$ so we write $\mathrm{O}(V, \langle -, - \rangle) = \mathrm{O}(n)$. As matrices we can express the norm or inner product preserving condition compactly as $A^T A = AA^T = I$, so, explicitly, we have

$$\mathrm{O}(n) = \{A \in \mathbb{R}^{n \times n} : AA^T = A^T A = I\}$$

Unlike other Lie groups we have discussed so far, $\mathrm{O}(n)$ is compact since it is a closed subspace of $(S^{n-1})^n$ (since the columns of a matrix have norm one by $AA^T = I$ and therefore lie in S^{n-1}).

Proposition 2.4.1

$\mathrm{O}(n) \subset \mathrm{GL}_n(\mathbb{R})$ is an embedded Lie subgroup.

PROOF : The idea is to consider the map ϕ given by $A \mapsto AA^T$ where $\mathrm{O}(n) = \phi^{-1}(I)$. There are two steps: show that I is a regular point of ϕ, and show that I is a regular value of ϕ (using the group structure, as above with $\mathrm{SL}_n(\mathbb{R})$).

For the first step, note that AA^T is a symmetric matrix, so $\phi : M_{n \times n}(\mathbb{R}) \to \mathrm{Sym}^2(\mathbb{R}^n)$. To see that I is a regular point of ϕ, we expand

$$\phi(I + tX) = (I + tX)(I + tX^T) = I + t(X + X^T) + O(t^2)$$

so $(D_I \phi)(X) = X + X^T$. Thus, surjectivity is clear since if X is symmetric, $(D_I \phi)(X) = 2X$ (if we do not know that ϕ is valued in symmetric matrices arguing surjectivity is impossible).

Thus, $\mathrm{O}(n)$ is an embedded Lie subgroup near I with Lie algebra

$$T_I \, \mathrm{O}(n) = \ker D_I \phi = \{X : X + X^T = 0\}$$

consisting of skew-symmetric matrices. To see that $\mathrm{O}(n)$ is an embedded Lie subgroup everywhere, we can argue as before, translating a neighborhood of the identity around via the left multiplication automorphism, which shows that I is a regular value of Φ. ■

We can also define $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}_n(\mathbb{R})$, but for $A \in \mathrm{O}(n)$, $\det A = \pm 1$ since $\det\left(AA^T\right) = \det(A)^2 = \det(I) = 1$, so $\mathrm{O}(n)$ has two components, of which $\mathrm{SO}(n)$ is the identity component. Thus, $T_I \,\mathrm{SO}(n) = T_I \,\mathrm{O}(n)$. $\dim \mathrm{O}(n) = \dim \mathrm{SO}(n) = \frac{1}{2}n(n-1)$ since orthonormality of the columns of a matrix amounts to enforcing $n + \binom{n}{2} = \frac{1}{2}n(n+1)$ linear conditions (n for norm one, and $\binom{n}{2}$ for orthogonality), and $n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$.

The philosophically correct way to calculate $\dim \mathrm{O}(n)$ is to instead calculate $\dim_{\mathbb{R}} \mathfrak{o}(n)$ which, as noted above, consists of skew-symmetric matrices, and a skew-symmetric $n \times n$ matrix has 0s on the diagonal and is determined by either its strictly upper or lower diagonal elements, of which there are precisely $\frac{1}{2}n(n-1)$. I frequently forget that I can/should do these calculations in the Lie algebra instead of the group itself.

Over $\mathbb{C}$, we can repeat all of this discussion with $\mathrm{U}(n)$, the group of matrices preserving the standard Hermitian inner product, which will similarly be an embedded Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$ with

$$T_I \,\mathrm{U}(n) = \{X \in \mathbb{C}^{n\times n} : X + X^\dagger = 0\}$$

where $X^\dagger := \overline{X}^T$.

We can show that $\dim \mathrm{U}(n) = n^2$; note that any complex $n \times n$ matrix can be written uniquely as $X = H + S$ where H is equal to its conjugate transpose (Hermitian) and S is equal to the negative of its conjugate transpose (skew-Hermitian), and the set of Hermitian and skew-Hermitian matrices are in bijection by multiplication by i. Setting $\mathfrak{u}(n) = T_I \,\mathrm{U}(n)$, we have

$$\mathbb{C}^{n\times n} = \mathfrak{u}(n) \oplus i\mathfrak{u}(n) = \mathfrak{u}(n) \otimes_{\mathbb{R}} \mathbb{C}$$

from which we see that $\dim_{\mathbb{C}} \mathbb{C}^{n\times n} = \dim_{\mathbb{R}} \mathrm{U}(n) = n^2$.

Writing $X = H + S$ uses the standard trick of

$$H = \frac{1}{2}(X + X^\dagger) \quad S = \frac{1}{2}(X - X^\dagger)$$

which also works for writing a real matrix as the sum of symmetric and antisymmetric matrices.

As with $\mathrm{O}(n)$, we can define $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}_n(\mathbb{C})$. Unlike with $\mathrm{O}(n)$, $\mathrm{U}(n)$ is connected and $\dim \mathrm{SU}(n) = n^2 - 1 \neq \dim \mathrm{U}(n)$. The reason for this is that going from $\det A = \pm 1$ to $\det A = 1$ amounts to picking one of two components, whereas, in $\mathrm{U}(n)$, going from $\|\det X\| = 1$ to $\det X = 1$ is a codimension one condition.

SU(2) and SO(3)

Recall that $\mathrm{SU}(2) = \{A = \left(\begin{smallmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{smallmatrix}\right) : \|\alpha\|^2 + \|\beta\|^2 = 1\}$ is diffeomorphic to S^3 via $A \mapsto (\alpha, \beta)$. Also, we have that $\mathrm{SO}(3)$ consists of rotations of $\mathbb{R}^3$ around some axis, so one might expect these groups to be closely related.

Proposition 2.4.2

$$\mathrm{SU}(2)/(\pm I) \xrightarrow[\rho]{\sim} \mathrm{SO}(3)$$

That is, $\mathrm{SU}(2) = S^3$ is the double cover of $\mathrm{SO}(3)$ so $\mathrm{SO}(3) \cong \mathbb{RP}^3$.

The standard proof of this fact is slick and uses the identification of S^3 with the unit quaternions but the one we present here is perhaps more geometric in nature.

PROOF : Consider

$$\mathfrak{su}(2) = T_I\,\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & -a \end{pmatrix} : a \in i\mathbb{R}, b \in \mathbb{C} \right\}$$

Then, there is a homomorphism $\rho : \mathrm{SU}(2) \to \mathrm{GL}(\mathfrak{su}(2))$ given by $\rho(U) = [A \mapsto UAU^{-1}]$ (one has to check that UAU^{-1} still lies in $\mathfrak{su}(2)$). We will show that this ρ determines the isomorphism claimed above.

Note that we haven't pulled this map out of a hat, this is the *Adjoint representation* which is an important construction in the study of Lie groups that we will see many more times throughout this course. The inner product below, similarly, is (I think) essentially the *Killing form*, which also has wider significance.

We define an inner product on $\mathfrak{su}(2)$ by $(A, B) = -\frac{1}{2}\operatorname{Tr}(AB)$; it is evidently symmetric bilinear and satisfies

$$(\rho(U)A, \rho(U)B) = (A, B)$$

This inner product is *a priori* complex valued, but we can show that it is in fact real and positive definite; note that $A \in \mathfrak{su}(2)$ is skew-Hermitian and therefore has has imaginary eigenvalues $ia, -ia$ so $(A, A) = -\frac{1}{2}\operatorname{Tr}(A^2) = a^2 \geq 0$ so $\rho : \mathrm{SU}(2) \to \mathrm{O}(\mathfrak{su}(2), (-,-)) \cong \mathrm{O}(3)$ is well-defined. Since $\mathrm{SU}(2)$ is connected, $\rho(SU(2))$ is as well so ρ lands in $\mathrm{SO}(3)$.

We can make this very explicit using the basis of Pauli matrices for $\mathfrak{su}(2)$:

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

which satisfy $\sigma_j^2 = -I$, $\sigma_1\sigma_2 = \sigma_3 = -\sigma_2\sigma_1$ and cyclic permutations thereof. Thus $\sigma_1, \sigma_2, \sigma_3$ form an orthonormal basis for $\mathfrak{su}(2) \cong \mathbb{R}^3$.

Recall the *matrix exponential* which is defined as follows and is well-defined for all A:

$$\exp A = \sum_{n \geq 0} \frac{A^n}{n!} = I + A + \frac{A^2}{2} + \cdots$$

Applying exp to the Pauli matrices, we have

$$\exp(t\sigma_k) = (\cos t)I + (\sin t)\sigma_k$$

since the σ_k square to $-I$. We want to consider the action of these matrices via ρ on $\mathfrak{su}(2)$:

$$\rho(\exp(t\sigma_1))\sigma_1 = \sigma_1 \quad \rho(\exp(t\sigma_1))\sigma_2 = (\cos t)\sigma_2 + (\sin t)\sigma_3 \quad \rho(\exp(t\sigma_1))\sigma_3 = (-\sin t)\sigma_2 + (\cos t)\sigma_3$$

i.e $\rho(\exp(t\sigma_1))$ acts by rotations of $\mathbb{R}^3$ about the σ_1 axis by angle t, and similarly for $\rho(\exp(t\sigma_k))$.

Since every element of $\mathrm{SO}(3)$ is a rotation about some axis, and we can effect an arbitrary rotation via a combination of rotations through the major axes, we see that ρ is surjective, and we may finish by noting that $\ker \rho = \pm I$. Clearly, $\pm I \in \ker \rho$, and note that if $U \in \ker \rho$, then U commutes with all $A \in \mathfrak{su}(2)$, hence can be simultaneously diagonalized with any such A, from which we can quickly conclude that $U = \pm I$. ∎

Lie Subgroups and Quotients

Lecture 5: September 9$^{\text{th}}$

I slept through this lecture, so this section of notes, transcribed from Tim's, is even more quixotic than usual.

In order to make sense of quotients of Lie groups, we need to formalize which subgroups of Lie groups are themselves Lie groups (or Lie subgroups). For example $\mathbb{Z}/n \subseteq S^1 = \mathrm{SO}(2)$ clearly should not be considered a Lie group. However, a subgroup itself having a Lie group structure is not sufficient for our purposes; consider for example the irrational line on the torus $\mathbb{R} \subseteq T^2$ given by $x \mapsto (e^{ix}, e^{i\alpha x})$ for some irrational number α. $\mathbb{R}$ is evidently a Lie group but $T^2/\mathbb{R}$ would not be a Lie group since (for example) the quotient topology is the trivial topology on a set of uncountably many orbits. Thus we need to be more careful about our notion of subobjects in order to get a "nice" category of Lie groups (e.g. having quotients).

There turns out to be a relatively straightforward "correct" condition to impose on subgroups as we will discover shortly.

Lemma 2.5.1

If H is a subgroup of a topological group G then the closure $\overline{H}$ is also a subgroup. Moreover, if H is locally closed, then H is closed.

Proof : To see that $\overline{H}$ is a subgroup, $e \in \overline{H}$ is immediate, so we need only show that for $x, y \in \overline{H}$, $xy^{-1} \in \overline{H}$. It will suffice to show that any open neighborhood U of xy^{-1} intersects H (and thus $xy^{-1} \in \overline{H}$ by the definition of closure). Consider the map

$$\theta = m \circ (\mathrm{id}_G \times i) : G \times G \to G$$

which is evidently continuous, so $\theta^{-1}(U)$ is an open neighborhood of $(x, y) \in G \times G$, so there is a product neighborhood $X \times Y \subseteq \theta^{-1}(U)$ with $x \in X$ and $y \in Y$ (by the definition of the product topology). X and Y then both intersect H, since any open neighborhood of x or y must intersect H, so let $x' \in X \cap H$ and $y' \in Y \cap H$. Thus, $x'y'^{-1} \in U \cap H$, from which the claim follows. If H is locally closed, then there is a neighborhood N of $e \in G$ s.t $N \cap H = N \cap \overline{H}$. We want to show that $H = \overline{H}$. Note that H is open in $\overline{H}$; for any $h \in H$, we can take hN to be an open neighborhood of h and note that

Could not be more unclear to me why we did xy^{-1} here instead of xy.

$$hN \cap \overline{H} = h(N \cap \overline{H}) = h(N \cap H) = hN \cap H \subseteq H$$

Thus H is an open, dense subgroup of $\overline{H}$.

To finish, note that for any $x \in \overline{H}$, the coset xH is also open and dense in $x\overline{H}\overline{H}$, so it follows that $H \cap xH$ is nonempty (since, otherwise, H would be an open subset of $\overline{H}$ disjoint from the dense subset xH) so there exists $h \in H$ with $x^{-1}h \in H$. Likewise, $h^{-1}x \in H$ as well, so $x = hh^{-1}x \in H$, so $H = \overline{H}$. ■

This gives the following corollary:

Corollary 2.5.2

An embedded Lie subgroup $H \subseteq G$ is closed in G.

Since embedded submanifolds are locally closed, this is immediate. To see that such an H is a Lie group we need the following lemma:

Proposition 2.5.3

The multiplication and inversion maps of an embedded Lie subgroup are smooth, hence it itself is a Lie group.

The proof is omitted.

It turns out that closedness is the essential property we want for our subgroups:

Theorem 2.5.4: Closed Subgroup Theorem

Suppose G is a Lie group and H a closed subgroup. Then H is also a submanifold of G, and is therefore an embedded Lie subgroup.

We will postpone our proof of this fact until we have set up the *exponential map*. This theorem simplifies our lives considerably, as we can now take any subgroup defined as the level set of a continuous map as (closed, hence) Lie subgroup (i.e. det $= 1$), without any calculations of derivatives to guarantee regular values. In order to identify the Lie algebra T_eG for groups cut out in such a way, we have the following:

Corollary 2.5.5

Suppose that $G \subseteq \mathrm{GL}_n(\mathbb{R})$ is cut out by a set of algebraic equations $f_k(A) = 0$ for $k \in \{1, \cdots, r\}$ where $f_k : \mathbb{R}^{n\times n} \to \mathbb{R}$ is polynomial in the entries of A. Then G is an embedded Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$ of dimension at least $n^2 - r$ and

$$T_eG \subseteq \{\xi \in \mathfrak{gl}_n(\mathbb{R}) : \forall k, f_k(I + t\xi) = O(t^2)\}$$

The usual $T_eG = \ker dF$ (where $F : \mathbb{R}^{n\times n} \to \mathbb{R}^r$ is the tuple of the f_k) fact doesn't work here since we don't enforce that 0 is a regular value of F.

PROOF : That G is an embedded Lie subgroup follows immediately by the preceding discussion.

For T_eG, given $\xi \in T_eG$ we can take a path $\gamma : (-\epsilon, \epsilon) \to G$ with $\gamma(0) = I$ and $\dot\gamma(0) = \xi$. Then $f_k \circ \gamma = 0$ so $(D_I f_k)(\xi) = 0$ which is the same as $f_k(I + t\xi) = O(t^2)$. ■

Tim notes that one can treat the t variable here as formal with $t^2 = 0$ whence the criterion becomes that $f_k(I+t\xi) = 0$.

The same argument applies for subgroups of $\mathrm{GL}_n(\mathbb{C})$ *mutatis mutandis*. Note that the above cannot be improved to an equality for T_eG: let $G \subseteq \mathbb{R}^2$ be the lower triangular subgroup of matrices of the form $\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right)$; we can define this via polynomial $f(A) = 0$ where $f(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = b^2$. Then $D_I f = 0$ so $\ker D_I f$ is strictly larger than T_IG.

Discrete Normal Subgroups

Lemma 2.5.6

Let G be a connected topological group. Then any discrete normal subgroup is central.

Recall that a subgroup is central if it is contained in the center of the group $Z(G)$.

PROOF : Suppose $N \subseteq G$ is a discrete normal subgroup. G acts on N by conjugation resulting in an action map $G \times N \to N$. Since this is continuous, the resulting homomorphism $G \to \operatorname{Aut} N$ is locally constant hence constant (since $\operatorname{Aut} N$ too must be discrete). Thus all conjugates of N are equal to N itself and N is central. ■

Quotients

Lemma 2.5.7

For a discrete normal subgroup N of a Lie group G, the quotient G/N is again a Lie group and the projection $G \to G/N$ a covering map.

PROOF : Because N is discrete, there is an open set U containing e in G such that $N \cap U = \{e\}$. We may assume that the translates gU are disjoint for $g \in N$ after perhaps shrinking U.

The quotient map $q : G \to G/N$ is open by the definition of the quotient topology, so $q|_U$ is a homeomorphism onto its image U' (since the translates of U are disjoint), with $q^{-1}(U') = \coprod_{g \in N} gU$. Thus, $q : q^{-1}(U') \to U'$ is a trivial covering map, and for $h \in G$ with image $h' \in G/N$, $q^{-1}(h'U') = \coprod_{g \in N} ghU$ so the projection map is a covering map locally everywhere. One can verify that G/N is Hausdorff, and then verify the consistency of the smooth structure on G/N which makes $q : hU \to h'U'$ a diffeomorphism for any $h \in G$. ■

Not sure what is meant here by trivial covering map, it's clearly not just the identity map which is what I would call a trivial covering map. Didn't really follow this proof.

Example 2.5.8

The quotient $\mathrm{PU}(n) := \mathrm{SU}(n)/Z(\mathrm{SU}(n))$ is a Lie group, and is centerless.

$Z(\mathrm{SU}(n))$ is equal to $\mathbb{Z}/n$ generated by the diagonal matrix with entries a primitive n^{th} root of unity.

Note that the projection $q : G \to G/Z$ induces a linear isomorphism $T_e G \to T_e(G/Z)$ i.e. an isomorphism of Lie algebras. For any connected Lie group G with discrete center Z (such as $\mathrm{SU}(n)$ as above and $\mathrm{SO}(n)$) we can pass to the (centerless) quotient $G' = G/Z$, which is sometimes referred to as the *adjoint* form of the group. In fact, there is a converse to this:

Lemma 2.5.9: Covering Criterion

A Lie group homomorphism $\rho : G \to G'$ with connected target G' such that $D_I\rho : T_IG \to T_IG'$ is an isomorphism, is a covering map.

PROOF : ρ is a diffeomorphism on a sufficiently small neighborhood of the identity by the inverse function theorem, $\rho|_U : U \to U'$. Since ρ is a homomorphism (i.e. it is left translation equivariant), it must be a local diffeomorphism everywhere, and therefore an open map. Let $K = \ker \rho$ which is necessarily discrete since $D_I\rho$ is an isomorphism, then $\rho^{-1}(U') = K \cdot U$, and the map $K \times U \to K \cdot U$ given by $(k, u) \mapsto ku$ is a diffeomorphism, so $\rho : K \cdot U \to U'$ is a trivial covering. Translating this around shows that ρ is a covering of its image, and since ρ is closed and open, and G' is connected, its image is all of G' and we are done. ■

A bunch of weird proofs today, very point-set-y in an uncomfortable way.

Covering Groups

Since some of our classification theorems will be concerned with simply connected Lie groups, we will want to consider covers (and in particular the universal cover) of Lie groups. It turns out that for G a Lie group, $\tilde{G} \to G$ a covering space, $\tilde{G}$ inherits a Lie group structure. Towards a proof of this fact, we need the following:

Proposition 2.6.1

Let G be a topological group, $a, b : (S^1, 0) \to (G, e)$. There is a based homotopy between ab (meaning concatenation) and $a \cdot b$ (meaning multiplication in G).

We give a proof based on explicit homotopies but in fact this follows by a very general argument known as an *Eckmann-Hilton argument*, which applies when there are two operations on a set that satisfy certain natural identities. The argument shows that the two operations must agree and are in fact commutative.

PROOF : Note that the loop ab is defined as

$$ab(t) = \begin{cases} a(2t) & 0 \leq t \leq \frac{1}{2} \\ b(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

which we can rewrite as $ab(t) = \alpha_{\frac{1}{2}}(t) \cdot \beta_{\frac{1}{2}}(t)$ where $\alpha_s(t)$ is defined as

$$\alpha_s(t) = \begin{cases} a(\frac{t}{s}) & 0 \leq t \leq s \\ e & s \leq t \leq 1 \end{cases}$$

with β_s defined similarly but starting with the identity until time s when it does the loop b in the remaining time. Then we may define a homotopy $\gamma_s(t) = \alpha_{\frac{s+1}{2}}(t) \cdot \beta_{\frac{1-s}{2}}(t)$ with endpoints $\gamma_0 = \alpha_{\frac{1}{2}} \cdot \beta_{\frac{1}{2}} = ab$ and $\gamma_1 = \alpha_1 \cdot \beta_0 = a \cdot b$. ■

Lecture 6: September 11$^{\text{th}}$

Then, we have:

Lemma 2.6.2

Let G be a topological group, path-connected and locally simply connected, $q : \tilde{G} \to G$ a covering space. Choose $\tilde{e} \in q^{-1}(e)$, then $\tilde{G}$ has a unique group structure with identity element $\tilde{e}$ covering the group structure in G. This makes $\tilde{G}$ a topological group and q a continuous homomorphism.

PROOF : The hypotheses on G mean that the lifting criteria from covering-space theory apply:

$$\begin{tikzcd} \tilde{G} \times \tilde{G} \arrow[r, dashed, "\tilde{m}"] \arrow[d, "q \times q"'] \arrow[dr, "\mu"] & \tilde{G} \arrow[d, "q"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

We want a map $\tilde{m}$ making the diagram commute. The only obstruction to lifting μ to $\tilde{m}$ is the requirement that $\mu_* \pi_1(\tilde{G} \times \tilde{G}) \subseteq q_* \pi_1(\tilde{G})$ in $\pi_1(G)$. If so, $\tilde{m}$ exists, sending $(\tilde{e}, \tilde{e})$ to $\tilde{e}$ and is unique.

For a reference, see Proposition 1.33 in Hatcher's Algebraic Topology.

To see that this holds, take based loops $\tilde{\alpha}, \tilde{\beta}$ in $\tilde{G}$ (with $\alpha, \beta = q(\tilde{\alpha}), q(\tilde{\beta})$). We want $\mu(\tilde{\alpha}, \tilde{\beta})$ to lift to a loop in $\tilde{G}$, where $\mu(\tilde{\alpha}, \tilde{\beta}) = m(\alpha, \beta)$. But by the above lemma, $m(\alpha, \beta)$ is homotopic to $\alpha\beta$ which lifts to $\tilde{\alpha}\tilde{\beta}$, so the lifting criterion is satisfied. We omit the checks of associativity, inverses, etc (the idea is to apply uniqueness of the lift relentlessly). ■

For Lie groups, we also have that the covering group is itself a Lie group:

Proposition 2.6.3

Let G be a connected Lie group, $q : \tilde{G} \to G$ a covering space. Pick $\tilde{e} \in q^{-1}(e)$, then there exists a unique Lie group structure on $\tilde{G}$ with identity $\tilde{e}$ such that q is a Lie group homomorphism and a *smooth* covering map.

PROOF : The group structure is constructed above and the smooth manifold structure comes from the fact that a covering space of a manifold is always uniquely a manifold. ■

Universal Covering Groups

A connected Lie group G has a universal covering group $\tilde{G}$.

Example 2.6.4

$\mathrm{SL}_2(\mathbb{R})$ has fundamental group $\mathbb{Z}$ and has $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ as a universal cover (this is an early example of a *non-matrix* Lie group).

Another ubiquitous example that we have seen before is $\widetilde{\mathrm{SO}(3)} = \mathrm{SU}(2)$.

In order to work with such groups, we recall the standard construction of the universal cover; let PG be the set of continuous paths $\gamma : [0, 1] \to G$ in

G such that $\gamma(0) = e$ with the compact open topology. There is a natural map $\mathrm{ev} : PG \to G$ given by $\gamma \mapsto \gamma(1)$, and we set $\tilde{G} = PG/\sim$ where the equivalence relation is given by homotopy relative to endpoints, and ev descends to a map $\tilde{G} \to G$.

There is an evident group structure on PG given by pointwise multiplication of paths which descends to $\tilde{G}$. This is a general construction, which can lead to some unpleasantness since PG is infinite dimensional, so in special cases we may choose alternate constructions when one is evident:

Example 2.6.5

Let $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{Aut}(S^2)$ be the set of Möbius maps $z \mapsto \frac{az+b}{cz+d}$, which contains $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{Aut}(\mathbb{H}^2) \cong \mathrm{PU}(1,1) = \mathrm{Aut}(\mathbb{D})$. $\mathrm{Aut}(\mathbb{D})$ acts on S^1 by diffeomorphisms, and the universal cover $\widetilde{\mathrm{Aut}(\mathbb{D})}$ is the group of pairs (μ, γ) with $\mu \in \mathrm{Aut}(\mathbb{D})$ and $\gamma : \mathbb{R} \to \mathbb{R}$ lifting the action of μ on the boundary.

$\widetilde{\mathrm{SO}(3)} = \mathrm{SU}(2)$ also has a natural construction without appeal to the path space via either our explicit construction far above or via the quaternions.

$\pi_1(G)$ and Fiber Bundles

It turns out that a useful way to study the fundamental groups of Lie groups is via the construction of a natural associated fiber bundle. Consider $\mathrm{SU}(n)$ with its natural (and transitive) action on $S^{2n-1} \subset \mathbb{C}^n$. Transitivity implies that any unit vector can appear in the first column of $A \in \mathrm{SU}(n)$ i.e. Ae_1 can be any unit vector. The stabilizer of e_1 is evidently given by matrices of the form

$$\left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right)$$

i.e. the stabilizer is isomorphic to $\mathrm{SU}(n-1)$. Then we have an orbit map $q : \mathrm{SU}(n) \to S^{2n-1}$ given by $A \mapsto Ae_1$ which gives rise to a fiber bundle:

$$\begin{tikzcd} \mathrm{SU}(n-1) \arrow[r, hook] & \mathrm{SU}(n) \arrow[d, "q"] \\ & S^{2n-1} \end{tikzcd}$$

The technical result we are using to assume that this is a fiber bundle is *Ehresmann's theorem*, that proper submersions give rise to fiber bundles. To see that q is a submersion, note that q is a surjection and is of constant rank by equivariance, so we only need to check that q is a submersion at a point.

To study $\pi_1(\mathrm{SU}(n))$ via this fiber bundle, we need to review homotopy groups and the long exact sequence of homotopy groups associated to a fiber bundle.

Homotopy Groups

Let (X, x) be a based topological space, then $\pi_n(X, x) = [(S^n, *), (X, x)]$ the set of based homotopy classes of maps from S^n. $\pi_0(X, x)$ is just the set of path-components of X with a distinguished point corresponding to the component containing the basepoint. For $n \geq 1$, $\pi_n(X, x)$ has a group structure, which is abelian for $n \geq 2$. The group operation for π_n, $n \geq 2$, is based on the *collapsing map* $S^n \xrightarrow{c} S^n \vee S^n$ which maps the upper and lower hemispheres to separate spheres and the entire equator to the wedge point. Then, given $f, g : (S^n, *) \to (X, x)$, we have $S^n \xrightarrow{c} S^n \vee S^n \xrightarrow{f \vee g} (X, x)$ and this gives a group operation on π_n.

One can show that π_n is abelian for $n \geq 2$ using an Eckmann-Hilton argument or geometrically via the definition of the group operation.

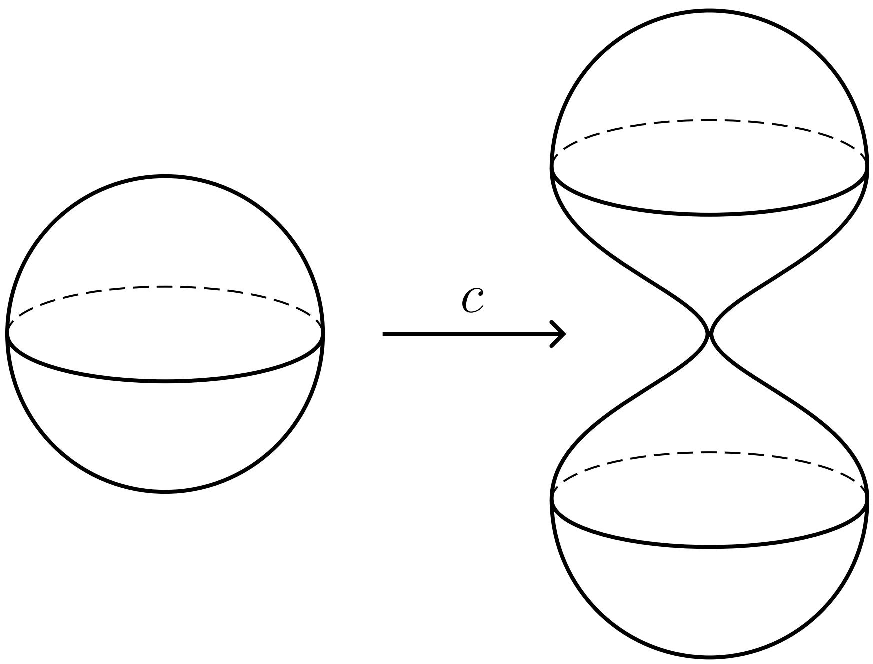


Figure 2.1: The collapse map.

A fiber bundle $F \overset{i}{\hookrightarrow} E \overset{q}{\twoheadrightarrow} B$ gives rise to a long exact sequence of homotopy groups

$$\cdots \xrightarrow{\partial} \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{q_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \to \cdots$$

terminating in

$$\cdots \to \pi_1(B) \to \pi_0(F) \to \pi_0(E) \to \pi_0(B) \to 0$$

The final entries in this sequence are no longer groups but *pointed set*, and we can make sense of exactness (in particular, we can make sense of kernels and cokernels) for pointed sets.

Lecture 7: September 13$^{\text{th}}$

And now, a somewhat out of place proposition that we forgot to cover last time, before returning to the discussion of higher homotopy groups.

Proposition 2.7.1

Let G be a connected Lie group, $\tilde{G} \xrightarrow{q} G$ its universal covering group. Then

$$\pi_1(G) \cong \mathrm{Aut}(\tilde{G}/G) \cong \ker(q) \hookrightarrow Z(\tilde{G})$$

This gives another proof of the fact that $\pi_1(G)$ is abelian.

We omit the proof, but the first isomorphism is from elementary covering space theory, and the second isomorphism follows by noting that $\pi_1(G) \curvearrowright \ker(q)$ and $\ker(q)$ has a distinguished point (the identity $\tilde{e} \in \tilde{G}$). The injection follows by noting that $\ker(q)$ is discrete and normal and therefore central by Lemma 2.5.6.

The other fact that we need from homotopy theory is the Hurewicz theorem, which gives the fundamental relationship between π_n and H_n.

Theorem 2.7.2: Hurewicz

There is a map $h_n : \pi_n(X, x) \to H_n(X)$ given by $[f : S^n \to X] \mapsto f_*[S^n]$. If $\pi_k(X, x) = 0$ for $k = 0, \cdots, n-1$, with $n \geq 2$, then h_n is an isomorphism. Moreover, h_{n+1} is an epimorphism. In the simply-connected case, this amounts to the statement that the first nontrivial homotopy and homology groups coincide. In the non-simply-connected case, h_1 is the abelianization map.

Note that Hurewicz gives us an almost certainly circular proof of the fact that $\pi_k(S^n) = 0$ for $k < n$ and $\pi_n(S^n) = \mathbb{Z}$.

Special Unitary Groups

Returning to our fibration $\mathrm{SU}(n-1) \overset{i}{\hookrightarrow} \mathrm{SU}(n) \xrightarrow{q} S^{2n-1}$, we inspect the corresponding long exact sequence at the end:

$$\cdots \to \pi_1(S^{2n-1}) \to \pi_0(\mathrm{SU}(n-1)) \to \pi_0(\mathrm{SU}(n)) \to \pi_0(S^{2n-1})$$

Since $\pi_0(\mathrm{SU}(1)) = 0$, we inductively prove that $\pi_0(\mathrm{SU}(n)) = 0$ for all n.

Now looking at π_1, the relevant section is

$$\cdots \to \pi_2(S^{2n-1}) \to \pi_1(\mathrm{SU}(n-1)) \to \pi_1(\mathrm{SU}(n)) \to \pi_1(S^{2n-1}) \to \cdots$$

Since $\pi_1(S^{2n-1}) = 0$ and $\pi_2(S^{2n-1}) = 0$ for $n \geq 2$ so $\pi_1(\mathrm{SU}(n)) = \pi_1(\mathrm{SU}(2)) = 0$ inductively for $n \geq 2$. One can also use this exact sequence to show that $\pi_2(\mathrm{SU}(n)) = 0$ for all n.

In fact, for all Lie groups G, $\pi_2(G) = 0$. The proof follows by reducing to the connected case (since π_2 only sees the connected component of the identity) and then to the compact case (by showing that Lie groups all deformation retract to a compact subgroup). There is also a proof via Morse theory on the path space PG.

Special Orthogonal Groups

There is a similar construction for the special orthogonal groups, via the natural action $\mathrm{SO}(n) \curvearrowright S^{n-1}$ given by $A \mapsto (x \mapsto Ax)$ which similarly gives rise to a fibration $\mathrm{SO}(n-1) \overset{i}{\hookrightarrow} \mathrm{SO}(n) \xrightarrow{q} S^{n-1}$. Arguing similarly via the long exact sequence, we find that $\pi_0(\mathrm{SO}(n)) = 0$ for all n, and $\pi_1(\mathrm{SO}(n)) = \pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2$ for all $n \geq 3$. $\pi_1(\mathrm{SO}(2)) = \mathbb{Z}$ since $\mathrm{SO}(2) \cong S^1$. As with $\mathrm{SU}(n)$, we find that $\pi_2(\mathrm{SO}(n)) = 0$ for all n.

I ask what can be said about π_3 because Tim said you could say "something," and now I have derailed the class slightly. We have that $\pi_3(\mathrm{SU}(n)) \cong \pi_3(\mathrm{SU}(2))$ for all $n \geq 2$ and $\pi_3(\mathrm{SU}(2)) = \pi_3(S^3) = \mathbb{Z}$. The generator for $\pi_3(\mathrm{SU}(n))$ is given by the inclusion of $\mathrm{SU}(2)$ into $\mathrm{SU}(n)$.

For the orthogonal groups, $\pi_3(\mathrm{SO}(n)) \cong \pi_3(\mathrm{SO}(5))$ for $n \geq 5$ which is also $\mathbb{Z}$, and $\pi_3(\mathrm{SO}(4)) = \mathbb{Z} \oplus \mathbb{Z}$ which apparently is due to the isomorphism $\mathrm{SO}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)/\pm(I, I)$.

Since $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2$ for $n \geq 3$, there is an interesting double cover $\mathbb{Z}/2 \hookrightarrow \mathrm{Spin}(n) \twoheadrightarrow \mathrm{SO}(n)$ for all n, which is the universal cover in the $n \geq 3$ case. This is a somewhat unsatisfying definition for $\mathrm{Spin}(n)$ since it does not give us much to work with, so we can ask for more concrete models for $\mathrm{Spin}(n)$, which we can construct directly at least for small n.

$\mathrm{Spin}(3) \cong \mathrm{SU}(2)$ since $\mathrm{SU}(2)$ is the universal (double) cover of $\mathrm{SO}(3)$, and $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. In fact, we can construct a "spinorial" representation of $\mathrm{SO}(n)$ given a complex vector space S and a Lie group homomorphism $\sigma : \mathrm{SO}(n) \to \mathrm{GL}(S)/\pm I$, take a generator γ of $\pi_1(\mathrm{SO}(n))$ and lift $\sigma(\gamma)$ to a path $\widetilde{\sigma(\gamma)}$ in $\mathrm{GL}(S)$, then $\widetilde{\sigma \circ \gamma}(1) = -I$.

If $n = 2m$ or $2m+1$, there is an essentially unique spinorial representation S with $\dim_{\mathbb{C}} 2^m$ called spinors, involving Clifford algebras. Consider

$$\Gamma = \{(g, \tilde{g}) : g \in \mathrm{SO}(n), \tilde{g} \in \mathrm{GL}(S) \text{ a lift of } \sigma(g)\}$$

This then gives a Lie group Γ equipped with an evident $2:1$ map to $\mathrm{SO}(n)$ which gives a concrete model for $\mathrm{Spin}(n)$.

The Classical Groups

Complex Classical Groups

Lecture 8: September 16th

We have seen many of the examples we will discuss today already, but we will write them down here somewhat systematically in order to have our standard set of examples going forward. There is a sense in which these lists will be exhaustive, but we won't be proving that.

$G_{\mathbb{C}}$	symmetries of	$T_e G_{\mathbb{C}} \subset \operatorname{End} \mathbb{C}^n$	$\dim_{\mathbb{C}} G_{\mathbb{C}}$
$\mathrm{GL}_n(\mathbb{C})$	$\mathbb{C}^n$	$\mathfrak{gl}_n(\mathbb{C})$: no condition	n^2
$\mathrm{SL}_n(\mathbb{C})$	$\mathbb{C}^n$, $e_1 \wedge \cdots \wedge e_n \in \det \mathbb{C}^n$,	$\mathfrak{sl}_n(\mathbb{C})$: trace-free	$n^2 - 1$
$\mathrm{SO}_n(\mathbb{C})$	$\mathbb{C}^n$, $e_1 \wedge \cdots \wedge e_n$, quadratic form	$\mathfrak{so}_n(\mathbb{C})$: skew-symmetric	$\frac{1}{2}n(n-1)$
$\mathrm{Sp}_{2m}(\mathbb{C})$,	$\mathbb{C}^{2m}$, symplectic form ω	$\mathfrak{sp}_{2m}(\mathbb{C})$: $\xi J + J\xi^T = 0$	$\frac{1}{2}n(n+1)$

The last of these is new, the *symplectic group* $\mathrm{Sp}_n(\mathbb{F})$ (distinguished from the *compact symplectic group* $\mathrm{Sp}(n)$ discussed below) and warrants some discussion: on $\mathbb{C}^2$ the standard skew form is $\omega(x, y) = x_1 y_2 - x_2 y_1$, so on $\mathbb{C}^{2m}$ we take the direct sum of copies of this, i.e.,

By $e_1 \wedge \cdots \wedge e_n \in \det \mathbb{C}^n$ in the second column, we mean that this form is preserved by the action of the corresponding group in the first column. Note that all these groups are noncompact.

$$\omega(x, y) = (x_1 y_2 - x_2 y_1) + (x_3 y_4 - x_4 y_3) + \cdots$$

and $A \in \mathrm{Sp}_{2m}(\mathbb{C})$ if $\omega(Ax, Ay) = \omega(x, y)$.

There is a matrix version of this; set $J_2 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $J_{2m} = J_2 \oplus \cdots \oplus J_2$ the block diagonal matrix built of such blocks. Note that $\omega(e_i, e_j) = (J_{2m})_{ij}$, then the above condition for $A \in \mathrm{Sp}_{2m}(\mathbb{C})$ can be rewritten as $AJ_{2m}A^T = J_{2m}$. Linearizing this condition produces the above defining condition on $X \in \mathfrak{sp}_{2m}(\mathbb{C})$, which is equivalent to JX being symmetric.

Real Classical Groups

The classification of real classical groups listed here is due to Cartan.

Definition 2.8.1: Real Structures

Let $G_\mathbb{C}$ be a complex Lie group, then a *real structure* on G is a map $\Theta : G_\mathbb{C} \to G_\mathbb{C}$ that is anti-holomorphic (meaning $D_g\theta \circ (iv) = -i \circ D_g\theta(v)$, derivative is $\mathbb{C}$-anti-linear), involutive (meaning $\Theta^2 = \mathrm{id}$), and a real Lie group homomorphism.

Set G^Θ to be the fixed point set of Θ, which is a closed subgroup of $G_\mathbb{C}$, and therefore an embedded Lie subgroup. Set $\mathfrak{g}_\mathbb{C} = T_I G_\mathbb{C}$ with $\mathfrak{g} := \mathfrak{g}_\mathbb{C}^\theta \subseteq \mathfrak{g}_\mathbb{C}$, where $\theta = D_I\Theta : \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$. It is automatic that $T_I G \subseteq \mathfrak{g}$ by Corollary 2.5.5, and in fact, $T_I G = \mathfrak{g}$ (the proof will be easy once we have the exponential map).

With this technology in place, we can construct an analogous table of real classical groups by picking real structures on the complex classical groups defined above, and defining G as the connected component of the identity in G^θ.

We get our first batch of real classical groups using the obvious involution $A \mapsto \overline{A}$:

G	$G_\mathbb{C}$	involution	G as symmetry group	T_eG	$\dim G$	G compact?
$\mathrm{GL}_n^+(\mathbb{R})$	$\mathrm{GL}_n(\mathbb{C})$	$A \mapsto \overline{A}$	$\mathbb{R}^n$, orientation	$\mathfrak{gl}_n(\mathbb{R}) = \mathrm{End}\,\mathbb{R}^n$	n^2	no
$\mathrm{SL}_n(\mathbb{R})$	$\mathrm{SL}_n(\mathbb{C})$	$A \mapsto \overline{A}$	$\mathbb{R}^n$, $e_1 \wedge \cdots \wedge e_n \in \det \mathbb{R}^n$	$\mathfrak{sl}_n(\mathbb{R})$: trace-free	$n^2 - 1$	no
$\mathrm{SO}(n)$	$\mathrm{SO}_n(\mathbb{C})$	$A \mapsto \overline{A}$	$\mathbb{R}^n$, $e_1 \wedge \cdots \wedge e_n$, pos.-def. quadratic form	$\mathfrak{so}(n)$: skew-symmetric	$\frac{1}{2}n(n-1)$	yes
$\mathrm{Sp}_{2m}(\mathbb{R})$,	$\mathrm{Sp}_{2m}(\mathbb{C})$	$A \mapsto \overline{A}$	$\mathbb{R}^{2m}$, symplectic form ω	$\mathfrak{sp}_{2m}(\mathbb{R})$: $\xi J + J\xi^T = 0$	$\frac{1}{2}n(n+1)$	no

To build the next example, we need to consider indefinite quadratic forms; consider the quadratic form

$$z \mapsto (z_1^2 + \cdots + z_p^2) - (z_{p+1}^2 + \cdots + z_n^2)$$

This quadratic form is called $Q_{p,q}$ where $p + q = n$ and with matrix

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

The symmetry group $\mathrm{SO}(\mathbb{C}^n, Q_{p,q}) \cong \mathrm{SO}_n(\mathbb{C})$ as $Q_{p,q} = Q_{n,0}$ over $\mathbb{C}$. However, over $\mathbb{R}$, we have $\mathrm{SO}(p,q) = \mathrm{SO}(\mathbb{R}^n, Q_{p,q})$ which is the fixed locus of

$A \mapsto \overline{A}$ on $\mathrm{SO}(\mathbb{C}^n, Q_{p,q})$. Note that $\mathrm{SO}(p,q) = \mathrm{SO}(q,p)$, so we may assume $p \geq q$.

By *Sylvester's law of inertia*, all non-degenerate quadratic forms on $\mathbb{R}^n$ are of the form $Q_{p,q}$.

G	$G_{\mathbb{C}}$	involution	G as symmetry group	T_eG	$\dim G$	G compact?
$\mathrm{SO}(p, n-p)_0$ $(1 \leq p \leq n-1)$	$\mathrm{SO}_{p,n-p}(\mathbb{C})$ $(\cong \mathrm{SO}_n(\mathbb{C}))$	$A \mapsto \bar{A}$	$\mathbb{R}^n$, $e_1 \wedge \cdots \wedge e_n$, quadratic form $Q_{p,n-p}$	$\mathfrak{so}(p, n-p)$: $\xi I_{p,n-p} + I_{p,n-p}\xi^T = 0$	$\frac{1}{2}n(n-1)$	no

We write $\mathrm{SO}(p,n-p)_0$ to denote the identity component, since $\mathrm{SO}(p,n-p)$ has two components for $0 < p < n$. As with $\mathrm{SO}(n)$, $\mathrm{SO}(p,q)$ is not simply connected and has interesting covering groups; the idea is that $S(\mathrm{O}(p) \times \mathrm{O}(q)) \hookrightarrow \mathrm{SO}(p,q)$ is a maximal compact subgroup of $\mathrm{SO}(p,q)$ and (therefore) a deformation retract, so the fundamental group can be understood in terms of the fundamental group of $\mathrm{SO}(p)$. For $p, q > 2$ we have $\pi_1 \mathrm{SO}(p,q) = \mathbb{Z}/2 \times \mathbb{Z}/2$. There is also a double cover of $\mathrm{SO}(p,q)$ called $\mathrm{Spin}(p,q)$, but, in general, it is itself no longer simply connected.

Next, we have the unitary and special unitary groups which arise from the involution $A \mapsto (A^\dagger)^{-1}$:

G	$G_{\mathbb{C}}$	involution	G as symmetry group	T_eG	$\dim G$	G compact?
$\mathrm{U}(n)$	$\mathrm{GL}_n(\mathbb{C})$	$A \mapsto (A^\dagger)^{-1}$	$\mathbb{C}^n$, pos.-def. hermitian form	$\mathfrak{u}(n)$: $\xi + \xi^\dagger = 0$	n^2	yes
$\mathrm{SU}(n)$	$\mathrm{SL}_n(\mathbb{C})$	$A \mapsto (A^\dagger)^{-1}$	$\mathbb{C}^n$, pos.-def. hermitian form, $e_1 \wedge \cdots \wedge e_n$	$\mathfrak{su}(n)$: $\xi + \xi^\dagger = 0$ trace-free	$n^2 - 1$	yes

As we know, $\mathrm{U}(n) \subseteq \mathrm{GL}_n(\mathbb{C})$ consists of *unitary* matrices which satisfy $AA^\dagger = A^\dagger A = I$. Linearizing this condition gives us that the Lie algebra consists of skew-hermitian matrices, analogously for $\mathrm{O}(n)$.

Since $\mathrm{U}(n)$ is essentially the correct analogue of $\mathrm{O}(n)$ for complex vector spaces (even though $\mathrm{O}(n,\mathbb{C})$ exists), we can consider an analogous "skew" construction as above with $\mathrm{SO}(p,q)$; note that the correct analogue for a bilinear inner product over $\mathbb{C}$ is a *Hermitian* inner product, which is *sesquilinear* ($\langle iu, v\rangle = i\langle u, v\rangle = -\langle u, iv\rangle$) and conjugate-symmetric ($\langle u, v\rangle = \overline{\langle v, u\rangle}$). Such a form determines and is determined by its Hermitian form $v \mapsto \langle v, v\rangle$. The standard Hermitian form on $\mathbb{C}^n$ is given by $v \mapsto \|v_1\|^2 + \cdots + \|v_n\|^2$, but we can instead consider the form

$$H_{p,q}(v) = (\|v_1\|^2 + \cdots + \|v_p\|^2) - (\|v_{p+1}\|^2 + \cdots + \|v_n\|^2)$$

where $p + q = n$ as above. This form gives us two more classical groups:

G	$G_{\mathbb{C}}$	involution	G as symmetry group	T_eG	$\dim G$	G compact?
$\mathrm{U}(p,q)$	$\mathrm{GL}_n(\mathbb{C})$	$A \mapsto I_{p,q}(A^\dagger)^{-1}I_{p,q}$	$\mathbb{C}^n$, $H_{p,q}$	$\mathfrak{u}(p,q)$: $I_{p,q}\xi = -\xi^\dagger I_{p,q}$	n^2	no
$\mathrm{SU}(p,q)$	$\mathrm{SL}_n(\mathbb{C})$	$A \mapsto I_{p,q}(A^\dagger)^{-1}I_{p,q}$	$\mathbb{C}^n$, $H_{p,q}$, $e_1 \wedge \cdots \wedge e_n$	$\mathfrak{su}(p,q)$: $I_{p,q}\xi = -\xi^\dagger I_{p,q}$, trace-free	$n^2 - 1$	no

Another direction to generalize is to consider the matrix groups for the quaternions $\mathbb{H}$ which naturally lie in $\mathrm{GL}_{2n}(\mathbb{C})$ since we can identify $\mathbb{C}^{2n}$ with $\mathbb{H}^n$. $\mathrm{GL}_n(\mathbb{H})$ is the subgroup of $\mathrm{GL}_{2n}(\mathbb{C})$ of those A which commute with the quaternion j, which leads to the involution $A \mapsto -jAj$ and the following two families of groups:

I imagine that $\mathrm{U}(p) \times \mathrm{U}(q)$ embeds into $\mathrm{U}(p,q)$ as a maximal compact so $\pi_1 \mathrm{U}(p,q)$ should be $\mathbb{Z}^2$. Hard to find a reference for this though.

G	$G_\mathbb{C}$	involution	G as symmetry group	T_eG	$\dim G$	G compact?
$\mathrm{GL}_m(\mathbb{H})$	$\mathrm{GL}_{2m}(\mathbb{C})$	$A \mapsto -jAj$	$\mathbb{H}^m$	$\mathfrak{gl}_m(\mathbb{H})$: $jA = Aj$	$4m^2$	no
$\mathrm{SL}_m(\mathbb{H})$	$\mathrm{SL}_{2m}(\mathbb{C})$	$A \mapsto -jAj$	$\mathbb{H}^m$, $e_1 \wedge_\mathbb{C} \cdots \wedge_\mathbb{C} e_{2m}$	$\mathfrak{sl}_m(\mathbb{H})$: $jA = Aj$, trace-free	$4m^2 - 1$	no

Note that, unlike for previous special versions of groups, $\mathrm{SL}_m(\mathbb{H})$ is not defined via $\det = 1$, but, rather, is defined as the intersection of $\mathrm{SL}_{2n}(\mathbb{C})$ with $\mathrm{GL}_n(\mathbb{H})$ inside of $\mathrm{GL}_{2n}(\mathbb{C})$. It is possible to define a real-valued quaternionic determinant such that $\mathrm{SL}_n(\mathbb{H})$ is its kernel, but this takes some work and we will not discuss it here. One source of difficulty in extending the definition from $\mathbb{R}$ or $\mathbb{C}$ is that expanding along different rows or columns result in different orderings of the terms of the products in the sum, and $\mathbb{H}$ is not commutative.

Applying this involution to the complex orthogonal and symplectic groups yields two new families:

G	$G_\mathbb{C}$	involution	G as symmetry group	T_eG	$\dim G$	G compact?
SO^*_{2m}	$\mathrm{SO}_{2m}(\mathbb{C})$	$A \mapsto -jAj$	$\mathbb{H}^n$, $\mathbb{C}$-quadratic form	$\mathfrak{so}^*_{2m}$: $jA = Aj$, $A^T = -A^T$	$m(2m-1)$	no
$\mathrm{Sp}(m)$	$\mathrm{Sp}_{2m}(\mathbb{C})$	$A \mapsto -jAj$	$\mathbb{H}^n$, symp. form ω	$\mathfrak{sp}_m$: $jA = Aj$, $AJ + JA^T = 0$	$m(2m+1)$	yes

The latter group is the compact symplectic group, mentioned above. To see that $\mathrm{Sp}(m)$ is compact, note that the Hermitian, symplectic, and quaternionic structures on $\mathbb{C}^{2n}$ are related by

$$\langle u, v\rangle = \omega(u, jv)$$

which leads to a *two-out-of-three principle* (as one encounters with almost complex, Riemannian, and symplectic structures in symplectic topology): a symmetry of two out of three of these structures is automatically a symmetry of the third. In terms of the symmetry groups, this implies that

$$\mathrm{Sp}(m) := \mathrm{Sp}_{2m}(\mathbb{C}) \cap \mathrm{GL}_m(\mathbb{H}) = \mathrm{Sp}_{2m}(\mathbb{C}) \cap \mathrm{U}(2m) = \mathrm{U}(2m) \cap \mathrm{GL}_m(\mathbb{H})$$

Since $\mathrm{Sp}(m)$ is a closed subgroup of $\mathrm{U}(2m)$ (which is compact), it is itself compact.

Since $\mathrm{Sp}(m) = \mathrm{Sp}_{2m}(\mathbb{C}) \cap \mathrm{U}(2m)$, we can also realize $\mathrm{Sp}(m)$ as the fixed point set of the involution $A \mapsto (A^\dagger)^{-1}$ acting on $\mathrm{Sp}_{2m}(\mathbb{C})$; consequently, we can also consider the indefinite versions of this involution used above for $\mathrm{U}(p,q)$:

G	$G_{\mathbb{C}}$	involution	G as symmetry group	T_eG	$\dim G$	G compact?
$\mathrm{Sp}(p,q)$	$\mathrm{Sp}_{2m}(\mathbb{C})$	$A \mapsto I_{2p,2q}(A^\dagger)^{-1}I_{2p,2q}$	$\mathbb{C}^{2m}$, ω, $H_{2p,2q}$	$\mathfrak{sp}(p,q)$: $AJ + JA^T = 0$,	$m(2m+1)$	no
$p+q=m$,				$I_{2p,2q}A + A^\dagger I_{2p,2q} = 0$		

This completes the classification of real classical groups due to Cartan. For reference, we include here a complete and contiguous version of the table we have built above:

I'm not sure what the definition of classical groups is that makes this list exhaustive since (for example) we don't say anything about Spin groups or covers of any of the new groups we've discovered, many of which are non-simply-connected.

G	$G_{\mathbb{C}}$	involution	G as symmetry group	T_eG	$\dim G$	G compact?
$\mathrm{GL}_n^+(\mathbb{R})$	$\mathrm{GL}_n(\mathbb{C})$	$A \mapsto \overline{A}$	$\mathbb{R}^n$, orientation	$\mathfrak{gl}_n(\mathbb{R}) = \operatorname{End}\mathbb{R}^n$	n^2	no
$\mathrm{SL}_n(\mathbb{R})$	$\mathrm{SL}_n(\mathbb{C})$	$A \mapsto \overline{A}$	$\mathbb{R}^n$, $e_1 \wedge \cdots \wedge e_n \in \det \mathbb{R}^n$	$\mathfrak{sl}_n(\mathbb{R})$: trace-free	n^2-1	no
$\mathrm{SO}(n)$	$\mathrm{SO}_n(\mathbb{C})$	$A \mapsto \overline{A}$	$\mathbb{R}^n$, $e_1 \wedge \cdots \wedge e_n$,	$\mathfrak{so}(n)$: skew-symmetric	$\frac{1}{2}n(n-1)$	yes
			pos.-def. quadratic form			
$\mathrm{Sp}_{2m}(\mathbb{R})$,	$\mathrm{Sp}_{2m}(\mathbb{C})$	$A \mapsto \overline{A}$	$\mathbb{R}^{2m}$, symplectic form ω	$\mathfrak{sp}_{2m}(\mathbb{R})$: $\xi J + J\xi^T = 0$	$\frac{1}{2}n(n+1)$	no
$\mathrm{SO}(p,n-p)_0$	$\mathrm{SO}_{p,n-p}(\mathbb{C})$	$A \mapsto \bar{A}$	$\mathbb{R}^n$, $e_1 \wedge \cdots \wedge e_n$,	$\mathfrak{so}(p,n-p)$:	$\frac{1}{2}n(n-1)$	no
$(1 \le p \le n-1)$	$(\cong \mathrm{SO}_n(\mathbb{C}))$		quadratic form $Q_{p,n-p}$	$\xi I_{p,n-p} + I_{p,n-p}\xi^T = 0$		
$\mathrm{U}(n)$	$\mathrm{GL}_n(\mathbb{C})$	$A \mapsto (A^\dagger)^{-1}$	$\mathbb{C}^n$, pos.-def. hermitian form	$\mathfrak{u}(n)$: $\xi + \xi^\dagger = 0$	n^2	yes
$\mathrm{SU}(n)$	$\mathrm{SL}_n(\mathbb{C})$	$A \mapsto (A^\dagger)^{-1}$	$\mathbb{C}^n$, pos.-def. hermitian form,	$\mathfrak{su}(n)$: $\xi + \xi^\dagger = 0$	n^2-1	yes
			$e_1 \wedge \cdots \wedge e_n$	trace-free		
$\mathrm{U}(p,q)$	$\mathrm{GL}_n(\mathbb{C})$	$A \mapsto I_{p,q}(A^\dagger)^{-1}I_{p,q}$	$\mathbb{C}^n$, $H_{p,q}$	$\mathfrak{u}(p,q)$: $I_{p,q}\xi = -\xi^\dagger I_{p,q}$	n^2	no
$p+q=n$,						
$\mathrm{SU}(p,q)$	$\mathrm{SL}_n(\mathbb{C})$	$A \mapsto I_{p,q}(A^\dagger)^{-1}I_{p,q}$	$\mathbb{C}^n$, $H_{p,q}$,	$\mathfrak{su}(p,q)$: $I_{p,q}\xi = -\xi^\dagger I_{p,q}$,	n^2-1	no
			$e_1 \wedge \cdots \wedge e_n$	trace-free		
$\mathrm{GL}_m(\mathbb{H})$	$\mathrm{GL}_{2m}(\mathbb{C})$	$A \mapsto -jAj$	$\mathbb{H}^m$	$\mathfrak{gl}_m(\mathbb{H})$: $jA = Aj$	$4m^2$	no
$\mathrm{SL}_m(\mathbb{H})$	$\mathrm{SL}_{2m}(\mathbb{C})$	$A \mapsto -jAj$	$\mathbb{H}^m$, $e_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} e_{2m}$	$\mathfrak{sl}_m(\mathbb{H})$: $jA = Aj$, trace-free	$4m^2-1$	no
SO^*_{2m}	$\mathrm{SO}_{2m}(\mathbb{C})$	$A \mapsto -jAj$	$\mathbb{H}^n$, $\mathbb{C}$-quadratic form	$\mathfrak{so}^*_{2m}$: $jA = Aj$, $A^T = -A^T$	$m(2m-1)$	no
$\mathrm{Sp}(m)$	$\mathrm{Sp}_{2m}(\mathbb{C})$	$A \mapsto -jAj$	$\mathbb{H}^n$, symp. form ω	$\mathfrak{sp}_m$: $jA = Aj$, $AJ + JA^T = 0$	$m(2m+1)$	yes
$\mathrm{Sp}(p,q)$	$\mathrm{Sp}_{2m}(\mathbb{C})$	$A \mapsto I_{2p,2q}(A^\dagger)^{-1}I_{2p,2q}$	$\mathbb{C}^{2m}$, ω, $H_{2p,2q}$	$\mathfrak{sp}(p,q)$: $AJ + JA^T = 0$,	$m(2m+1)$	no
$p+q=m$,				$I_{2p,2q}A + A^\dagger I_{2p,2q} = 0$		

Left-Invariant {Vector Fields, Differential Forms, $\cdots$}

Lecture 9: September 18th

One of the handy features of Lie groups is that they are equipped with an obvious transitive left action by themselves; this gives rise to a certain type of theorem or principle: a "structure" at $e \in G$ is equivalent to a left-invariant structure on all of G, usually given by translating the structure around using the aforementioned left action.

Smooth Actions on $\mathcal{O}_M$ and $\mathrm{Vect}(M)$

Let M be any manifold, $\mathrm{Diff}(M)$ its group of diffeomorphisms, and $\mathcal{O}_M = C^\infty(M)$ the $\mathbb{R}$-algebra of functions from M to $\mathbb{R}$. $\mathrm{Diff}(M) \curvearrowright \mathcal{O}_M$ via $\phi \cdot f := (\phi^{-1})^* f = f \circ \phi^{-1}$ where the inverse is required to get a left action.

Let $\mathrm{Vect}(M)$ denote the $\mathcal{O}_M$-module of vector fields on M, which we can think of in the following equivalent ways:

Derivations: A *derivation* of $\mathcal{O}_M$ is an $\mathbb{R}$-linear map $v : \mathcal{O}_M \to \mathcal{O}_M$ satisfying a Leibniz rule as follows:

$$v(fg) = fv(g) + gv(f)$$

In this picture, $\text{Diff}(M) \curvearrowright \text{Vect}(M)$ by

$$(\phi \cdot v)f = (v(f \circ \phi^{-1})) \circ \phi$$

Sections of TM: Let v be a global section of $\mathcal{O}_M$, then the action of $\text{Diff}(M)$ is given by

$$(\phi \cdot v)_x = D\phi^{-1}(v_{\phi(x)})$$

If G acts on M via a homomorphism $a : G \to \text{Diff}(M)$ which then induces an action of G on $\text{Vect}(M)$; our preferred such homomorphism will be $g \mapsto L_g$ where $L_g : G \to G$ is given by $h \mapsto gh$. Whenever G acts on M, we get a subspace $\text{Vect}(M)^G$ of invariant vector fields ξ satisfying $g \cdot \xi = \xi$, so, in particular, we have $\text{Vect}(G)^G$ the subspace of left-invariant vector fields.

Lemma 2.9.1

Let $\text{ev} : \text{Vect}(G)^G \to T_eG$ be given by $\xi \mapsto \xi_e$. Then ev is an isomorphism of vector spaces.

Proof : To construct the inverse, fix $\xi_e \in T_eG$. We want to extend this to a vector field on G; set $\xi_g = DL_g(\xi_e)$ the pushforward of ξ_e by left translation. ξ is then left-invariant by construction. ■

Thus elements of the Lie algebra $\mathfrak{g}$ are naturally in bijection with left-invariant vector fields on G. Say that $G \subseteq \text{GL}_n(\mathbb{R})$, so $T_eG \subseteq \mathfrak{gl}_n(\mathbb{R}) = \mathbb{R}^{n^2}$. Then $X \in T_eG$ corresponds to a left invariant vector field on G given by $G \ni A \mapsto AX$ where A on the right hand side is equal to $D_I L_A$ (this follows from a quick calculation).

We may take a basis $(\xi_e^1, \cdots, \xi_e^n)$ for T_eG and extend these to left-invariant vector fields $(\xi^1, \cdots, \xi^n)$ which then form a basis for T_gG for all $g \in G$. Thus we have a trivialization of TG, i.e., all Lie groups G are *parallelizable*.

The tangent bundle being trivializable gives one relatively easy obstruction to a given manifold having a Lie group structure. In the compact setting, by Poincaré-Hopf, if $\chi(M)$ is nonzero, then M does not even have one non-vanishing vector field so M certainly cannot be parallelizable.

Left-Invariant Integration

Let M^n be a smooth manifold, Ω_M^n the space of n-forms. $\text{Diff}(M) \curvearrowright \Omega_M^n$ by $\phi \cdot \omega = (\phi^{-1})^*\omega$ so when G acts on M, we have a space of G-invariant forms $(\Omega_M^n)^G \subseteq \Omega_M^n$. In particular, when $G = M$ and with our favorite action of left translation, we have $(\Omega_G^n)^G \subseteq \Omega_G^n$. Then (as above) there is a linear isomorphism $(\Omega_G^n)^G \xrightarrow{\text{ev}} \wedge^n (T_eG)^*$ given by restricting to the fiber above the identity of $\wedge^n T^*G$. Thus, given $\omega_e \neq 0 \in \wedge^n (T_eG)^*$ we get a left-invariant *volume form* (a nowhere vanishing top degree form) $\omega \in (\Omega_G^n)^G$, hence we can integrate compactly supported functions $f \in \mathcal{O}_{G,c}$ $f \mapsto \int_G f\omega$. If G is itself compact, then we can normalize so that $\int_G \omega = 1$.

Recall that a measure being Borel just means that the σ-algebra is generated by open sets.

We can think of this as coming from a left-invariant Borel measure μ_G on G. In fact, on any locally compact topological group G, there is a left-invariant Borel measure on G called the *Haar measure*, which is unique up to scale after the imposition of some additional regularity criteria.

In particular, Haar's theorem states that there is a unique (up to a constant) additive measure on G that is left-invariant, assigns finite values to every compact set, is *outer regular*, meaning that for a Borel set $S \subseteq G$

$$\mu(S) = \inf_{U \text{ open}} \mu(U)$$

and *inner regular*, meaning that for an open set $U \subseteq G$

$$\mu(U) = \sup_{K \text{ compact}} \mu(K)$$

One Parameter Subgroups

Definition 2.10.1: Immersed Lie subgroups

A *immersed Lie subgroup* H in a Lie group G is a pair (H, ϕ) of a Lie group H and an immersive Lie homomorphism $\phi : H \to G$.

Example 2.10.2: Irrational Line on a Torus

Let $T^n = \mathbb{R}^n/\mathbb{Z}^n$, $0 \neq v \in \mathbb{R}^n$, and consider the map $\mathbb{R} \to T^n$ given by $t \mapsto tv$. If all entries of v are rational, then the image will eventually wrap around and repeat itself periodically (i.e. the image is a circle). If there is an irrational slope (e.g. $v = (1, \pi) \in \mathbb{R}^2$) then the image will have no self intersections and will have instead have a dense image in T^n (this is nonobvious and will be shown later).

Recall that a smooth map is an immersion if its derivative is everywhere injective. For this case, it suffices that $D_e\phi$ is injective.

Problem Session

Isaac and I are running the problem session this week, so I've written up my solution for the problem I'm talking about.

Exercise 2.10.3

Let G be a compact Lie group with a linear representation on a finite-dimensional complex vector space V giving rise to a Lie group homomorphism $\rho : G \to \mathrm{GL}(V)$.

1. Prove that there exists a Hermitian inner product on V such that $\rho(G) \in \mathrm{U}(V)$ for all $g \in G$.
2. Prove that if $U \subseteq V$ is a G-invariant vector subspace of V then there is a complementary G-invariant subspace U'. Deduce that V decomposes as a direct sum of irreducible representations V_i (where a representation is irreducible if the only G-invariant subspaces are 0 and V itself).
3. If G is abelian, show that the irreducible summands V_i are lines.

PROOF : 1. Recall that $\rho(g) \in \mathrm{U}(V)$ if $\langle \rho(g)u, \rho(g)v \rangle = \langle u, v \rangle$ for all $g \in G$, all u, v.

The idea is to pick a random Hermitian inner product $\langle -, - \rangle_{\text{old}}$ on V and construct a new one $\langle -, - \rangle$ by averaging using the left-invariant volume form on G. In particular, we have

$$\langle u, v \rangle = \int_G \langle \rho(g)u, \rho(g)v \rangle_{\text{old}} \operatorname{vol}_G(g)$$

Then, we have that

$$\langle \rho(h)u, \rho(h)v \rangle = \int_G \langle \rho(gh)u, \rho(gh)v \rangle_{\text{old}} \operatorname{vol}_G(g) = \int_G \langle \rho(g)u, \rho(g)v \rangle_{\text{old}} \operatorname{vol}_G(gh^{-1}) = \int_G \langle \rho(g)u, \rho(g)v \rangle_{\text{old}} \operatorname{vol}_G(g) = \langle u, v \rangle$$

I called this *Weyl's Unitary Trick* and Tim disagrees; it seems that people commonly (almost universally) call this Weyl's unitary trick but Weyl's actual trick is replacing a noncompact group with a compact one, using *Hurwitz's* averaging trick.

2. Let $U \subseteq V$ be G-invariant, and let U' be the orthogonal subspace to U with respect to $\langle -, - \rangle$. $V = U \oplus U'$ as a vector space; to extend this to a direct sum of representations we need only show that G sends U' to itself. Since U' is defined as $0 \neq u' \in U \iff \langle u', u \rangle = 0$ for all $u \in U$, it follows that for all $g \in G$,

$$\langle \rho(g)u', \rho(g)u \rangle = \langle u', u \rangle = 0$$

so $\rho(g)u'$ is orthogonal to $\rho(g)u \in U$ and therefore $\rho(g)u' \in U'$. It follows that one can keep peeling off subrepresentations until the remaining factors are irreducible.

3. Here we use a key fact from linear algebra: commuting matrices are simultaneously diagonalizable. Since the $\rho(g)$ for $g \in G$ are obviously commuting they have a common eigenbasis $v_1, \cdots, v_n$ such that $\rho(g)v_i = \lambda_i^g v_i$. Then it is clear that each $\mathbb{C}v_i$ is an irreducible subrepresentation of G. ■

In fact it is true that A and B (unitary) are simultaneously diagonalizable iff they commute. To see the forward direction, note that diagonal matrices commute so

$$AB = (PD_AP^{-1})(PD_BP^{-1}) = PD_AD_BP^{-1} = PD_BD_AP^{-1} = BA$$

For the other direction, note that unitary matrices are diagonalizable so diagonalize A; then for any of its eigenvectors v we have

$$A(Bv) = BAv = \lambda Bv$$

i.e Bv is also an eigenvector of A with eigenvalue λ. If we assume that A has distinct eigenvalues (and therefore one-dimensional eigenspaces), then $Bv = \lambda' v$ and we are done. The general case is somewhat more involved, see these notes by Keith Conrad for the details.

One Parameter Subgroups and the Exponential Map

Lecture 10: September 23rd

Definition 2.11.1: One Parameter Subgroups

A *one parameter subgroup* of a Lie group G is a Lie homomorphism $\theta : \mathbb{R} \to G$.

θ will be an immersed Lie subgroup if and only if $\frac{d\theta}{dt}|_{t=0} \neq 0$.

Example 2.11.2

For a vector space V, $t \mapsto tv$ for any $v \in V$ is a one parameter subgroup as above. The same is true in the quotient of V by a lattice, also as above.

Example 2.11.3

In $\mathrm{GL}(V)$, $t \mapsto \exp(tA)$ is a one parameter subgroup for all $A \in \mathrm{End}(V)$ where exp is the matrix exponential. This is a homomorphism since $\exp(tA)\exp(sA) = \exp((t+s)A)$.

Note that if θ is a one parameter subgroup, $\xi_e = \frac{d\theta}{dt}|_{t=0} \in T_eG$, then $T_{\theta(t)}G \ni \dot{\theta}(t) = \xi_{\theta(t)}$ where ξ is the unique left-invariant vector field extending ξ_e with $\xi_g = (D_eL_g)\xi_e$. The idea of the proof is that, since θ is a homomorphism, $\theta(s+t) = \theta(s)\theta(t)$, so

$$\dot{\theta}(t) = \frac{\partial}{\partial s}\bigg|_{s=0} \theta(s+t) = \frac{\partial}{\partial s}\bigg|_{s=0} \theta(s)\theta(t) = \text{“}\dot{\theta}(0)\theta(t)\text{”}$$

The scare quotes are to indicate that one has to find the correct interpretation to make the terminal expression meaningful.

Conversely, given $\xi_e \in T_eG$, there exists a solution θ to $\dot{\theta}(t) = \xi_{\theta(t)}$ which is a one parameter subgroup. To see this, note that this is the equation for the flow of a vector field ξ on G which automatically has short-time existence (and full existence for G compact), and uniqueness for any connected interval containing 0. The initial conditions are $\theta(0) = e$ and $\dot{\theta}(0) = \xi$, so given ξ, we get a solution on an interval $(-a, a) \subseteq \mathbb{R}$, and we can extend it to $(-2a, 2a)$ by setting $\theta(t) = \theta(\frac{t}{2})^2$ so we can extend to a solution on all of $\mathbb{R}$. Thus, there is a bijective correspondence between one parameter subgroups of G and T_eG by $\theta \mapsto \dot{\theta}(0)$.

If $G \subseteq \mathrm{GL}(V)$ is an embedded Lie subgroup, $\xi \in T_eG \subseteq \mathrm{End}(V)$, then the inverse to this bijection is given by the matrix exponential since

$$\frac{d}{dt}\exp(t\xi) = \xi\exp(t\xi)$$

clearly exhibits a solution to $\dot{\theta}(t) = \xi_{\theta(t)}$. We need to show that $\exp(t\xi) \in G$; the idea is to use the fact that $\xi\exp(t\xi) \in (T_eG)\cdot\exp(t\xi)$ which we can use to check that the path $t \mapsto \exp(t\xi)$ remains tangent to G.

The Exponential Map

Collectively, the one parameter subgroups of G form a map $\Theta : T_eG \times \mathbb{R} \to G$ characterized by

$$\frac{\partial}{\partial t}\Theta(\xi_e, t) = \xi_{\Theta(\xi_e,t)} \qquad \Theta(\xi_e, 0) = e \implies \frac{\partial}{\partial t}\bigg|_{t=0} \Theta(\xi_e, t) = \xi_e$$

Note that we are thinking of flows not as diffeomorphisms of the underlying manifold but as flows of a specific point.

Flows of vector fields on a manifold vary smoothly as the initial data is perturbed (the vector field itself and the starting point) so Θ is smooth.

Definition 2.11.4: The Exponential Map

The *exponential map* of G is the map $\exp_G : T_eG \to G$ given by $\xi \mapsto \Theta(\xi, 1)$ the time one flow of the vector field induced by ξ.

Evidently, the exponential map is smooth since Θ is. Note that if $\rho : G \to H$ is a Lie homomorphism, then $\rho \circ \exp_G = \exp_H(D_e\rho)$, i.e, the exponential construction is *natural* with respect to Lie homomorphisms. $t \mapsto \exp_G(t\xi_e)$ is the unique one parameter subgroup with derivative ξ_e characterized by $\frac{d}{dt}|_{t=0} \exp_G(t\xi) = \xi$.

Moreover, $D_0 \exp_G : T_eG \to T_eG$ is equal to the identity, so by the inverse function theorem, $\exp_G$ is a local diffeomorphism i.e. $\exp_G$ restricts to a diffeomorphism from some neighborhood of $0 \in T_eG$ to a neighborhood of $e \in G$. Thus, G comes with a distinguished choice of chart near the identity and therefore distinguished charts everywhere.

Note that $\exp_G$ is usually not injective; for example we may consider $T^n = \mathbb{R}^n/\mathbb{Z}^n$ as above, and $\exp^{-1}(0) = \mathbb{Z}^n$. However, $\exp_G$ is surjective when certain adjectives are available for G:

G connected and compact suffices for $\exp_G$ to be surjective, but there are other sufficient conditions as well. $\exp_{\mathrm{GL}_n(\mathbb{C})}$ is surjective so compactness is not necessary, but connectedness clearly is necessary.

Example 2.11.5

For SO(3), consider $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$. $\mathfrak{so}(3)$ consists of traceless skew-symmetric matrices and SO(3) consists of rotations of angle α around an axis defined by some unit vector v denoted $R_{v,\alpha}$. Pick $A \in \mathrm{SO}(3)$ with $Ae_3 = v$, so we can write

$$R_{v,\theta} = A \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} A^{-1} = \exp\big(A \cdot \theta L \cdot A^{-1}\big) = A\exp(\theta L)A^{-1}$$

where $L = \left(\begin{smallmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ and therefore $\exp(\theta L) = R_{e_3,\theta}$. Thus $\exp_{\mathrm{SO}(3)}$ is surjective.

Lecture 11: September 25th

Missed this lecture as well. Animals living in my walls prevented me from getting sleep.

Note however that exp is not surjective for $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$:

Example 2.11.6

$\mathfrak{sl}_2(\mathbb{C})$ consists of traceless 2×2 complex matrices, and any such matrix is conjugate to either a diagonal matrix (with eigenvalues a and $-a$) or the Jordan block $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ (this is just the Jordan canonical form theorem). Thus, $\exp A$ for $A \in \mathfrak{sl}_2(\mathbb{C})$ is conjugate to one of the two following forms:

$$\begin{pmatrix} e^{-a} & 0 \\ 0 & e^{a} \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Thus all diagonalizable matrices are in the image of exp, and it turns out that there are only two other conjugacy classes of matrices in $\mathrm{SL}_2(\mathbb{C})$:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

The latter type is not in the image of exp so exp is not surjective.

Naturality of exp

The naturality (i.e. equivariance with respect to Lie group homomorphisms) of exp is enormously consequential:

Theorem 2.11.7

The functor T_e (which acts on maps via D_e) from the category of connected Lie groups to the category of real vector spaces is faithful, i.e., if $\rho, \rho' : G \to H$ have the same derivative at the identity, they are equal.

PROOF : By naturality, $\rho \circ \exp_G = \exp_H(D_e\rho) : T_eG \to H$ and since $D_e\rho = D_e\rho'$, $\rho \circ \exp_G = \rho' \circ \exp_G$. Therefore, $\rho = \rho'$ on the subgroup generated by the image of $\exp_G$, but the image of $\exp_G$ is a neighborhood of e and therefore generates G, so $\rho = \rho'$. ■

Note that we are not claiming that $\exp_G$ is surjective.

Corollary 2.11.8

If two embedded Lie subgroups $H, H' \subseteq G$ share the same tangent space, then their identity components coincide.

The relevant maps here are the two inclusion maps which agree on T_e by assumption. Note that T_e is not full, i.e, surjective on Hom spaces; it will turn out that a subspace $V \subseteq T_e\,\mathrm{GL}_n(\mathbb{R})$ is realized as the tangent space of some Lie immersion iff V is closed under conjugation by $\mathrm{GL}_n(\mathbb{R})$. More importantly, T_e as described is not full due to omitting some important structure in the target category, the Lie bracket. We will later augment T_e with this data in order to get an equivalence of categories (if we assume our Lie groups are 1-connected).

Another consequence of naturality is the following:

Lemma 2.11.9: Surjectivity Criterion

Suppose $\dim G = \dim H$, with H connected, and suppose there exists a Lie group homomorphism $\rho : G \to H$ with discrete kernel. Then ρ is surjective, and so induces an isomorphism $G/\ker\rho \to H$.

PROOF : It suffices by Lemma 2.5.9 to show that $D_e\rho$ is an isomorphism. Since $\dim G = \dim H$, it then suffices to show that $\ker D_e\rho = 0$. Since $\rho \circ \exp = \exp \circ D_e\rho$, if $v \in \ker D_e\rho$, $\exp(v) \in \ker\rho$, and the one-parameter subgroup generated by v also lies in $\ker\rho$; but $\ker\rho$ is discrete, so the 1-parameter subgroup must be the trivial one, and $v = 0$. ■

The Closed Subgroup Theorem

We are now ready to prove the closed subgroup theorem:

Theorem 2.12.1: Closed Subgroup Theorem

Let H be a closed subgroup of G. Then H is an embedded Lie subgroup.

Set $\mathfrak{g} = T_eG$; the strategy will be to identify the subspace $\mathfrak{h} \subseteq \mathfrak{g}$ which will become T_eH. One guess might be $\mathfrak{h} = \exp^{-1}(H)$ but for example for $G = S^1 \times S^1$, $\exp^{-1}(e)$ is a lattice in $\mathfrak{g}$ when we should expect it to be a point. The non-origin lattice points in $\exp^{-1}(e)$ are therefore to be avoided, and one way to tell them apart from $(0,0)$ is by the fact that the one-parameter subgroup generated by any of them is nontrivial. Thus, the correct notion to consider is

$$\mathfrak{h} = \{v \in \mathfrak{g} : \exp(\mathbb{R}v) \subseteq H\}$$

This definition will end up working, but we will have to massage it a little first:

Definition 2.12.2

A unit vector $v \in S(\mathfrak{g})$ (i.e. of norm one for some randomly selected norm) is called a *limited H-direction* if there exists a subsequence (v_n) in $\mathfrak{g}$ with $\exp v_n \in H$, $v_n \to 0$, and $\frac{v_n}{\|v_n\|} \to v$. The limiting H-directions will form a subspace $L_H \subseteq S(\mathfrak{g})$.

I'm following the notes here but I assume that L_H is meant to be a subset rather than a subspace.

Lemma 2.12.3

For $v \in S(\mathfrak{g})$, v is a limiting H-direction iff $\exp(tv) \in H$ for all $t \in \mathbb{R}$ (so $\mathfrak{h} = \mathbb{R} \cdot L_H$).

PROOF : If $\exp(tv) \in H$ for all t, then v is the limiting direction of the sequence $\frac{v}{n} \to 0$ so $v \in S(\mathfrak{g})$. Conversely, for $t \in \mathbb{R}$ and $v \in L_H$, and a sequence (x_n) of positive reals converging to 0, there exists a sequence (m_n) such that $m_n x_n \to t$ (e.g. $m_n = \lfloor t/x_n \rfloor$). Thus, we may pick m_n so that $m_n \|v_n\| \to t$ and therefore $m_n v_n \to tv$, so $\exp(m_n v_n) \to \exp(tv)$. Since $\exp(m_n v_n) = \exp(v_n)^{m_n} \in H$, and H is closed, $\exp(tv)$ also lies in H. ■

Lemma 2.12.4

$\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$.

PROOF : Above, we established that $\mathfrak{h} = \mathbb{R} \cdot L_H$, so $\mathfrak{h}$ is closed under scalar multiplication. It remains only to show that $\mathfrak{h}$ is closed under addition.

To that end, consider a chart $\phi : (U, e) \to (U', 0)$ based around the identity $e \in G$ where V is a normed vector space. Group multiplication extends to this chart (provided U is sufficiently small as to be closed under multiplication) as $m : U' \times U' \to U'$ with $m(0,0) = 0$, $m(x, 0) = x$, and $m(0, y) = y$. One can show (by considering $D_0 m$) that $m(x, y) = x + y + o(\|x\| + \|y\|)$, so, applying this principle to an exponential chart (i.e, $U' = \exp^{-1}(U)$) we have

$$\log(\exp tu \cdot \exp tv) = t(u + v) + o(t)$$

Suppose we have limiting H-direction u, v whose sum is nonzero, and set $w := \frac{u+v}{\|u+v\|}$. We want w to be a limiting H-direction as well. TO that end, set $\gamma(t) = \log(\exp(tu)\exp(tv)) \in \mathfrak{g}$ for small t, then $\exp \gamma(t) \in H$, while $\frac{\gamma(t)}{t} \to u + v$ as $t \to 0$ by the above. Finally, $\|\gamma(t)\| = \|t\|\|u + v\| + \eta(t)$ where $\eta(t) = o(t)$ so

$$\frac{\gamma(t)}{\|\gamma(t)\|} = \frac{\gamma(t)}{t\|u + v\| + \eta} = \frac{\gamma(t)}{t\|u + v\|} + o(t) \implies \frac{\gamma(t)}{\|\gamma(t)\|} \to w$$

as $t \to 0^+$. Thus, setting $w_n = \gamma(\frac{1}{n})$, $\frac{w_n}{\|w_n\|} \to w$ and $\exp w_n \in H$, so w is a limiting H-direction as well. ■

Lemma 2.12.5

Set $U_r := \{x \in \mathfrak{g} : \|x\| < r\}$. For any $r > 0$, the subset $\exp(U_r \cap \mathfrak{h}) \subseteq H$ is a neighborhood of e.

PROOF : Fix r and a splitting $\mathfrak{g} = \mathfrak{h} \oplus V$ with elements $x \in \mathfrak{g}$ written as $x = u + v$, $u \in \mathfrak{h}$, $v \in V$. Consider the map $\phi : \mathfrak{g} \to G$ given by $\phi(u + v) = \exp(u)\exp(v)$ with $D_e\phi = \mathrm{id}$. By the inverse function theorem, for small $r > 0$, $\phi : U_r \to U'_r := \phi(U_r)$ is a diffeomorphism.

Assuming the lemma is false, there is some sequence $x_n = u_n + v_n$ with $x_n \in U_{\frac{1}{n}}$ and v_n not converging to 0 such that $\exp(x_n) \in H$ (i.e. there is a sequence $x_n \in U_{\frac{1}{n}}$ not inside the intersection with $\mathfrak{h}$ whose exponential lands in H, contradicting the fact that $\exp(U_r \cap \mathfrak{h}) \subseteq H$ is a neighborhood of e). But then $\exp(u_n)\exp(v_n) \in H$ and $\exp(u_n) \in H$ hence $\exp(v_n) \in H$, so, passing to a subsequence, we may assume that $\frac{v_n}{\|v_n\|}$ has a limit $v \in S(\mathfrak{g})$. Then $v \in L_H \cap S(V) = \emptyset$ which is a contradiction. ■

This is enough to give us the closed subgroup theorem:

PROOF : Choose r sufficiently small such that $\exp : U_r \to U_r'$ is a diffeomorphism, and thus $\exp(U_r \cap \mathfrak{h}) \subseteq U_r' \cap H$. By the above, there exists a neighborhood U'' of $e \in G$ s.t $U'' \subseteq U_r'$, and $\exp(U_r \cap \mathfrak{h})$ contains $U'' \cap H$. We can set $W = \exp^{-1}(U'')$ so that $\exp|_W$ carries $W \cap \mathfrak{h}$ diffeomorphically onto $U'' \cap H$, and therefore exp is a submanifold chart centered on $e \in H$. For any other $h \in H$, $L_h(W)$ is an open subset admitting the submanifold chart $\exp \circ L_h^{-1}$. ■

There is an analogous result for complex Lie groups:

Corollary 2.12.6

If H is a closed subgroup of the complex Lie group G, and $\mathfrak{h}$ constructed as above is a complex subspace of $\mathfrak{g}$, then H is an embedded complex Lie subgroup.

PROOF : We constructed H as a submanifold above using submanifold charts, so it remains only to show that $H \subseteq G$ is a complex submanifold. By complex geometry, it in fact suffices to show that for all $h \in H$, $T_hH \subseteq T_hG$ is a complex subspace, but since $T_hH = (D_eL_h)(T_eH)$, and $L_h : G \to G$ is holomorphic, D_eL_h is $\mathbb{C}$-linear so T_hH is a complex subspace of T_hG. ■

I'm not sure what particular result is being referenced when stating that a submanifold is a complex submanifold iff its tangent spaces are complex subspaces.

Abelian Lie Groups

Lecture 12: September 27th

Missed *this* class because I was sick. I will probably attend another lecture for this class one day.

As a sort of capstone to our study of the basic objects and results about Lie groups, we will use the exponential map to give a classification of abelian Lie groups.

Theorem 2.12.7

Let A be a connected, abelian Lie group; then $\exp : T_eA \to A$ is a covering homomorphism.

Corollary 2.12.8

A connected abelian Lie group is isomorphic to a product $\mathbb{R}^m \times T^n$ where $T = S^1 = \mathbb{R}/\mathbb{Z}$; in particular, a connected and compact abelian Lie group is a torus.

This follows immediately by the above since exp induces an isomorphism $T_eA/\ker\exp \cong A$ where $\ker\exp$ is a discrete additive subgroup of T_eA (discrete since it is a covering map). A discrete subgroup D of a d-dimensional vector space V is of the form $D = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_n$ for some $n \leq d$ and $\mathbb{Z}$-linearly independent vectors v_i. $D \otimes_{\mathbb{Z}} \mathbb{R}$ is some vector subspace V' and $V'/D \cong T^n$; fixing V'' s.t $V = V' \oplus V''$ we then have that

$$V/D = V'/D \oplus V''/D = V'/D \oplus V'' = \mathbb{R}^{D-n} \times T^n$$

The proof of the theorem itself is omitted, but requires only the fact that $\exp a \exp b = \exp(a+b)$ for $a, b \in T_eA$, via the identity $\exp a \exp b = (\exp \frac{a}{n})^n (\exp \frac{b}{n})^n$. This also gives us the following:

Theorem 2.12.9

Let G be a compact Lie group, A an abelian Lie subgroup. The identity component of the closure, $\overline{A_0}$, is then an embedded torus, and we can then conclude that every compact Lie group contains non-trivial torus subgroups.

The maximal torus in a compact Lie group will become important later when we discuss classification results.

The proof of this theorem is also omitted.

Lie Groups **Fall 2024**

Lie Algebras

PROFESSOR TIM PERUTZ ABHISHEK SHIVKUMAR

We will now discuss more formally and abstractly the properties of the Lie algebra T_eG of a Lie group G. Lie algebras and their correspondence with Lie groups once sufficient adjectives are enforced are the key to proving the famous classification result for semisimple Lie groups via Dynkin diagrams.

Let k denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition 3.0.1: Representations

A k-linear *representation* of a Lie group G is a k-vector space V and a Lie homomorphism $\rho : G \to \mathrm{GL}(V)$ that acts by $g \cdot v := \rho(g)v$. A map of representations is a natural transformation, or a map $\lambda : V_1 \to V_2$ such that the following diagram commutes:

$$\begin{tikzcd} V_1 \arrow[r, "\rho_1"] \arrow[d, "\lambda"'] & V_1 \arrow[d, "\lambda"] \\ V_2 \arrow[r, "\rho_2"'] & V_2 \end{tikzcd}$$

A representation (V, ρ) (often we will abusively refer to the representation as V) has a dual, $V^* := \mathrm{Hom}_k(V, k)$ with $g \in G$ acting by

$$g \cdot \lambda := \lambda(g^{-1} \cdot -)$$

where the inverse is required to make this a left action. We can also form the direct sum of representations (V_1, ρ_1) and (V_2, ρ_2) with the action given by

$$(\rho_1 \oplus \rho_2)(g)(= \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}$$

We can also form the tensor product of representations $(V_1 \otimes_k V_2, \rho_1 \otimes \rho_2)$ in the obvious way, and thus (using tensors and duals) $\mathrm{Hom}_k(V_1, V_2)$ has a natural G-action given by $g \cdot \lambda = \rho_2 \circ \lambda \circ \rho_1^{-1}$. The subspace of $\mathrm{Hom}_k(V_1, V_2)$ on which G acts trivially consists of precisely those $\lambda : V_1 \to V_2$ which are maps of representations $(V_1, \rho_1) \xrightarrow{\lambda} (V_2, \rho_2)$. Finally, for a representation V, G acts on $\wedge^k(V)$ as in the case of tensor products.

I suppose $\mathrm{Sym}^n(V)$ also carries a natural G-representation but I don't think it will come up here.

Definition 3.0.2: Decomposable and Reducible

A representation V is called *decomposable* if $V = V_1 \oplus V_2$ with $V_i \neq 0$. V is *reducible* if there exists $U \subset V$ a proper nonzero subspace s.t. $\rho(G)(U) \subseteq U$ for all $g \in G$. V is *irreducible* if there are no such subspaces.

Note that irreducible implies indecomposable.

Lemma 3.0.3: Schur's Lemma

If (V_1, ρ_1) and (V_2, ρ_2) are irreducible representations of G, then a G-map $\lambda : V_1 \to V_2$ is either zero or an isomorphism. Thus if (V, ρ) is irreducible, then any G-map $\mu : V \to V$ is in the center of $\mathrm{GL}(V)$ (by equivariance) and hence scalar (a multiple of the identity matrix).

Proof : Let $\lambda : V_1 \to V_2$ be as above and consider $\ker \lambda$; since λ is G-equivariant (by the definition of a map of representations), for any $v \in \ker \lambda$, $\lambda(g \cdot v) = g \cdot \lambda(v) = 0$ so $\ker \lambda$ is a (possibly trivial) subrepresentation. Since V_1 is irreducible, either $\ker \lambda = 0$ or $\ker \lambda = V_1$. ■

For finite-dimensional representations (all we're going to care about) of *finite groups*, there is no difference between irreducible and indecomposable essentially because we can simultaneously diagonalize all of the matrices. The existence of nontrivial Jordan blocks obstructs an indecomposable representation from being irreducible: let $G = \mathbb{R}$ and $V = \mathbb{R}^2$ acting by

$$\rho(\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

ρ is indecomposable since for V to be a direct sum of representations, $\rho(\lambda)$ must diagonalize (or break up into block diagonal matrices in general), but this is clearly impossible. On the other hand, ρ is clearly reducible since $g \cdot e_1 = e_1$ for all $g \in G$.

We need not have a Lie group in order to run into this trouble; an infinite discrete group such as $\mathbb{Z}$ mapping a generator to some Jordan block as above would suffice.

As noted in Exercise 2.10.3, we can use a hermitian inner product to take break up a representation V into the direct sum of a subrepresentation and its orthogonal complement with respect to the inner product; the same is true for a real inner product over $\mathbb{R}$.

Also noted in the discussion following Exercise 2.10.3, we may find an inner product on which G acts by isometries by averaging:

$$\langle v_1, v_2 \rangle = \int_G (\rho(g)v_1, \rho(g)v_2) d\mu_G$$

where $d\mu_G$ is the left-invariant volume form, and $(-,-)$ is some randomly chosen real or hermitian inner product. Hence, in the compact case, irreducibility and indecomposability coincide.

Tim's notes advise us to prove that μ is scalar by extending scalars to the algebraic closure and considering $\mu - xI$; perhaps this is useful to prove that $Z(\mathrm{GL}_n)$ consists of scalars?

Again, this is basically just about the ability to diagonalize our matrices, which we can do for a unitary or orthogonal representation, and we can force our representations to be unitary or orthogonal by building such an inner product as above.

The Adjoint Representation

Suppose a Lie group G acts on a manifold M i.e. we have a smooth action map $\alpha : G \times M \to M$ that satisfies $\alpha(g_2 g_1, x) = \alpha(g_2, \alpha(g_1, x))$ and $\alpha(e, x) = x$. Taking the "partial derivative" D_2 of α with respect to $x \in M$, i.e. computing $D\alpha$ along $\{g\} \times M$, we get a linear map

$$\delta_g(x) := (D_2\alpha)_{(g,x)} = (D\alpha|_{\{g\}\times M})_x : T_x M \to T_{\alpha(g,x)} M$$

Linearizing naïvely, in a coordinate chart near x we have $\alpha(g, x + tv) = \alpha(g, x) + t\delta_g(x)(v) + O(t^2)$; the properties of the action together with the chain rule then imply that

$$\delta_{g_2 g_1}(x) = \delta_{g_2}(\alpha(g_1, x)) \circ \delta_{g_1}(x)$$

The case we will be interested in is when G acts on itself by conjugation with action map $c : G \times G \to G$ given by $c(g,h) = ghg^{-1}$ and $c(g_2 g_1) = c(g_2) \circ c(g_1)$. Fixing g, the derivative with respect to h is

$$\delta_g(h) = (D_2 c)_{(g,h)} : T_h G \to T_{ghg^{-1}} G$$

Definition 3.1.1: The Adjoint Representation

We define

$$\mathrm{Ad}(g) = \delta_g(e) : T_e G \to T_e G$$

to be the *adjoint representation* of G on its Lie algebra.

By the above definition, $\mathrm{Ad}(g_2 g_1) = \mathrm{Ad}(g_2) \circ \mathrm{Ad}(g_1)$, and $\mathrm{Ad}(e) = \mathrm{id}$, so $\mathrm{Ad} : G \to \mathrm{GL}(T_e G)$ is a homomorphism, i.e., a linear representation of G.

Example 3.1.2

For G abelian, conjugation is the trivial action, so $\mathrm{Ad}(g) = \mathrm{id}$ is also trivial.

There's some mildly confusing usage of $o(t)$ here and above where I think $O(t^2)$ is more appropriate; perhaps I am introducing mistakes into Tim's correct notes but so it goes. Since t is a small deformation parameter here $o(t)$ and $O(t^2)$ are conveying the same information but I still find $O(t^2)$ easier to parse since only one of these is still true when t is large.

For matrix Lie groups, the adjoint representation has a simple explicit formula, which we can derive as follows: viewing tangent vectors at a point $x \in M$ as tangency-classes of paths $\gamma : (-\epsilon, \epsilon) \to M$ mapping 0 to x, and the derivative of a map Φ at a point x as $D_x \Phi = \frac{d}{dt}|_{t=0} \Phi \circ \gamma$ for some such curve γ, note that we have a distinguished set of curves through $e \in G$ given by the one-parameter subgroups. In particular, for $\xi \in T_e G$, $t \mapsto \exp(t\xi)$ is a path through e whose derivative at $t = 0$ is precisely ξ, so we may compute $\mathrm{Ad}(g)\xi$ as

$$\mathrm{Ad}(g)\xi = \left.\frac{d}{dt}\right|_{t=0} g \exp(t\xi) g^{-1}$$

Since exp is locally invertible, we may pass to a log-chart (in order to *actually* compute a derivative, we must do this) that maps a neighborhood $U \ni e$ diffeomorphically to a neighborhood of $0 \in T_e G$:

$$\mathrm{Ad}(g)\xi = \left.\frac{d}{dt}\right|_{t=0} \log\bigl(g \exp(t\xi) g^{-1}\bigr)$$

The operator norm $\|\cdot\|_{\mathrm{op}}$ is defined as

$$\|g\|_{\mathrm{op}} = \sup_{\|v\|=1} \|gv\|$$

So we have that $\log\bigl(g \exp(t\xi) g^{-1}\bigr) = \mathrm{Ad}(g)\xi + O(t^2)$. This is as much as we can do for an abstract Lie group G, but if $G \subseteq \mathrm{GL}(V)$, then

$$\exp \xi = I + \xi + \frac{\xi^2}{2!} + \frac{\xi^3}{3!} + \cdots \text{ and } \log g = (g - I) - \frac{(g-I)^2}{2} + \frac{(g-I)^3}{3} - \cdots$$

when $\|g\|_{\mathrm{op}} < 1$, so, for $\xi \in T_e G \subseteq \mathfrak{gl}(V)$ we have

$$\mathrm{Ad}(g)\xi = \left.\frac{d}{dt}\right|_{t=0} \log\bigl(g \exp(t\xi) g^{-1}\bigr) = \left.\frac{d}{dt}\right|_{t=0} \log\bigl(g(I + t\xi + O(t^2)) g^{-1}\bigr) =$$

$$\left.\frac{d}{dt}\right|_{t=0} t g \xi g^{-1} + O(t^2) = g \xi g^{-1}$$

Thus, for matrix Lie groups, $\mathrm{Ad}(g)$ is simply conjugation by g.

Example 3.1.3

For $\mathrm{SO}(n)$, $\mathrm{Ad}(g)\xi = g\xi g^{-1}$. As a sanity check, we want to make sure that $g\xi g^{-1}$ is skew-symmetric when g is orthogonal:

$$(g\xi g^{-1})^T = (g^{-1})^T \xi^T g^T = g(-\xi)g^{-1} = -g\xi g^{-1}$$

using the fact that $g \in \mathrm{SO}(n)$.

The Lie Bracket

Lecture 13: September 30[th]

Out sick again. Last time, I promise.

Another piece of structure on the Lie algebra T_eG comes from examining the commutator map $C : G \times G \to G$ given by $C(g,h) = ghg^{-1}h^{-1}$ in a neighborhood of (e,e).

Lemma 3.2.1

The multiplication map $m : G \times G \to G$ has derivative at the identity $(D_{(e,e)}m)(\xi,\eta) = \xi + \eta$ for $\xi, \eta \in T_eG$. The inversion map $i : G \to G$ has derivative at the identity $D_e i(\xi) = -\xi$.

Proof : Since $m(g,e) = g$, $(D_{(e,e)}m)(\xi,0) = \xi$ and similarly $(D_{(e,e)}m)(0,\eta) = \eta$ and the first assertion follows by linearity. For inversion, note that $m \circ (\mathrm{id}_G \times i) : G \to G$ is the constant function $g \mapsto e$, so, by the chain rule, $D_{(e,e)}m \circ (\mathrm{id}_{T_eG} \times D_e i) = 0$; using the above, this implies that $\xi + D_e i(\xi) = 0$ from which the result follows. ■

Note that C is just the following composite:

$$C : G \times G \xrightarrow{(m, m\circ(i\times i))} G \times G \xrightarrow{m} G$$

where, explicitly, we have $(g,h) \mapsto (gh, g^{-1}h^{-1}) \mapsto ghg^{-1}h^{-1}$ as desired. Therefore, we may compute $D_{(e,e)}C$ using the above result:

$$D_{(e,e)}C : T_eG \times T_eG \xrightarrow{(\xi,\eta)\mapsto(\xi+\eta,-\eta-\xi)} T_eG \times T_eG \xrightarrow{(\alpha,\beta)\mapsto\alpha+\beta} T_eG$$

whence it is easily seen that this composite is the zero map.

Lecture 14: October 2[nd]

Apparently there was some trouble with the proofs on the 30[th], so we did a lot of recap/redos today. As such, not really clear what the demarcating line is for each lecture, so I'm sticking it here.

Setting c to be the composite

$$c : T_eG \times T_eG \xrightarrow{\exp\times\exp} G \times G \xrightarrow{C} G \xrightarrow{\log} T_eG$$

which is well-defined on some small neighborhood of $(0,0)$, we may Taylor expand to second order:

$$c(tx,ty) = c(0,0) + t(D_{(0,0)}c)\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}t^2(D^2_{(0,0)}c)\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) + O(t^3)$$

By the above, $c(0,0) = 0$ (since $\log(I) = 0$) and $D_{(0,0)}c = 0$, so only the quadratic term survives, and we therefore want to inspect the term $\frac{1}{2}t^2(D^2_{(0,0)}c)\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right)$. By linearity, we have

$$(D^2_{(0,0)}c)\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = (D^2_{(0,0)}c)\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}\right) + \\ 2(D^2_{(0,0)}c)\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}\right) + (D^2_{(0,0)}c)\left(\begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}\right)$$

Since $c(x,0) = 0$ and $c(0,y) = 0$, the first and third terms above both vanish (since traveling along, say, y while fixing $x = 0$ yields no change), so

$$c(tx,ty) = t^2(D^2_{(0,0)}c)\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}\right) + O(t^3)$$

We can then define:

I think the preceding calculation was the source of contention in the lecture I missed on the 30^{th}, some minor details missing or incorrect that ruined the calculation. This does not seem to be a particularly natural way to come up with the Lie bracket.

Definition 3.2.2: Lie Bracket

The *Lie bracket* on T_eG is defined as the map

$[-.-] : T_eG \times T_eG \to T_eG$ given by $[x,y] = (D^2_{(e,e)}c)\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}\right)$

When we want to consider T_eG together with its bracket operation, we write it as $\mathfrak{g}$.

Note that the Taylor expansion of c is now

$$c(tx,ty) = t^2[x,y] + O(t^3)$$

The bracket is bilinear by construction since it arises as a derivative. The symmetry of mixed partial derivatives tells us that $(D^2_{(e,e)}c)$ is invariant under interchange of $\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right)$. Since $c(x,0) = 0$, $[x,x] = 0$, so by bilinearity, $[-,-]$ is *skew-symmetric*, i.e.,

$$[x,y] + [y,x] = 0$$

In terms of c, this means that $(D^2_{(e,e)}c)$ picks up a sign if you replace its arguments above with $\left(\begin{smallmatrix} y \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right)$.

Example 3.2.3

For an abelian Lie group G, the commutator map is constant, hence the bracket on $\mathfrak{g}$ is trivial.

Example 3.2.4

For a matrix Lie group $G \subseteq \mathrm{GL}(V)$ we can explicitly compute the bracket using the Taylor expansions for exp and log as in our above computation of Ad for matrix Lie groups:

$$c(t\xi, t\eta) = \log\left(e^{t\xi}e^{t\eta}e^{-t\xi}e^{-t\eta}\right) = \\ \log\left(I + t^2(\xi\eta - \eta\xi) + O(t^3)\right) = t^2(\xi\eta - \eta\xi) + O(t^3)$$

hence

$$[\xi, \eta] = \xi\eta - \eta\xi$$

Example 3.2.5

Let $\mathrm{Aff}(V)$ denote the set of affine transformations of a vector space V i.e. self maps of the form $g(x) = Ax + v$, which are constructed as semidirect products born from the short exact sequence

$$1 \to V \to \mathrm{Aff}(V) \to \mathrm{GL}(V) \to 1$$

with the splitting section given by the inclusion $\mathrm{GL}(V) \hookrightarrow \mathrm{Aff}(V)$, and with the corresponding short exact sequence of vector space

$$0 \to T_0V \to T_e \mathrm{Aff}(V) \to \mathfrak{gl}(V) \to 0$$

split by the inclusion $\mathfrak{gl}(V) \hookrightarrow T_e \mathrm{Aff}(V)$. Since $V \subseteq \mathrm{Aff}(V)$ is evidently abelian, its bracket is trivial. The bracket of A coincides with the bracket on $\mathfrak{gl}(V)$ i.e. the commutator. It remains, then to compute the bracket of $v \in V$ with $A \in \mathfrak{gl}(V)$.

Fix $v \in V = T_0V$ and $A \in \mathfrak{gl}(V)$ and consider

$$C(v, A)(x) = e^{-tA}(e^{tA}x + tv) - tv = x - t^2Av + O(t^3)$$

hence $[v, A] = -Av$.

In the problem session, Isaac proved that $\mathrm{Aff}_G(V) = V \rtimes G$ which has an evident Lie group structure via an embedding into $\mathrm{GL}(V \times \mathbb{R})$, and that $T_e \mathrm{Aff}_G(V) = T_eG \times V$ for G a closed subgroup of $\mathrm{GL}(V)$.

Another interpretation/definition of the Lie bracket comes from Taylor expanding the multiplication of near-identity elements in a Lie group G:

Proposition 3.2.6

$$\log(\exp(t\xi)\exp(s\eta)) = t\xi + s\eta + \frac{1}{2}st[\xi, \eta] + O(3)]$$

where $O(3)$ denotes all third order and higher terms in both variables.

Proof : Taking a Taylor expansion to second order, we have

$$\log(\exp(t\xi)\exp(s\eta)) = t\xi + s\eta + As^2 + Bst + Ct^2 + O(3)$$

where the constants A, B, C absorb the $\frac{1}{2!}$ Taylor coefficients, and depend on ξ and η but not t and s, so

$$\log(\exp(-t\xi)\exp(-s\eta)) = -t\xi + -s\eta + As^2 + Bst + Ct^2 + O(3)$$

and therefore

$$\log(\exp(t\xi)\exp(s\eta)\exp(-t\xi)\exp(-s\eta)) = 2As^2 + 2Bst + 2Ct^2 + O(3)$$

Comparing with the above quadratic Taylor expansion for the commutator, $A = C = 0$ and $2B = [\xi, \eta]$. ∎

There is one other important (and defining) fact about the Lie bracket:

Proposition 3.2.7

The Lie bracket satisfies the *Jacobi identity*:

$$[[\alpha, \beta], \gamma] + [[\beta, \gamma], \alpha] + [[\gamma, \alpha], \beta] = 0$$

In the case of matrix Lie groups, this is a straightforward calculation using the fact that the bracket is a commutator. We will (over the coming weeks and months) give three proofs of this result, the first of which is as follows:

Lemma 3.2.8: Hall-Witt Identity

For a group G, let $(x, y) = xyx^{-1}y^{-1}$ and $x^y = yxy^{-1}$. Then we have

$$((x, y), z^y)((y, z), x^z)((x, z, y^x)) = e$$

I am omitting the proof because it amounts to "write it out and you can see that it is true." In search of the actual meaning of this identity I found this blog post by Danny Calegari with some geometric motivation that I don't yet find particularly convincing.

The Jacobi identity for $[-, -]$ then follows by applying the Hall-Witt identity to the triple $(x, y, z) = (\exp t\alpha, \exp t\beta, \exp t\gamma)$ and taking the leading (cubic) term in the Taylor expansion. Then, we are finally ready to define:

Definition 3.2.9: Lie Algebras

A *Lie algebra* over a base commutative ring k is a k-module V together with a k-bilinear map $V \times V \to V$ $(x, y) \mapsto [x, y]$ that is skew-symmetric and satisfies the Jacobi identity. A map of Lie algebras is a linear map respecting the bracket.

Evidently, by the above discussion, $\mathfrak{g} = (T_eG, [-, -])$ is a Lie algebra. Note that the bracket is natural, in the sense that it commutes with $D_e\rho$ for Lie group homomorphisms $\rho : G \to H$:

$$D_e\rho[\xi, \eta] = [D_e\rho(\xi), D_e\rho(\eta)]$$

To conclude, we have a functor $\mathcal{L}$: LieGroups $\to$ LieAlgebras$_{\mathbb{R}}$. This functor is the central feature of Lie theory, and with mild adjectives is an equivalence of categories.

Here Tim recounts an anecdote from an interview with Deligne where he (Deligne) recounts learning group theory from Jacques Tits, who starts to prove that the center of a group is normal (which is a trivial calculation), stops himself, and says something to the effect of "A normal subgroup is one that is stable under inner automorphisms. Since I have been able to define the center, it is invariant under *all* automorphisms of the data, including inner ones."

The point being that there's nothing really to say about naturality of the bracket; since it is well-defined without any arbitrary choices, it has no choice but to be natural. Tim says "nothing can happen!"

Deligne follows this anecdote by saying that "That Tits did not need to go through a step-by-step proof, but instead could just say that symmetry makes the result obvious, has influenced me a lot. I have a very big respect for symmetry, and in almost every one of my papers there is a symmetry-based argument."

Adjoint Representation Redux

We now have two ways of thinking about $[-,-]$ on $\mathfrak{g}$:

- it is the leading (quadratic) term in the Taylor expansion of the commutator map for G, in logarithmic coordinates
- it is (twice) the quadratic term in the expansion of group multiplication near the identity, in logarithmic coordinates

There is another perspective on the Lie bracket coming from the adjoint representation: differentiate the map $g \mapsto \mathrm{Ad}(g) : G \to \mathrm{GL}(\mathfrak{g})$ defining

$$\mathrm{ad} = D_e \,\mathrm{Ad} : \mathfrak{g} \to T_I \,\mathrm{GL}(\mathfrak{g}) = \mathrm{End}(\mathfrak{g})$$

Since Ad is a Lie group homomorphism, its derivative is a map of Lie algebras (by naturality), so

$$\mathrm{ad}\,\xi \circ \mathrm{ad}\,\eta - \mathrm{ad}\,\eta \circ \mathrm{ad}\,\xi = \mathrm{ad}[\xi, \eta]$$

Lecture 15: October 7^{th}

My first guess for how to calculate $\mathrm{ad}(\eta)\xi$ was to set $g = I + t\eta$ which would give you the same answer (agrees with exp to first order), but does not actually define a path in G. Would have to mess around with coordinates probably. This means I have not really internalized the exponential map.

As always, ad will take a special form for matrix groups $G \subseteq \mathrm{GL}(V)$; we already know that $\mathrm{Ad}(g)\xi = g\xi g^{-1}$ so we may compute $\mathrm{ad}(\eta)\xi$ by expanding $e^{t\eta}\xi e^{-t\eta}$ to first order in t:

$$e^{t\eta}\xi e^{-t\eta} = (I+t\eta)\xi(I-t\eta)+O(t^2) = \xi+t(\eta\xi-\xi\eta)+O(t^2) \implies \mathrm{ad}(\eta)\xi = \eta\xi-\xi\eta$$

Thus $\mathrm{ad}\,\eta = [\eta, -]$, and in fact, this holds in general:

Proposition 3.3.1

For all Lie groups G, the adjoint representation of $\mathfrak{g}$ is given by

$$\mathrm{ad}(\eta) = [\eta, -]$$

where $[-,-]$ is the Lie bracket on $\mathfrak{g}$.

PROOF : We may realize $\mathrm{ad}(\eta)$ as a mixed partial derivative using the definition of Ad:

$$\mathrm{ad}(\eta)\xi = \frac{\partial^2}{\partial s \partial t}\bigg|_{(0,0)} \log(\exp(s\eta)\exp(t\xi)\exp(-s\eta))$$

in logarithmic coordinates. On the other hand, we have

$$[\eta, \xi] = \frac{\partial^2}{\partial s \partial t}\bigg|_{(0,0)} \log(\exp(s\eta)\exp(t\xi)\exp(-s\eta)\exp(-t\xi))$$

and we may expand

$$\exp(s\eta)\exp(t\xi)\exp(-s\eta)\exp(-t\xi) = \exp\big(st[\eta, \xi] + o(t^2, s^2)\big)$$

so

$$\exp(s\eta)\exp(t\xi)\exp(-s\eta) = \exp\big(st[\eta,\xi] + o(s^2,t^2)\big)\exp(t\xi)$$

and the two expressions have the same st term in their expansion. ■

This definition of $[-,-]$ gives another proof of the Jacobi identity:

$$[[\xi,\eta],\zeta] = [\xi,[,\eta,\zeta]] - [\eta,[\xi,\zeta]]$$

via $\operatorname{ad}\xi \circ \operatorname{ad}\eta - \operatorname{ad}\eta \circ \operatorname{ad}\xi = \operatorname{ad}[\xi,\eta]$, and then apply skew-symmetry.

Differential Geometry Perspectives

Yet another perspective on the Lie bracket comes from differential geometry; in particular, it arises as a specialization of the *Lie derivative*, which has a very general definition which we will explore in some specific cases.

Let M be a manifold, $\mathcal{O}_M$ the RR-algebra of smooth functions $M \to \mathbb{R}$. Let X be a vector field on M, then X generates a (short-time, at least) flow $\phi_t^X \in \mathrm{Diff}(M)$ via the ODE

$$\frac{d}{dt}\phi_t^X(x) = X(\phi_t^X(x))$$

where $x \mapsto \phi_t^X(x)$ is an integral curve for X through the point x.

Short time depends on your definition of small (and on M, obviously). When M is compact (without boundary?) we have flows for all time. Tim refers to "small-time" as "weasel words."

$\mathrm{Diff}(M)$ then acts on $\mathcal{O}_M$ by pullbacks (today, a right action):

$$f \cdot \phi = \phi^* f = f \circ \phi$$

where $\phi \in \mathrm{Diff}(M)$ and $f \in \mathcal{O}_M$. Then, the Lie derivative along the vector field X of the function f is given by

$$\mathcal{L}_X f = \frac{d}{dt}\bigg|_{t=0} f \cdot \phi_t^X$$

which one can check is well-defined and natural with respect to diffeomorphisms.

In fact, one can show that $\mathcal{L}_X f = df(X)$ (where the previous, abstract definition is a special case of the broader notion of a Lie derivative). To see this, we work in coordinates at $0 \in U \subseteq \mathbb{R}^n$ so that we can write $X : U \to \mathbb{R}^n$ and $\phi_t^X(0) = 0 + tX_0 + O(t^2)$ where X_0 is the value of X at the point 0, from which the result follows after some calculation.

Tim asks us here if we want to see the calculation, but no one speaks up. I did want to see it but I was a coward.

$\mathrm{Diff}(M)$ also acts on $\mathrm{Vect}(M)$ on the right via

$$(X \cdot \phi)_x = D\phi^{-1}(X_{\phi(x)})$$

so we have a Lie derivative on vector fields Y is given by

$$\mathcal{L}_X Y = \frac{d}{dt}\bigg|_{t=0} Y \cdot \phi_X^t$$

On the other hand, we have the bracket of vector fields; thinking of vector fields X, Y as derivations of $\mathcal{O}_M$ (i.e., locally partial differential operators or directional derivatives) we may define

$$[X,Y]f = X(Y(f)) - Y(X(f))$$

In a coordinate chart with coordinates x^i, X and Y are $\mathbb{R}$-linear combinations of the $\frac{\partial}{\partial x^i}$, and by a similar calculation to the above, one can show that $\mathcal{L}_X Y = [X,Y]$.

Tim offers an abstract perspective on this; for k a commutative ring, A an associative k-algebra, we may define $\mathrm{Der}_k(A)$ to be the vector space of *derivations*, i.e., k-linear maps $\delta : A \to A$ satisfying the Leibniz rule

$$\delta(ab) = \delta(a)b + a\delta(b)$$

The commutator of derivations is itself a derivation, so $\mathrm{Der}_k(A)$ is a Lie algebra. When $k = \mathbb{R}$ and $A = \mathcal{O}_M$, $\mathrm{Der}_{\mathbb{R}}(\mathcal{O}_M) = \mathrm{Vect}(M)$.

I guess the punchline of this is that the fact that our derivations happen to correspond to vector fields is not the fundamental reason why they come with a bracket operation? Abstract derivations form a Lie algebra.

There is a nice geometric interpretation of the Lie bracket of vector fields:

Theorem 3.4.1

$$[X,Y] = \frac{1}{2}\frac{d^2}{dt^2}\bigg|_{t=0} \Phi_t^X \circ \Phi_t^Y \circ \Phi_{-t}^X \circ \Phi_{-t}^Y$$

Lecture 16: October 9th

In other words, $[X,Y]$ measures the failure of the local flows of X and Y to commute. If X and Y are coordinate vector fields locally at some point i.e. $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$ for x, y coordinates in a local chart, then $[X,Y] = 0$ by equality of mixed partial derivatives. In general however, flowing along X for some small time t, then Y, then $-X$, and finally $-Y$, we obtain some small nonzero vector which is exactly the value of the vector field $[X,Y]$ at the given starting point.

Note that $\frac{d}{dt}\Big|_{t=0}$ of the above expression vanishes; this is evident algebraically but can be seen geometrically since taking a single time derivative does not give you the latitude to flow along X *then* Y, there are only terms corresponding to flowing along X then $-X$ (or Y and $-Y$) and this obviously cancels.

Sketch : In a chart centered at $0 \in \mathbb{R}^n$, we may set $\phi_t := \Phi_t^X$ and $\psi_t := \Phi_t^Y$, and define

$$A(t) := \psi_{-t}(0) \quad B(t) := \phi_{-t}(A(t)) \quad C(t) := \psi_t(B(t)) \quad E(t) := \phi_t(C(t)) = \Phi_t^X \Phi_t^Y \Phi_{-t}^X \Phi_{-t}^Y(0)$$

Evaluating the second time derivative of this final expression obtains our result. ■

Specializing to the case $M = G$ a Lie group, we have $\mathrm{Vect}(G)^G \subseteq \mathrm{Vect}(G)$ the space of G-invariant vector fields, which is closed under the bracket $[-,-]$, so we get another bracket on $\mathfrak{g} = \mathrm{Vect}(G)^G$ which coincides with the others by the above theorem. Since exp describes the flow associated to some left-invariant vector fields ξ, η, we have

$$\xi \circ \eta - \eta \circ \xi = \frac{1}{2} \frac{d^2}{dt^2}\bigg|_{t=0} \exp(t\xi)\exp(t\eta)\exp(-t\xi)\exp(-t\eta)$$

which matches the above expression of the bracket and the previous description of $[-,-]$ as the quadratic approximation to the commutator in logarithmic coordinates.

Thus, we now have five different points of view on the Lie bracket:

- it is the leading (quadratic) term in the Taylor expansion of the commutator map for G, in logarithmic coordinates
- it is (twice) the quadratic term in the expansion of group multiplication near the identity, in logarithmic coordinates
- it is related to ad the derivative of the adjoint action via naturality
- it is the bracket (commutator) of left-invariant vector fields on G
- it is the Lie derivative of one vector field along another

Associated to these different points of view, we also have three proofs of the Jacobi identity; from the Hall-Witt identity, from the naturality of ad, and from the fact that the bracket of vector fields automatically satisfies the Jacobi identity and it turns out that all of our Lie algebras can be identified with spaces of vector fields.

Invariant Bilinear Forms

Set $\mathfrak{g} := \mathcal{L}G = (T_eG, [-,-])$.

Definition 3.5.1: Invariant Bilinear Forms

A *bilinear form* $(-,-) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is called G-invariant if

$$(\mathrm{Ad}(g)x, \mathrm{Ad}(g)y) = (x, y)$$

and is called $\mathfrak{g}$-invariant if

$$(\mathrm{ad}(\xi)x, y) + (x, \mathrm{ad}(\xi)y) = 0$$

Note that G invariance implies $\mathfrak{g}$-invariance by differentiating the first equation. If G is connected, the converse is true.

Example 3.5.2: The (Cartan-)Killing Form

The Cartan-Killing or (more commonly) Killing form is a $\mathfrak{g}$-invariant form on any finite dimensional Lie algebra $\mathfrak{g}$ given by

$$\kappa_{\mathfrak{g}}(x, y) = \mathrm{Tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y)) = \mathrm{Tr}(z \mapsto [x, [y, z]])$$

$\kappa_{\mathfrak{g}}$ is evidently bilinear and symmetric since $\mathrm{Tr}\, AB = \mathrm{Tr}\, BA$. $\kappa_{\mathfrak{g}}$ is also natural with respect to Lie algebra isomorphisms.

In the case where $\mathfrak{g} = \mathcal{L}G$ and our Lie algebra isomorphism $\theta = \mathrm{Ad}(g) : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$, we see that naturality implies that $\kappa_{\mathfrak{g}}$ is G-invariant.

Example 3.5.3: $\mathfrak{su}(2)$ Killing Form

$\mathfrak{su}(2)$ has a classical basis given by the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ which satisfy commutation relations

$$[\sigma_i, \sigma_j] = \sum_k \varepsilon_{ijk}\sigma_k$$

for ε_{ijk} the *Levi-Civita symbol* that gives the sign of the permutation ijk; note that the sum on the right hand side contains only one term when $i \neq j$. A quick calculation in these coordinates (using symmetry of the indices) gives us that

$$\mathrm{Tr}(\mathrm{ad}\,\sigma_i \circ \mathrm{ad}\,\sigma_j) = -2\delta_{ij}$$

so

$$\kappa_{\mathfrak{su}(2)}\left(\sum_i a_i\sigma_i, \sum_j b_j\sigma_j\right) = -2a \cdot b$$

which is negative definite.

We used a slightly modified version of $\kappa_{\mathfrak{su}(2)}$ in our proof of Proposition **??** where the coefficient of $\mathrm{Tr}(AB)$ is $-\frac{1}{2}$ rather than -2 in order to make the Pauli matrices orthonormal with respect to the form.

Lecture 17: October 11th

One definition of a *nilpotent* Lie algebra is that $\mathrm{ad}\,x$ is nilpotent endomorphism for all $x \in \mathfrak{g}$, so $\kappa(x, x) = 0$ for all x in a nilpotent Lie algebra, so $\kappa = 0$. Moreover, If G is compact with discrete center then $\kappa_{\mathfrak{g}}$ is negative definite. Thus, the Killing form reflects some interesting properties of $\mathfrak{g}$.

Example 3.5.4: $\mathfrak{sl}_2(\mathbb{R})$ Killing Form

We give the following basis for $\mathfrak{sl}_2(\mathbb{R})$:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In this basis, we may check that $\kappa_{\mathfrak{sl}_2(\mathbb{R})}$ has matrix

$$\kappa_{\mathfrak{sl}_2(\mathbb{R})} = 4\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

which is non-degenerate and indefinite (note that $\mathrm{SL}_2(\mathbb{R})$ is non-compact).

Generalizing these examples, we have the following:

Theorem 3.5.5

For G a compact Lie group with Lie algebra $\mathfrak{g}$ with $Z(\mathfrak{g}) = 0$ (where $Z(\mathfrak{g}) := \{x \in \mathfrak{g} : \operatorname{ad} x = 0\}$), $\kappa_{\mathfrak{g}}$ is negative definite.

There is a moderate converse to this theorem that we will develop.

PROOF : G acts on $\mathfrak{g}$ via Ad, and since G is compact, we can find a G-invariant positive definite inner product $(-,-)$ on $\mathfrak{g}$ (by averaging, as in the unitary trick). Let $e_1, \cdots, e_n$ be an orthonormal basis with respect to this inner product. We want to show that for $0 \neq x \in \mathfrak{g}$, $\operatorname{Tr}\operatorname{ad}(x)^2 < 0$:

$$\operatorname{Tr}\operatorname{ad}(x)^2 = \sum_i (\operatorname{ad}(x)^2 e_i, e_i) = \sum_i (\operatorname{ad}(x)e_i, \operatorname{ad}^*(x)e_i) = -\sum_i (\operatorname{ad}(x)e_i, \operatorname{ad}(x)e_i) = -\sum_i \|\operatorname{ad}(x)e_i\|^2$$

where we use the fact that, since $(-,-)$ is Ad-invariant, it is ad-invariant i.e. $(\operatorname{ad}(x)e, e') + (e, \operatorname{ad}(x)e') = 0$ by definition, so $\operatorname{ad}^* = -\operatorname{ad}$.

The final term above is clearly negative if $\operatorname{ad}(x) \neq 0$, but $Z(\mathfrak{g}) = 0$ so $x \neq 0 \implies \operatorname{ad}(x) \neq 0$, so we are done. ■

Normal Embedded Subgroups

Given an embedded Lie subgroup $H \subseteq G$ we have $\mathfrak{h} \subseteq \mathfrak{g}$ a Lie subalgebra i.e. one which is closed under the bracket, $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. If H is normal in G, then conjugation by $g \in G$ preserves H, i.e., $\operatorname{Ad}(G)$ preserves $\mathfrak{h}$. Differentiating this action we find that $\operatorname{ad}(\mathfrak{g})$ also preserves $\mathfrak{h}$, so $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$. This motivates the following definition:

Definition 3.6.1: Ideals

Let $\mathfrak{g}$ be a k-Lie algebra, then an *ideal* $\mathfrak{a} \subseteq \mathfrak{g}$ is a k-submodule $\mathfrak{a}$ such that $[\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{a}$.

Tim remarks that something with no ideals should be called perhaps *amoral* or *cynical* rather than simple.

Thus, if $\mathfrak{g}$ has no nontrivial ideals then G has no normal subgroups of positive dimension. A Lie algebra with no nontrivial ideals is called *simple*, analogously to the adjective for groups.

For example, one can check that $\mathfrak{su}(2)$ is simple so $\mathrm{SU}(2)$ has no positive dimensional closed normal subgroups. Note that this does not imply that $\mathrm{SU}(2)$ is simple (it is not) since the Lie algebra does not detect e.g. discrete subgroups of $\mathrm{SU}(2)$.

By contrast, consider a composition series

$$\langle e \rangle = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$$

for a solvable Lie group G where $G_i \triangleleft G_{i+1}$ and G_{i+1}/G is an abelian Lie group. Then the corresponding series of Lie algebras

$$0 = \mathfrak{g}_0 \subseteq \mathfrak{g}_1 \subseteq \cdots \subseteq \mathfrak{g}_n = \mathfrak{g}$$

with $[\mathfrak{g}_{i+1}, \mathfrak{g}_i] \subseteq \mathfrak{g}_i$ and $\mathfrak{g}_{i+1}/\mathfrak{g}_i$ abelian (i.e. with trivial bracket). A Lie algebra admitting such a chain is called solvable (as is a Lie group admitting the corresponding chain).

The notions of solvability, nilpotence, along with the related notions of simplicity and semisimplicity are key to the study of Lie algebras, and will occupy the next few lectures.

Nilpotent and Solvable Lie Algebras

Definition 3.7.1: Nilpotence

A Lie algebra $\mathfrak{g}$ is *nilpotent* if there exists a natural number m such that

$$\mathrm{ad}(x_1) \circ \cdots \circ \mathrm{ad}(x_m) = 0$$

for any $x_i \in \mathfrak{g}$. Setting $C^1\mathfrak{g} = \mathfrak{g}$ and $C^{k+1}\mathfrak{g} = [\mathfrak{g}, C^k\mathfrak{g}]$ to be the *lower central series* for $\mathfrak{g}$, a decreasing sequence of ideals, then an equivalent formulation for nilpotence is that the lower central series terminates.

Somehow, a flag here doesn't seem more elegant to me than a matrix computation since the complete flag in question is coming from an ordered basis.

Our basic model for a nilpotent Lie algebra will be the following:

Example 3.7.2

Let $N \subseteq \mathrm{GL}_n(\mathbb{R})$ be the group of upper triangular matrices with 1s on the diagonal, whose Lie algebra $\mathfrak{n}$ consists of strictly upper triangular matrices; $\mathfrak{n}$ is evidently nilpotent either by explicit matrix calculations or by consideration of the standard complete flag in $\mathbb{R}^n$ associated to the ordered basis $e_1, \cdots, e_n$, where $F^i = \mathrm{span}(e_1, \cdots, e_n)$ and by noting that $\mathrm{ad}(\mathfrak{n})F^i \subseteq F^{i-1}$.

Note that the adjoint representation of $\mathfrak{n}$ is here indecomposable — not the direct sum of invariant subspaces — but not irreducible.

In fact, many of the features of a generic nilpotent algebra are present in this basic example:

Theorem 3.7.3: Engel

The following are equivalent:

1. $\mathfrak{g}$ is nilpotent
2. $\mathrm{ad}(x)$ is a nilpotent endomorphism for all $x \in \mathfrak{g}$
3. There exists a complete flag $0 = F^0 \subset F^1 \subset \cdots \subset F^n = \mathfrak{g}$ with $\dim F^i = i$ and $\mathrm{ad}(g)(F^i) \subseteq F^{i-1}$.

Sketch : (1) $\implies$ (2) is trivial, and (3) $\implies$ (1) is clear since it shows that $\mathfrak{g}$ has nilpotence degree at most n. The difficult part is (2) $\implies$ (3); one can easily find a filtration preserved by $\mathrm{ad}(x)$ once x is specified, but the point is to find a filtration that works for all x. The complete proof can be found in Serre's *Lie Algebras and Lie Groups*, Chapter V. ■

Lecture 18: October 14^{th}

Remark 3.7.4

Using the *Baker-Campbell-Hausdorff formula*

$$\log(\exp x \cdot \exp y) = x + y + \frac{1}{2}[x,y] + \frac{1}{12}([x,[x,y]] + [y,[y,x]]) + \cdots$$

one gets for $\mathfrak{g}$ nilpotent, a polynomial formula for a Lie group law on the *Lie algebra* $\mathfrak{g}$, i.e., we may realize a Lie group G with given nilpotent Lie algebra by an explicit formula $G \cong \mathfrak{g}$.

Definition 3.7.5: Solvability

Let $D^1\mathfrak{g} = \mathfrak{g}$, $D^{k+1}\mathfrak{g} = [D^k\mathfrak{g}, D^k\mathfrak{g}]$ be the *derived series* for $\mathfrak{g}$. Note that $D^k\mathfrak{g} \subseteq C^k\mathfrak{g}$ and the $D^{k+1}\mathfrak{g}$ are ideals in $D^k\mathfrak{g}$. We say that $\mathfrak{g}$ is *solvable* if the derived series terminates.

Since $D^k\mathfrak{g} \subseteq C^k\mathfrak{g}$, nilpotence implies solvability. Just as Engel's theorem tells us that all nilpotent Lie algebras are not too far from our basic example of upper triangular matrices, the standard example for a Lie algebra that is solvable but not nilpotent is the *Borel* subalgebra $\mathfrak{b}$ of $\mathfrak{gl}_n(\mathbb{R})$ which consists of all upper triangular matrices (with diagonal entries not necessarily 0). Since $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{u}$, $\mathfrak{b}$ is evidently solvable.

An equivalent condition for solvability is the existence of a chain of subalgebras

$$\mathfrak{g} = a_1 \supseteq a_2 \supseteq \cdots \supseteq a_{n-1} \supseteq a_n = 0$$

where a_{i+1} is ideal in a_i and a_i/a_{i+1} is abelian (take $a_i = D^i\mathfrak{g}$); this gives us our counterpart to Engel's theorem for solvable Lie algebras:

Theorem 3.7.6: Lie's Theorem on Solvability

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $k = \overline{k}$ of characteristic 0. Then, $\mathfrak{g}$ is solvable iff there exists a complete flag F^i in $\mathfrak{g}$ such that $\mathrm{ad}(\mathfrak{g})F^i \subseteq F^i$.

Morally, this shows that solvable Lie algebras are basically upper triangular matrices as in our first example.

The proof that $\mathfrak{g}$ is solvable if such a flag exists is essentially the proof that $\mathfrak{b}$ is solvable, the hard direction is that if $\mathfrak{g}$ is solvable, such a flag exists. The idea is similar to the difficult step of Engel's theorem, to find a simultaneous eigenvector $v \in \mathfrak{g}$ for $\mathrm{ad}(x)$ as x varies over $\mathfrak{g}$.

There is another criterion for solvability that is captured by the (Cartan-)Killing form:

Theorem 3.7.7: Cartan's Criterion for Solvability

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field k of characteristic 0; then $\mathfrak{g}$ is solvable iff $\kappa_{\mathfrak{g}}(x,[y,z]) = 0$ for all $x, y, z \in \mathfrak{g}$.

SKETCH : If $\mathfrak{g}$ is solvable, it preserves some complete flag F^i (over $\overline{k}$) by Lie's theorem, so ad induces an action of $\mathfrak{g}$ on each line F^i/F^{i-1} which is necessarily an action by scalars, so $\operatorname{ad}[y,z] = \operatorname{ad} y \operatorname{ad} z - \operatorname{ad} z \operatorname{ad} y$ necessarily annihilates each such line, i.e., $\operatorname{ad}[y,z](F^i) \subseteq F^{i-1}$ which implies that $\operatorname{ad}(x) \circ \operatorname{ad}[y,z](F^i) \subseteq F^{i-1}$. Picking a basis $v_1, \cdots, v_n$ for $\mathfrak{g}$ so that $F^i/F^{i-1} = \operatorname{span}(v_i)$, we see that for an endomorphism A preserving the flag, $\operatorname{Tr} A$ is the sum of the scalars by which A acts on F^i/F^{i-1}, hence $\operatorname{Tr}(\operatorname{ad}(x) \circ \operatorname{ad}[y,z]) = 0$ for all x, y, z.

The converse is trickier; suppose $\operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad}[y,z])$ is identically 0, then we want to prove that $\operatorname{ad}[y,z]$ is always nilpotent in $\operatorname{End}\mathfrak{g}$ which implies that $[\mathfrak{g},\mathfrak{g}]$ is nilpotent by Engel, hence $\mathfrak{g}$ is solvable. The tricky point is to show that $\operatorname{ad}[y,z]$ is nilpotent; a reference as above is Serre (op. cit.). ■

Allegedly the linear algebra lemma used to prove the converse is the following:

Lemma 3.7.8

Let V be a finite dimensional vector space over a field k of characteristic 0, $V \supseteq V' \supseteq V''$ subspaces. Let $\mathfrak{g} = \operatorname{End} V$, $\mathfrak{h} \subseteq \mathfrak{g}$ given by $\mathfrak{h} = \{X \in \mathfrak{g} : X(V') \subseteq V''\}$. If $X \in \mathfrak{h}$ and $\operatorname{Tr}(XY) = 0$ for all $Y \in \mathfrak{h}$, then X is nilpotent.

Simple, Semisimple, and Reductive Lie algebras

Note that if $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ where the $\mathfrak{g}_i$ are ideals of $\mathfrak{g}$, then for $i \neq j$ $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_i \cap \mathfrak{g}_j = 0$.

Definition 3.8.1: (Semi)Simplicity and Reductivity

Let $\mathfrak{g}$ be a Lie algebra over a field k, then $\mathfrak{g}$ is *simple* if its only ideals are 0 and $\mathfrak{g}$ and it is not abelian (i.e, the bracket does not vanish identically). $\mathfrak{g}$ is *semisimple* if it is the finite direct sum of simple ideals. $\mathfrak{g}$ is *reductive* if $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}$ where $\mathfrak{s}$ is semisimple and $\mathfrak{a}$ is abelian.

Thus an abelian Lie algebra is reductive but not semisimple.

Example 3.8.2

$\mathfrak{gl}_n(k) = \mathfrak{sl}_n(k) \oplus kI$ where kI refers to the scalar matrices, so $\mathfrak{gl}_n(k)$ is reductive, as it turns out that $\mathfrak{sl}_n(k)$ is semisimple. Similarly, $\mathfrak{u}(n) = \mathfrak{su}(n) \oplus \mathbb{R}I$ so $\mathfrak{u}(n)$ is reductive, and $\mathfrak{su}(n)$ is *simple.*

Someone asks about reductive *groups* (there is a running joke about the algebraic geometers/Langlands theorists being unwilling to define a reductive group other than by giving various examples) and Tim says that if a Lie group G is connected, any adjective of its Lie algebra is an adjective of the group, but consciously refuses to explain what happens when G is disconnected.

Lecture 19: October 16th

The notions of simplicity, semisimplicity and reductivity can be rephrased in terms of representations; $\mathfrak{g}$ is simple iff ad is a nontrivial irreducible representation (recall that a Lie algebra representation is a map $\theta : \mathfrak{g} \to \operatorname{End} V$ that respects brackets). $\mathfrak{g}$ is semisimple iff ad is completely reducible (a direct sum of irreducibles) with no trivial summands. $\mathfrak{g}$ is reductive iff ad is completely reducible (possibly with some trivial summands).

Here we just list some basic and easy results about these adjectives. I have changed some of the lemmas to propositions to make the coloring more interesting.

Proposition 3.8.3

If K is a compact connected Lie group, then K is reductive.

PROOF : We showed in Exercise 2.10.3 that every representation of K is completely reducible, so Ad is and therefore ad is, and we are done (the key point is compactness). ■

Lemma 3.8.4

If $\mathfrak{g}$ is semisimple, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

PROOF : As noted above, if $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ where the $\mathfrak{g}_i$ are ideals of $\mathfrak{g}$, then for $i \neq j$ $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_i \cap \mathfrak{g}_j = 0$, so we may pass to the case where $\mathfrak{g}$ is simple, in which case $[\mathfrak{g}, \mathfrak{g}]$ is an ideal, and thus equal to 0 or $\mathfrak{g}$. If $[\mathfrak{g}, \mathfrak{g}] = 0$ then $\mathfrak{g}$ is abelian, so $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. ■

Proposition 3.8.5

For $\mathfrak{g}$ semisimple, the decopmosition into simple ideals is unique.

The proof is omitted.

Lemma 3.8.6

If $\mathfrak{g}$ admits and ad-invariant non-degenerate symmetric bilinear form $\sigma : \mathfrak{g} \times \mathfrak{g} \to k$, then $\mathfrak{g}$ is reductive (and remains reductive after extension of scalars to a larger field).

The bit about extending scalars was not in lecture, but added to the notes after the fact as it became necessary when we were discussing the semisimplicity of $\mathfrak{sl}_n(\mathbb{C})$.

PROOF : The key idea is that we may use σ to give us orthogonal complements to decompose $\mathfrak{g}$ into irreducible summands $\mathfrak{g}_i$. Extend scalars to K/k; σ remains non-degenerate and ad-invariant. Note that any ideal $\mathfrak{a}$ in $\mathfrak{g}$ has an orthogonal complement from σ and non-degeneracy implies that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$, so induction on the dimension of $\mathfrak{g}$ gives the desired decomposition of $\mathfrak{g}$ into simple an abelian ideals. ■

We have a preferred ad-invariant form as above ($\kappa_\mathfrak{g}$) and, in fact, one can show that its non-degeneracy gives us something stronger than reductivity:

Theorem 3.8.7: Cartan's Criteria for Semisimplicity

Let k be a field of characteristic 0, $\mathfrak{g}$ a Lie algebra over k. The following are equivalent:

1. $\mathfrak{g}$ is semisimple
2. The radical $\mathrm{rad}(\mathfrak{g})$ is zero
3. The Cartan-Killing form $\kappa_\mathfrak{g}$ is non-degenerate

PROOF : Note that $\mathrm{rad}(\mathfrak{g})$ is defined as the (unique) largest solvable ideal in $\mathfrak{g}$ (it is an exercise to show that this is well-defined).

In the step 1. $\implies$ 2., we use the fact that the unique decomposition of a semisimple Lie algebra $\mathfrak{g} = \oplus_i \mathfrak{g}_i$ into simple ideals implies that any ideal in $\mathfrak{g}$ is a sum of the $\mathfrak{g}_i$ and therefore itself semisimple.

1. $\implies$ 2. Suppose $\mathfrak{s} \subseteq \mathfrak{g}$ is solvable; one can show that any ideal in a semisimple Lie algebra is itself semisimple, so $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$ by Lemma 3.8.4, so the derived series cannot terminate unless $\mathfrak{s} = 0$.

2. $\implies$ 3. Assuming $\mathrm{rad}(\mathfrak{g}) = 0$, set $\mathfrak{r} = \{x \in \mathfrak{g} : \kappa_{\mathfrak{g}}(x, -) = 0\}$, which is the radical of the Cartan-Killing form. We want $\mathfrak{r} = 0$, which is equivalent (over a field) to non-degeneracy of $\kappa_{\mathfrak{g}}$. First, we claim that $\mathfrak{r}$ is an ideal: if $\kappa_{\mathfrak{g}}(x, -) = 0$ (i.e. $x \in \mathfrak{r}$), $y \in \mathfrak{g}$, then we want $[y, x] \in \mathfrak{r}$. By ad-invariance, $\kappa_{\mathfrak{g}}(x, \mathrm{ad}(y)z) + \kappa_{\mathfrak{g}}(\mathrm{ad}(y)x, z) = 0$ and the former term vanishes by assumption so $\kappa_{\mathfrak{g}}(\mathrm{ad}(y)x, z) = \kappa_{\mathfrak{g}}([y, x], z) = 0$ with z arbitrary, so $\mathfrak{r}$ is an ideal.

   Next, we use the fact (this is easy to check) that for any ideal $\mathfrak{h}$, $\kappa_{\mathfrak{g}}|_{\mathfrak{h}} = \kappa_{\mathfrak{h}}$ so $\kappa_{\mathfrak{g}}|_{\mathfrak{r}} = \kappa_{\mathfrak{r}}$. But $\kappa_{\mathfrak{g}}|_{\mathfrak{r}} = 0$ by construction, so we may apply Cartan's criterion for solvability (Theorem 3.7.7) which implies that $\mathfrak{r}$ is solvable hence 0 since $\mathrm{rad}(\mathfrak{g}) = 0$.

Apparently the radical of a bilinear form B is defined by $\{x : B(x, -) = 0\}$.

3. $\implies$ 1. By the above lemma, non-degeneracy of $\kappa_{\mathfrak{g}}$ implies that $\mathfrak{g}$ is at least reductive. If $\mathfrak{g}$ has no nontrivial abelian ideals, then it is semisimple and one can quickly check that the existence of non-trivial abelian ideal makes $\kappa_{\mathfrak{g}}$ degenerate. ■

The deep part of this theorem is that a semisimple Lie algebra always has a non-degenerate Cartan-Killing form, since this result depends both on Cartan's criterion for solvability and the theorems of Engel and Lie. The non-degeneracy of the Cartan-Killing form is essential to the classification of semisimple complex Lie algebras.

Corollary 3.8.8

If K/k is a field extension of characteristic zero fields, and $\mathfrak{g}$ is a Lie algebra over k, then $\mathfrak{g}$ is semisimple iff $\mathfrak{g} \otimes_k K$ is semisimple over K.

We use the above fact that extension of scalars preserves κ and its non-degeneracy.

Example 3.8.9

On $\mathfrak{sl}_n(\mathbb{R})$ it is straightforward to check that there are no nontrivial abelian ideals. Let $\sigma(X, Y) = \mathrm{Tr}(XY)$ (essentially the Killing form up to a constant) which is Ad-invariant hence ad-invariant. We may calculate that

$$\sigma(X, X^T) = \mathrm{Tr}\big(XX^T\big) = \sum_{i,j} x_{ij}^2 > 0$$

for $X \neq 0$ so $\mathfrak{sl}_n(\mathbb{R})$ is semisimple.

In fact $\mathfrak{sl}_n(\mathbb{R})$ is *simple* for $n \geq 2$.

Example 3.8.10

$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{R}) \otimes \mathbb{C}$ is semisimple over $\mathbb{C}$ since $\mathfrak{sl}_n(\mathbb{R})$ is semisimple over $\mathbb{R}$.

Example 3.8.11

$\mathfrak{su}(n)$ is semisimple since $\mathfrak{su}(n) \otimes \mathbb{C} = \mathfrak{sl}_n(\mathbb{C})$ is semisimple.

Compact Lie Groups

By Proposition 3.8.3, we know that a compact Lie group K has reductive Lie algebra, $\mathfrak{k} = \mathfrak{s} \oplus \mathfrak{a}$ and $\kappa(\mathfrak{a}, -) = 0$. In fact, we can say a little more:

Theorem 3.8.12

If $\mathfrak{k}$ is the Lie algebra of a compact Lie group K, then $\mathfrak{k} = \mathfrak{s} \oplus \mathfrak{a}$ is reductive and the Cartan-Killing form on $\mathfrak{s}$ is negative-definite.

The proof is similar to that of Theorem 3.5.5 and is omitted.

There is a converse to this as follows:

Theorem 3.8.13

If G is a connected Lie group whose Cartan-Killing form is definite, then $Z(G)$ is discrete (and in fact finite), G is semisimple, and $G/Z(G)$ is compact (and therefore $\kappa_{\mathfrak{g}}$ is in fact negative-definite).

I missed yet another lecture so I'm not sure what that's in the notes was actually covered in class.

SKETCH : We have a homomorphism $\mathrm{Ad} : G \to \mathrm{O}(\mathfrak{g}, \kappa_{\mathfrak{g}})$ whose kernel $Z(G)$ is discrete (since, otherwise ad would have non-trivial kernel and $\kappa_{\mathfrak{g}}$ would therefore be degenerate), and we may deduce semisimplicity of G from Cartan's criteria. The tricky part is to show that $G/Z(G) = \operatorname{Im}\mathrm{Ad}$ is compact.

Since $\mathrm{O}(\mathfrak{g}, \kappa_{\mathfrak{g}})$ is compact, its closed subgroups are compact, so we need to show that $\operatorname{Im}\mathrm{Ad}$ is closed. One shows this by identifying $\operatorname{Im}\mathrm{Ad}$ with $(\operatorname{Aut}\mathfrak{g})_0$ (which denotes the identity component of the automorphisms of $\mathfrak{g}$) using derivations. ■

The Lie Group-Lie Algebra Correspondence

Lecture 20: October 21st

We are now finally ready to give the equivalence of categories between Lie algebras and Lie groups, decorated with the right adjectives.

Local Lie Groups

This definition is cooked up to capture the structure of the "germ of a Lie group" near the identity. I suppose germ of a Lie group should just mean the germs of the multiplicative and invertible structures as maps.

Definition 3.9.1: Local Lie Groups

A *local Lie group* is a finite-dimensional manifold Γ together with a point $e \in \Gamma$, an open neighborhood $U \ni e$ and smooth maps $\mu : U \times U \to \Gamma$ (multiplication) and $\iota : U \to U$ (inversion) such that the following identities hold for some neighborhood V of e (with $\mu(V \times V) \subseteq V$):

1. $\mu(x, e) = x = \mu(e, x)$ for all $x \in V$ (identity)
2. $\mu(x, \iota(x)) = e = \mu(\iota(x), x)$ for all $x \in V$ (inverses)
3. $\mu(x, \mu(y, z)) = \mu(\mu(x, y), z)$ for all $x, y, z \in V$ (associativity)

A homomorphism of local Lie groups $f : \Gamma \dashrightarrow \Gamma'$ is a smooth map defined on some open neighborhood of $e \in \Gamma$ such that $f(e) = e'$ and $f(\mu(x, y)) = \mu'(f(x), f(y))$ holds near e. Two such homomorphisms are considered equivalent if they agree in some neighborhood of e.

A *local isomorphism* of local Lie groups is a homomorphism $f : \Gamma \to \Gamma'$ such that there exists a local homomorphism $g : \Gamma' \to \Gamma$ so that fg and gf are equivalent to the appropriate identity maps.

A lie group G is evidently a local Lie group with no need to fuss about neighborhoods. Local Lie groups form a category $\text{LieGroups}_{\text{loc}}$ and we can evidently still define the Lie algebra functor $\mathcal{L} : \text{LieGroups}_{\text{loc}} \to \text{LieAlgebras}_{\mathbb{R}}$ where the target category is the category of finite-dimensional real Lie algebras.

Categorical Interlude

An alternative characterization of an equivalence of categories (which is technically a lemma, but is often given as a definition) is that F is fully faithful (injective and surjective, respectively, on Hom-sets) and essentially surjective on objects (for every object $b \in B$ there exists $a \in A$ so that $F(a) \cong b$).

Definition 3.9.2: Equivalence of Categories

A functor $F : A \to B$ is called an *equivalence of categories* if there exists a functor $G : B \to A$ so that FG and GF are both naturally isomorphic to the appropriate identity functors.

The Correspondence Theorems

We have three categories in play: $\text{LieGroups}_{\text{sc}}$ of simply connected Lie groups, $\text{LieGroups}_{\text{loc}}$, and $\text{LieAlgebras}_{\mathbb{R}}$ (everything is finite dimensional). Our main result is that all these categories are equivalent:

Theorem 3.9.3

The following functors are equivalences of categories:

local	The functor $\mathcal{L} : \text{LieGroups}_{\text{loc}} \to \text{LieAlgebras}_{\mathbb{R}}$
local-global	The forgetful functor $\text{LieGroups}_{\text{sc}} \to \text{LieGroups}_{\text{loc}}$
global	The functor $\mathcal{L} : \text{LieGroups}_{\text{sc}} \to \text{LieAlgebras}_{\mathbb{R}}$

Since the third functor is the composite of the first two, if they are equivalences, so is it (this is the reason to introduce local Lie groups in the first place, as a stepping stone in the proof of the global correspondence, which is the one we care about).

These assertions apply to real Lie groups and real Lie algebras, but the same general principles will give an analogous result for complex Lie groups and complex Lie algebras, *mutatis mutandis.*

The Baker-Campbell-Hausdorff Formula

We will vaguely invoke the BCH formula below so we are introducing it here, though it will receive a more full treatment later in the course (hopefully).

Theorem 3.9.4: Baker-Campbell-Hausdorff

For any Lie group G there is an identity

$$\log_G(\exp_G x \cdot \exp_G y) = \sum_{n\geq 1} m_n(x,y)$$

valid for x, y in a neighborhood of $0 \in \mathfrak{g}$ on which $\exp_G$ is a diffeomorphism onto its image and on which the series $\sum_{n\geq 1} m_n$ is absolutely convergent with respect to a chosen norm on $\mathfrak{g}$. $m_n(x,y)$ is a $\mathbb{Q}$-linear combination of terms of the form

$$(\operatorname{ad} x)^{i_1} \circ (\operatorname{ad} y)^{j_1} \circ \cdots \circ (\operatorname{ad} x)^{i_k} \circ (\operatorname{ad} y)^{j_k}(x)$$
$$\text{or } (\operatorname{ad} x)^{i_1} \circ (\operatorname{ad} y)^{j_1} \circ \cdots \circ (\operatorname{ad} x)^{i_k}(y)$$

where $k \leq n$, $i_a, j_a \geq 1$ (except i_1 can be 0), and $\sum_a i_a + \sum_a j_a = n$. Moreover, the coefficients of m_n are independent of the group G.

The first few terms for m_n are as follows:

$$m_1(x,y) = x + y \quad m_2(x,y) = \frac{1}{2}[x,y] \quad m_3(x,y) = \frac{1}{12}([x,[x,y]] + [y,[y,x]])$$

Thus, to second order, $\exp_G x \cdot \exp_G y = \exp_G(x + y + \frac{1}{2}[x,y])$, so there are correction terms for the naïve non-identity $\exp x \cdot \exp y = \exp(x+y)$ which holds for $x, y \in \mathbb{C}$. There are explicit formulae for the m_n, one of which is called *Dynkin's formula* (which we will cover in this course, time permitting).

Sketching the Correspondences

Lecture 21: October 23rd

Proposition 3.9.5

The functor $\mathcal{L} : \text{LieGroups}_{\text{loc}} \to \text{LieAlgebras}_{\mathbb{R}}$ is full.

Sketch : I.e., every map of real Lie algebras is the derivative of a map of local Lie groups. The idea is (as always) to use the exponential map: let $\theta : \mathcal{L}\Gamma \to \mathcal{L}\Gamma'$ be any such map in the target category, and note that $\exp_\Gamma$ maps a neighborhood U of $0 \in \mathfrak{g}$ diffeomorphically onto a neighborhood of $e \in \Gamma$, and set $\Theta = \exp_\Gamma \circ \theta \circ \log_\Gamma$. This is a smooth map and its derivative at e is precisely θ (it's not entirely obvious that this is a homomorphism, but this is where we use the BCH formula, as the terms m_n in the BCH series are preserved by Θ). ∎

Note that for nilpotent Lie algebras, the BCH series terminates i.e. is a polynomial, and we get a global Lie group structure via this construction, not just a local one.

We can convert the "inverse" functor $\mathrm{BCH} : \mathrm{LieAlgebras}_\mathbb{R} \to \mathrm{LieGroups}_{\mathrm{loc}}$ using Dynkin's explicit formula for the Baker-Campbell-Hausdorff expansion by viewing the formula as the definition of a local Lie group structure for a given Lie algebra. One shows the resulting structure is associative, unital, and admits inverses. Proving absolute convergence of the series is required for this, which requires Dynkin's formula. The exponential map for the resulting local Lie group is then precisely $\exp(x) = \sum_{n \geq 0} \frac{x^n}{n!}$ and given a map θ of Lie algebras, we can define Θ a map of local Lie groups by $\Theta(\exp x) = \exp(\theta(x))$ as above.

Proposition 3.9.6

The forgetful functor $\mathrm{LieGroups}_{\mathrm{sc}} \to \mathrm{LieGroups}_{\mathrm{loc}}$ is full.

Tim gives a sketch of the proof of this result, but I have omitted it because I honestly don't care if we're not going to finish proving it.

Corollary 3.9.7

The functor $\mathcal{L} : \mathrm{LieGroups}_{\mathrm{sc}} \to \mathrm{LieAlgebras}_\mathbb{R}$ is full.

There's also some discussion about the "flight recorder" method for establishing an inverse to $\mathcal{L} : \mathrm{LieGroups}_{\mathrm{sc}} \to \mathrm{LieAlgebras}_\mathbb{R}$ in which the Lie group is constructed as a path space in the Lie algebra modulo a certain equivalence relation. We may see this later in the semester, time permitting.

For essential surjectivity, one strategy is the closed subgroup method; i.e., for $\mathfrak{g} = \mathcal{L}G$, $\mathfrak{h}$ a subalgebra, one wishes to prove that $\mathfrak{h}$ is the Lie algebra of a Lie immersion $H \to G$ (and indeed this is true). The first approximation to H is $\exp \mathfrak{h}$ which is almost correct, but it may not be closed (as we have seen, it could even be dense), so one has to construct H from $\exp \mathfrak{h}$ in a slightly more complicated way (this is related to the way in which one argues that the leaves of a foliation are immersed submanifolds). One then applies a hard result known as Ado's theorem which says that any Lie algebra $\mathfrak{g}$ embeds as a Lie subalgebra of a linear Lie algebra $\operatorname{End} V$.

For faithfulness, one of the cases is handled by Theorem 2.11.7 by naturality of exp, and the local version is more or less the same since the derivative of a given map only sees the germ of the Lie group anyway. In any case, the functor we are interested in is faithful so the local versions are less important, as, again, for us, the local Lie groups are only of interest as they make proving the correspondence easier.

Exceptional Isomorphisms

Lecture 22: October 25th

We have the following 3-dimensional Lie groups: $\mathrm{SO}(3)$, $\mathrm{SU}(2)$, $\mathrm{Sp}(1)$ (which are compact), and $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{Sp}_2(\mathbb{R})$, $\mathrm{SL}_1(\mathbb{H})$, $\mathrm{SO}(2,1)_0$, and $\mathrm{SU}(1,1)$ (which are non-compact). Unsurprisingly, there are some redundancies in this list ($\mathrm{Sp}(1) = \mathrm{SU}(2)$ which double covers $\mathrm{SO}(3)$), which will be the focus of our discussion today.

One can pursue the search for exceptional isomorphisms by starting with dimension (as we do here), although this is very coarse; one can do much better using Dynkin diagrams (we should see some of this later).

Theorem 3.10.1

We have the following isomorphisms of complex classical groups:

- $\mathrm{Spin}_3(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{C}) \cong \mathrm{Sp}_2(\mathbb{C})$
- $\mathrm{Spin}_4(\mathbb{C}) \cong \mathrm{Spin}_3(\mathbb{C}) \times \mathrm{Spin}_3(\mathbb{C})$
- $\mathrm{Spin}_5(\mathbb{C}) \cong \mathrm{Sp}_4(\mathbb{C})$
- $\mathrm{Spin}_6(\mathbb{C}) \cong \mathrm{SL}_4(\mathbb{C})$

We have the following isomorphisms of compact classical groups:

- $\mathrm{Spin}(3) \cong \mathrm{SU}(2) \cong \mathrm{Sp}(1)$
- $\mathrm{Spin}(4) \cong \mathrm{Spin}(3) \times \mathrm{Spin}(3)$
- $\mathrm{Spin}(5) \cong \mathrm{Sp}(2)$
- $\mathrm{Spin}(6) \cong \mathrm{SU}(3)$

Recall that $\pi_1 \mathrm{SO}(n) = \mathbb{Z}/2$ for $n > 2$ and $\mathbb{Z}$ for $n = 2$, and $\mathrm{Spin}(n)$ is the unique double cover of $\mathrm{SO}(n)$. Recall also that $\mathrm{SO}(n) \hookrightarrow \mathrm{SO}_n(\mathbb{C})$ is a homotopy equivalence (i.e. $\mathrm{SO}(n)$ is a deformation retract of $\mathrm{SO}_n(\mathbb{C})$) by polar decopmosition, so $\pi_1 \mathrm{SO}(n) = \pi_1 \mathrm{SO}_n(\mathbb{C})$ and thus $\mathrm{SO}_n(\mathbb{C})$ has a unique double cover called $\mathrm{Spin}_n(\mathbb{C})$. It will essentially suffice to prove the isomorphisms of the complex spin groups, and the latter list will follow by arguing that isomorphic complex Lie groups have isomorphic real forms. For the former list, it will in fact essentially suffice to show that $\mathrm{Spin}_6(\mathbb{C}) \cong \mathrm{SL}_4(\mathbb{C})$, and filter this isomorphism down to the lower Spin groups.

$\mathrm{Spin}_3(\mathbb{C})$ is the double cover of $\mathrm{SO}_3(\mathbb{C})$, which we can show is isomorphic to $\mathrm{SL}_2(\mathbb{C})/\pm I$ from which it follows that $\mathrm{Spin}_3(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{C})$. Most of the proofs of these exceptional isomorphisms are of this flavor.

Theorem 3.10.2

The Lie algebras of the complex classical groups are all simple except for $\mathrm{SO}_4(\mathbb{C})$; moreover the simple Lie algebras $\mathfrak{sl}_n(\mathbb{C})$ for $n \geq 2$, $\mathfrak{sp}_{2n}(\mathbb{C})$ for $n \geq 2$, and $\mathfrak{so}_n(\mathbb{C})$ for $n \geq 7$ are all distinct.

Therefore the above list of isomorphisms is essentially exhaustive.

$\mathrm{Spin}_8(\mathbb{C})$ has an exceptional symmetry; an outer automorphism of order 3 called "triality." This is witnessed in an evident order 3 symmetry of its Dynkin diagram. Some googling tells me that $\mathrm{Spin}(n)$ comes with two half-spinor representations and a vector representation (one which restricts to a representation of $\mathrm{SO}(n)$); triality permutes these three for $n = 8$.

$\mathrm{Spin}_6(\mathbb{C}) \cong \mathrm{SL}_4(\mathbb{C})$

Let V be a 4-dimensional $\mathbb{C}$-vector space with volume form $\mathrm{vol} \in \det V$; fixing a basis $e_1, \cdots, e_4$ we can take $\mathrm{vol} = e_1 \wedge e_2 \wedge e_3 \wedge e_4$. $\mathrm{SL}_4(\mathbb{C})$ has an evident action on V, and from V we would like to produce a 6-dimensional vector space on which $\mathrm{Spin}_6(\mathbb{C})$ acts, namely, $\wedge^2 V$ which is spanned by $e_{ij} := e_i \wedge e_j$ for $i < j$. $\wedge^2 V$ carries a quadratic form $q : \wedge^2 V \to \mathbb{C}$ defined by $\alpha \wedge \alpha = q(\alpha)\,\mathrm{vol}$. That $\mathrm{SL}_4(\mathbb{C})$ respects this form is the content of the claimed isomorphism.

We will not show the other isomorphisms in our list but they are generally variations on a theme. For example, to show $\mathrm{Spin}_5(\mathbb{C}) \cong \mathrm{Sp}_4(\mathbb{C})$, we equip V a 4-dimensional $\mathbb{C}$-vector space with a symplectic form and there is a similar calculation ultimately using Lemma 2.5.9. Tim claims the remaining isomorphisms are in fact *restricted* versions of the one we prove here, but I don't see this.

If $A \in \mathrm{SL}(V)$, then

$$(\wedge^2 A)\alpha \wedge (\wedge^2 A)\alpha = (\wedge^4 A)(\alpha \wedge \alpha) = (\det A)(\alpha \wedge \alpha) = \alpha \wedge \alpha$$

Thus, $q((\wedge^2 A)\alpha)\,\mathrm{vol} = q(\alpha)\,\mathrm{vol}$, so $\wedge^2 A$ preserves q and therefore we have a homomorphism $\mathrm{SL}(V) \to \mathrm{SO}(\wedge^2 V, q)$. We will show below that $\ker\lambda = \{\pm I\}$.

Inspecting $\wedge^2 V$ more closely, we have the following basis:

$$\frac{1}{\sqrt{2}}(e_{12} \pm e_{34}) \qquad \frac{1}{\sqrt{2}}(e_{13} \pm e_{24}) \qquad \frac{1}{\sqrt{2}}(e_{41} \pm e_{23})$$

Left to right, the elements corresponding to the $+$ sign are labeled the α_i and the $-$ sign are the β_i, where one can check that the α_i, β_j are wedge-orthogonal and that $\alpha_i^2 = 1$, $\beta_i^2 = -1$. Note that, since q is non-degenerate, restricting λ to the real form induces $\mathrm{SL}_4(\mathbb{R}) \to \mathrm{SO}(3,3)$ (since the α_i square to 1 and the β_i square to -1). Thus we have a homomorphism $\lambda : \mathrm{SL}(V) \to \mathrm{SO}(\wedge^2 V, q)$ given by $\lambda(A) = \wedge^2 A$. Both Lie groups are 15-dimensional, so we should expect a discrete kernel:

We didn't say this in class but this implies that $\mathrm{Spin}(3,3) = \mathrm{SL}_4(\mathbb{R})$.

Proposition 3.10.3

$\ker\lambda = \{\pm I\}$.

PROOF : Suppose $A \in \ker\lambda$ so $\wedge^2 A = \mathrm{id}$, so for any u, v, $(\wedge^2 A)(u \wedge v) = Au \wedge Av = u \wedge v$. Let P be the plane spanned by u, v, and note that $u \wedge v$ fully determines P (either by general geometric algebra or by setting $P = \{x \in V : x \wedge u \wedge v = 0\}$). Thus $A \in \ker\lambda$ preserves every plane $P \subseteq V$, so it preserves all intersections of planes i.e. all lines, hence A is scalar, so $A = \pm I$. ∎

Thus, by Lemma 2.5.9, we have that $\mathrm{SL}_4(\mathbb{C})/\{\pm I\} \xrightarrow{\sim} \mathrm{SO}_6(\mathbb{C})$, and lifting to universal covers gives the desired isomorphism.

Lie Groups **Fall 2024**

Maximal Tori and Root Systems

Professor Tim Perutz Abhishek Shivkumar

Overview

Lecture 23: October 28th

Let G be a compact connected Lie group, then G has an important subgroup T, its *maximal torus subgroup* (as we will show).

Theorem 4.1.1

For T a maximal torus in G compact connected,

$$G = \bigcup_{g \in G} gTg^{-1}$$

This generalizes e.g. the fact from linear algebra that unitary matrices are diagonalizable by unitary matrices. This easily implies that any other maximal torus T' is conjugate to T, so T is an invariant up to conjugacy of G.

Towards these results we will examine the action of $\mathrm{Ad}|_T : T \to \mathrm{GL}(\mathfrak{g})$ (perhaps after complexification) which will give us a decomposition $\mathfrak{g} = \mathfrak{t} \oplus_j P_j$ where the P_j are two-dimensional subspaces called the "root planes," such that for all $t \in T$, $\mathrm{Ad}(t)P_j = P_j$ and $\mathrm{Ad}(t)$ acts by rotation (with respect to the Killing form) so we have homeomorphisms $\rho_j : T \to S^1$ s.t. $\mathrm{Ad}(t)|_{P_j}$ is rotation by $\rho_j(t)$. Explicitly, there exists an oriented orthonormal (with respect to the Killing form) basis u, v that is rotated by ρ_j (perhaps up to sign).

The homomorphisms $\pm\rho_j : T \to S^1 = \mathbb{R}/\mathbb{Z}$ are called *global roots* of (G, T), and the computation of these roots for the compact classical groups will be one of the major focuses for this chapter. We will introduce the *Weyl group* of the pair (G, T) which is the group of inner automorphisms of G that preserve T. The following chapter will examine the geometry of roots in further detail.

We won't get to any representation theory this semester, unfortunately, but Ben-Zvi and Mason-Brown are teaching a two part course in representation theory in the next academic year, which is exciting.

There are four categories of interest in this chapter:

- compact simply-connected Lie groups K
- real negative-definite (referring to κ) Lie algebras $\mathfrak{k}$

- simply connected semisimple complex Lie groups $G_{\mathbb{C}}$
- semisimple complex Lie algebras $\mathfrak{g}_{\mathbb{C}}$

We will have closely related classification results for all four of these categories, and these classifications are essentially all "the same." The first step to this classification is a product decomposition:

Proposition 4.1.2

If $\kappa_{\mathfrak{k}}$ is negative-definite then $\mathfrak{k}$ decomposes as a direct sum of its simple ideals $\mathfrak{k} = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_n$. The same is true for a complex semisimple Lie algebra.

There's some discussion about why we don't call real negative-definite Lie algebras semisimple since they are direct sums of simples. No clear answer is given. The structure theory seems to be worse in general over non-algebraically-closed fields, perhaps this is the reason.

Proposition 4.1.3

If K is a compact, simply connected Lie group, then there exist subgroups $K_1, \cdots, K_n$ that are closed, simple, simply connected, normal, and mutually commuting such that the map $K_1 \times \cdots \times K_n \to K$ is an isomorphism, and the K_i are unique up to ordering. The same is true for a complex semisimple (simply connected) Lie group.

SKETCH : The Lie algebra of K_i must be $\mathfrak{k}_i$ so we can realize $\mathfrak{k}_i$ as the Lie algebra of an immersion $\rho_i : \tilde{K}_i \to K$ with $\tilde{K}_i$ simply connected by our correspondence theorems from the previous chapter. Then $\prod_i \rho_i : \tilde{K}_1 \times \cdots \times \tilde{K}_n \to K$ is a homomorphism since $[\mathfrak{k}_i, \mathfrak{k}_j] = 0$ so the factors commute, and is in fact an isomorphism of Lie groups via the correspondence theorem, and everything else now follows for $K_i = \rho_i(\tilde{K}_i)$. ■

Theorem 4.1.4

The Lie algebra functors $\mathcal{L}$ on compact simply connected and complex simply connected semisimple Lie groups are both equivalences of categories; moreover, $\mathcal{L}$ respects product decompositions into simple factors.

Proof omitted since the only new fact beyond our established correspondence theorems is that a simply connected Lie group is compact iff its Lie algebra has negative-definite Killing form.

Lemma 4.1.5

There is a complexification functor $- \otimes_{\mathbb{R}} \mathbb{C}$ which takes negative-definite real algebras to complex semisimple Lie algebras. This functor respects product decompositions into simple factors, and any complex semisimple Lie algebra $\mathfrak{g}$ has a negative-definite real Lie subalgebra $\mathfrak{k}$ so that $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ so $\mathfrak{g} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$, and $\mathfrak{k}$ is unique up to $(\operatorname{Aut} \mathfrak{g})_0$.

Proof omitted but only for the time being.

Theorem 4.1.6: Classification of simple Lie Algebras

Any complex simple Lie algebra $\mathfrak{g}$ is isomorphic to one of the following:

- $A_n = \mathfrak{sl}_{n+1}(\mathbb{C})$ for $n \geq 1$
- $B_n = \mathfrak{so}_{2n+1}(\mathbb{C})$ for $n \geq 2$
- $C_n = \mathfrak{sp}_{2n}(\mathbb{C})$ for $n \geq 3$
- $D_n = \mathfrak{so}_{2n}(\mathbb{C})$ for $n \geq 4$
- one of E_6, E_7, E_8, F_4, or G_2 (the exceptional Lie algebras)

Note that we have not yet constructed the exceptional Lie algebras.

Passing to Lie groups via the correspondences gives us the classification result for compact Lie groups, once we have constructed the exceptional Lie groups and their Lie algebras. We could also consider similar classification problems, such as:

- Compact Lie groups K up to local isomorphism: a compact connected Lie group is locally isomorphic to $A \times S$ where A is abelian and S is semisimple (for example, consider $\mathrm{Spin}^c(4) = \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)\times U(1)}{\pm(1,1,1)}$ which is locally isomorphic to $\mathrm{SU}(2) \times \mathrm{SU}(2) \times U(1)$ with obvious abelian and semisimple factors).
- Reductive complex Lie groups $G_{\mathbb{C}}$ up to local isomorphism: the classification will follow from the classification of reductive complex Lie algebras.
- Reductive complex Lie algebras $g_{\mathbb{C}}$: reductive complex lie algebras are by definition of the form $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{s}$ where $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{a} = Z(\mathfrak{g})$ so the above classification extends easily.

We can say slightly more about the first case of the above:

Proposition 4.1.7

Any compact connected Lie group K has a finite covering group $A \times S \to K$ where A is a torus and S is compact semisimple.

Maximal Tori

Lecture 24: October 30th

Recall that a torus is a Lie group isomorphic to $T^k = \mathbb{R}^k/\mathbb{Z}^k$ which is equivalent to a Lie group that is compact, connected, and abelian (non-compact connected Lie groups are simply tori $\times\mathbb{R}^k$). We have already discussed tori that are *topologically cyclic* or *monogenic* i.e. G a topological group is topologically cyclic if there exists $g \in G$ s.t $\overline{\langle g \rangle} = G$.

There are two natural lattices (free abelian groups of maximal rank) we can associate to an r-dimensional torus T: the lattice of characters $\mathrm{Hom}(T, S^1)$, and the lattice of cocharacters $\mathrm{Hom}(S^1, T)$. For example, let V be a vector space, Λ a maximal rank lattice, and set $T = V/\Lambda$. Any $V^\vee \ni \lambda : V \to \mathbb{R}$ descends to a homomorphism $T \to S^1$ iff $\lambda(\Lambda) \subseteq \mathbb{Z}$, so we have a map $\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z}) \hookrightarrow \mathrm{Hom}(T, S^1)$ into the character lattice. In fact, this map is a bijection by naturality of the exponential map (or via a covering space argument).

The character lattice is also known as the *weight lattice*.

We can build an inverse map $\mathrm{Hom}(T, S^1) \to \Lambda^* \subseteq V^*$ by $\rho \mapsto (D_e\rho : T_eT = V \to \mathbb{R})$ and show that this lands in Λ^*. There is a similar strategy for the cocharacter lattice which we can identify with Λ.

The natural pairing $\Lambda^* \times \Lambda \to \mathbb{Z}$ is realized intrinsically (for T) as $\mathrm{Hom}(T, S^1) \times \mathrm{Hom}(S^1, T) \to \mathrm{Hom}(S^1, S^1) = \mathbb{Z}$ by composing the two homomorphisms.

Tim relates this to a very specific case of Langlands duality; in particular, semisimple Lie groups G come with a Langlands dual ${}^LG$ and one basic feature of Langlands duality is that the character lattice of one is the cocharacter lattice of the other.

Remark 4.2.1

Any sub-torus $S \subseteq T$ can be realized as the intersection of the kernels of those characters $\chi \in \mathrm{Hom}(T, S^1) = \Lambda^*$ with $\chi(S) = 0$.

Proposition 4.2.2

A vector $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$ generates a 1-parameter subgroup Π^n of T^n of dimension S where $S = \dim_{\mathbb{Q}} \mathbb{Q}\{v_1, \cdots, v_n\}$.

Sketch : The key step is to represent $S = \exp(\mathbb{R}v)$ as an intersection of kernels of characters as in the above remark. ■

Definition 4.2.3: Maximal Tori

A *maximal torus* T in a Lie group G is a torus subgroup such that if $T \subseteq T'$ for T' another torus subgroup, then $T = T'$.

Lemma 4.2.4

A torus $T \subseteq G$ with G compact is maximal iff its Lie algebra $\mathfrak{t}$ is a maximal abelian Lie subalgebra of $\mathfrak{g}$.

Proof : Suppose $\mathfrak{t}$ is maximal. If we have $T' \supseteq T$ with $\mathfrak{t}' \supseteq \mathfrak{t}$, then $\mathfrak{t}' = \mathfrak{t}$ so $T' = T$. If T is maximal, and we have $\mathfrak{t}' \supseteq \mathfrak{t}$, set $A = \exp(\mathfrak{t}')$. $\overline{A}$ (its closure) is then a closed subgroup, so an embedded connected (since it is submanifold with a dense path-connected subset) Lie subgroup of G. G compact implies $\overline{A}$ is compact, and we know that $\overline{A}$ is abelian and connected, and therefore a torus. $T \subseteq \overline{A}$ so $T = \overline{A}$. ■

Proposition 4.2.5

Let G be a compact Lie group. Then, a maximal torus T exists and if $\dim G > 0$ then $\dim T > 0$.

Why are we specifying $\dim G > 0$? What is a zero-dimensional Lie group?

PROOF : Recall that for G of positive dimension, Theorem 2.12.9 implies that G admits an embedded torus subgroup of positive dimension. Take some such torus T of maximal dimension; then T is maximal since if $T' \supseteq T$, they must have the same Lie algebra and are therefore equal. ■

Lemma 4.2.6: Maximality Criterion

For a subgroup $H \subseteq G$, define the invariant subspace of the Lie algebra $\mathfrak{g}^H = \{x \in \mathfrak{g} : \mathrm{Ad}_H x = x\}$. A torus $T \subseteq G$ with G compact is maximal iff $\mathfrak{g}^T = \mathfrak{t}$.

The proof is omitted but not too difficult.

The Road Not Taken

Tim quotes the Robert Frost poem.

We have taken the branch in the road towards compact Lie groups and their maximal tori. The other branch considers complex semisimple/reductive Lie algebras whose corresponding subobjects are maximal *toral* subalgebras. The stories are analogous but there are many technical differences.

Definition 4.2.7: Toral Subalgebras

Let $\mathfrak{g}$ be a Lie algebra over a field k. A subalgebra $\mathfrak{t}$ is called *toral* if for every $x \in \mathfrak{t}$, $\mathrm{ad}\, x \in \mathrm{End}\, \mathfrak{g}$ is a semisimple endomorphism (i.e. it is diagonalizable over $\overline{k}$ iff its minimal polynomial has no repeated roots).

Toral implies abelian, and if $k = \overline{k}$ and $\mathfrak{g}$ is reductive, then a non-zero toral subalgebra exists. The latter fact is somewhat nontrivial to show; one uses the fact from linear algebra that for $\mathfrak{g}$ semisimple over $k = \overline{k}$, any $x \in \mathfrak{g}$ decomposes uniquely as $x = x_s + x_n$ where $\mathrm{ad}\, x_s$ is a semisimple endomorphism and $\mathrm{ad}\, x_n$ is nilpotent, which comes from a sort of Jordan decomposition (the Jordan-Chevalley decomposition).

Roots

Lecture 25: November 4th

Lemma 4.3.1

Let G be a compact abelian Lie group, $\rho : G \to \mathrm{GL}(E)$ a representation on a finite dimensional complex vector space E. Then

$$E = \bigoplus_i E_i$$

where each E_i is a complex line invariant under G and G acts on E_i through a homomorphism $\rho_i : G \to U(1)$ i.e. $\rho(g)e_i = \rho_i(g)e_i$ for $e_i \in E_i$. The ρ_i are called the *weights* of ρ.

We want to apply this to $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}_{\mathbb{C}})$ restricted to the maximal torus $T \subseteq G$, which is evidently compact abelian, and where $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification. We may write $\mathfrak{g}_{\mathbb{C}}$ as the direct sum of $\mathfrak{t}_{\mathbb{C}}$, on which T acts trivially, and lines $E_i = \mathbb{C}\eta_i \subseteq \mathfrak{g}_{\mathbb{C}}$ on which $\mathrm{Ad}(t)\eta_i = \rho_i(t)\eta_i$ with weights $\rho_i : T \to U(1)$.

Recalling our maximality criterion Lemma 4.2.6, $T \subseteq G$ is maximal iff $\mathfrak{g}_{\mathbb{C}}^T = \mathfrak{t}_{\mathbb{C}}$, so for our maximal torus T, the homomorphisms ρ_i are nonconstant. Conversely, if we have such a decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_i E_i$ where the ρ_i are nontrivial, then the torus T must be maximal.

$\mathfrak{g}_{\mathbb{C}}^T = \mathfrak{t}_{\mathbb{C}}$ implies that the ρ_i are nonconstant since if some ρ_i is constant then E_i is outside of $\mathfrak{t}_{\mathbb{C}}$ but fixed by $\mathrm{Ad}(T)$.

The E_i are called *root spaces* in $\mathfrak{g}_{\mathbb{C}}$, and the homomorphisms $\rho_i : T \to U(1)$ are called *global roots*. We call these ρ_i *global* roots as we could also consider $\mathrm{ad} : \mathfrak{t} \to \mathrm{End}\, \mathfrak{g}_{\mathbb{C}}$ which preserves the same lines E_i and acts as $\mathrm{ad}(\xi)\eta_i = \theta_i(\xi)\eta_i$ for $\xi \in \mathfrak{t}$ and $\theta_i : \mathfrak{t} \to i\mathbb{R} \subseteq \mathbb{C}$ the *infinitesimal roots*, and where $\rho_i \circ \exp = \exp(\theta_i)$ via the naturality of exp applied to Ad.

The terminology *roots* for the ρ_i comes from the fact that they are literally the roots of the characteristic polynomial of $\mathrm{Ad}(t)$, $t \in T$.

Real Root Planes

If we do not complexify, $\mathrm{Ad}|_T : T \to \mathrm{GL}(\mathfrak{g})$ no longer decomposes into eigenlines but *does* decompose into eigenplanes. The idea is that if $\rho_i : T \to U(1)$ is a root of the complexification acting on the eigenline $\mathbb{C}\eta_i$, so is $\rho_i^{-1} = \overline{\rho_i}$ acting on $\mathbb{C}\overline{\eta_i}$. Set $u = \eta_i + \overline{\eta_i}$ and $v = i(\eta_i - \overline{\eta_i})$ both of which are evidently invariant under conjugation and therefore lie in $\mathfrak{g} \subseteq \mathfrak{g}_{\mathbb{C}}$, and set $P_i = \mathbb{R}u \oplus \mathbb{R}v \subseteq \mathfrak{g}$ called a *root plane*.

The action is defined as follows: $\mathrm{Ad}(t)\eta_i = e^{i\theta_i(t)}\eta_i$ with $\theta_i : T \to \mathbb{R}$, then

$$\mathrm{Ad}(t)u = u\cos\theta_i(t) + v\sin\theta_i(t) \qquad \mathrm{Ad}(t)v = u\sin\theta_i(t) + v\cos\theta_i(t)$$

There are a handful of natural objections to this construction:

1. The root planes are not well-defined if the eigenvalues of $\mathrm{Ad}\, t$ have multiplicity greater than 1; this can be quickly resolved by verifying that the eigenvalues in fact always have multiplicity 1.

2. Rotation by $\theta_i(t)$ without reference to a form does not make sense; since V is a two-dimensional real vector space, $V \otimes_{\mathbb{R}} \mathbb{C} = L \oplus \overline{L}$, then rotation of V by θ is meaningful as the rotation that induces the automorphism $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ on $V \otimes_{\mathbb{R}} \mathbb{C}$.

3. θ versus $-\theta$: we are taking two roots and calling them ρ_i and $\overline{\rho_i}$ but the assignment of one of these as the actual root and the other as its conjugate is arbitrary. This one turns out to be real, so we only get θ up to sign.

Roots of SU(2)

The maximal torus T of SU(2) consists of diagonal matrices of the form $D(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ so we can diagonalize the action of T on $\mathfrak{su}(2) \otimes \mathbb{C} = \mathfrak{sl}_2(\mathbb{C})$. $\mathfrak{sl}_2(\mathbb{C})$ has an obvious basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with the following relations:

$$[e, f] = h \qquad [h, e] = 2e \qquad [h, f] = -2f$$

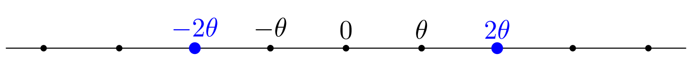
Note that $h \in \mathfrak{t} \otimes \mathbb{C}$, and the relations above show that e and f are eigenvectors for $\mathrm{ad}(h)$ with eigenvalues ± 2 respectively. $D = \exp(i\theta h)$, so via the naturality of exp, we have that $\mathrm{Ad}(D)e = \exp(2i\theta)e$ and $\mathrm{Ad}(D)f = \exp(-2i\theta)f$. Thus, we see that e and f each span a complex root space with corresponding global roots $\rho(D) = e^{2i\theta}$ and $\overline{\rho}(D) = e^{-2i\theta}$. The maximality criterion then implies that T is maximal.

There is one real root plane $P = \mathbb{R}\langle u, v\rangle$ with $u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $\mathrm{Ad}(D)$ rotates P by $\pm 2\theta$.

Lecture 26: November 6th

Set $\mathfrak{t}^* = \mathrm{Hom}(\mathfrak{t}, \mathbb{R})$ which contains the character lattice $\Lambda = \mathrm{Hom}(T, \mathbb{R}/\mathbb{Z})$ (the inclusion is given by taking the derivative) which we can draw as follows:

All subsequent root lattice figures are stolen (with permission) from Tim's notes.



$\mathfrak{t}^*$ is the line, containing the lattice Λ (whose generator θ is really the map $\mathrm{Diag}(i\theta, -i\theta) \mapsto \theta$, i.e., we are essentially decomplexifying) and the real roots are $\pm 2\theta$ by the above calculation. Thus, we can see that $\Lambda/\mathbb{Z}\{\text{roots}\} = \mathbb{Z}/2$.

Note that SU(2) double covers SO(3).

Roots of SO(3)

SO(3) contains an evident torus $T = \mathrm{SO}(2)$ given by rotations about the z-axis, i.e., given by matrices of the form $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. In $\mathfrak{so}(3)$ we may take the following generators, which are the infinitesimal rotations about the coordinate axes:

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where, since $\exp(\theta L_k)$ is rotation by angle θ about the e_k-axis (so $\exp(\theta L_3) = R_\theta$, $L_3 \in \mathfrak{t}$.

This is the basis for which $\mathfrak{so}(3)$ as a Lie algebra is just $\mathbb{R}^3$ with the cross product, i.e.,

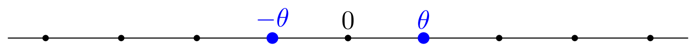
$$[L_i, L_j] = \sum_k \varepsilon_{ijk} L_k$$

$\mathrm{Ad}(\exp_G x) = e^{\mathrm{ad}\, x}$ by naturality, so we may compute

$$\mathrm{Ad}(R_\theta)L_1 = L_1 \cos\theta + L_2 \sin\theta \qquad \mathrm{Ad}(R_\theta)L_2 = -L_1 \sin\theta + L - 2\cos\theta$$

so L_1 and L_2 span a root plane P. The global roots are therefore $R_\theta \mapsto e^{\pm i\theta}$.

As above, we may draw $\mathfrak{t}^*$ containing the character lattice Λ:



Note that, here, $\Lambda/\mathbb{Z}\{\text{roots}\} = 1$.

This surely has something to do with π_1 and covering spaces but I can't yet formulate a conjecture.

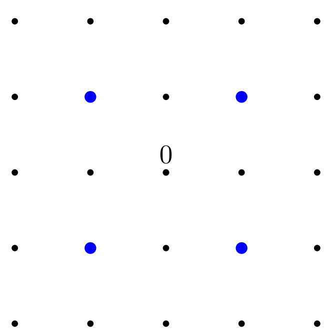
Roots of SO(4)

SO(4) also contains an evident torus $T = \mathrm{SO}(2) \times \mathrm{SO}(2)$ given by block matrices of the form $R_{\theta_1,\theta_2} = \begin{pmatrix} R_{\theta_1} & 0 \\ 0 & R_{\theta_2} \end{pmatrix}$. Define the following 2×2 matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

One root plane is given by $P = \mathrm{span}\left(\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \right)$ with global root $R_{\theta_1,\theta_2} \mapsto e^{\pm i(\theta_1 - \theta_2)}$ and the other is given by $P' = \mathrm{span}\left(\begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}, \begin{pmatrix} 0 & L \\ -L & 0 \end{pmatrix} \right)$ with global root $R_{\theta_1,\theta_2} \mapsto e^{\pm i(\theta_1+\theta_2)}$.

Thus, as above, we may draw the character lattice:



The covolume of the lattice generated by the roots is 2 so $\Lambda/\mathbb{Z}\{\text{roots}\} = \mathbb{Z}/2$ as with SU(2).

No longer confident that the covolume is saying anything about π_1.

Roots of U(n)

U(n) contains an evident torus T consisting of matrices of the form $\mathrm{Diag}(e^{i\theta_1}, \cdots, e^{i\theta_n})$. $\mathfrak{u}(n) \otimes \mathbb{C} = \mathfrak{gl}_n(\mathbb{C})$ since any matrix can be written as the sum of a hermitian and skew-hermitian matrices. We want to diagonalize $\mathrm{Ad}\,|_T$; to do so, we utilize the obvious basis E_{jk} for $\mathfrak{gl}_n(\mathbb{C})$ which is given by $(E_{jk})_{lm} = \delta_{jl}\delta_{km}$ (i.e. E_{jk} has a single nonzero entry at index (j,k), which is equal to 1).

Then, if $D = \mathrm{Diag}(e^{i\theta_1}, \cdots, e^{i\theta_n})$, $D^{-1} = \mathrm{Diag}(e^{-i\theta_1}, \cdots, e^{-i\theta_n})$ so

$$(DE_{jk}D^{-1})_{lm} = \sum_{a,b} D_{la}(E_{jk})_{ab}(D^{-1})_{bm} = \sum_{a,b} e^{i\theta_l}\delta_{al}\delta_{ja}\delta_{kb}e^{-i\theta_b}\delta_{bm} = e^{i(\theta_l-\theta_m)}(E_{jk})_{lm}$$

Thus, $\mathrm{Ad}(D)E_{jk} = e^{i(\theta_j-\theta_k)}E_{jk}$ and E_{jk} is a simultaneous eigenvector for all $\mathrm{Ad}(D)$ with $D \in T$.

One can show that this torus T is in fact maximal by showing that it is the invariant subspace of $\mathfrak{u}(n)$ under the action of T, and that (therefore) the global roots are the functions $D \mapsto e^{i(\theta_j-\theta_k)}$ for $i \neq j$. One can then use this to calculate Killing form $\kappa_{\mathrm{U}(n)}(X,Y) = -2n\,\mathrm{Tr}(XY) + 2\,\mathrm{Tr}(X)\,\mathrm{Tr}(Y)$.

Roots of $\mathrm{SU}(n)$

Lecture 27: November 8th

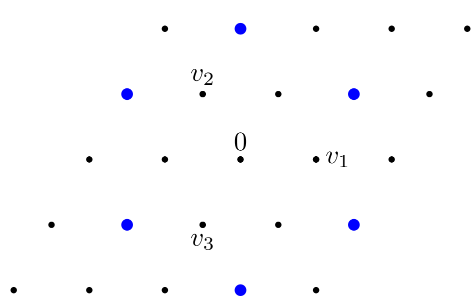
$\mathrm{SU}(n)$ contains the restriction of the above torus, which consists of matrices of the form $\mathrm{Diag}(e^{i\theta_1}, \cdots, e^{i\theta_n})$ where the θ_i satisfy $e^{i(\theta_1+\cdots+\theta_n)} = 1$. The corresponding Lie algebra consists of matrices of the form $\mathrm{Diag}(i\theta_1, \cdots, i\theta_n)$ satisfying $\sum_i \theta_i = 0$. To compute the roots, we consider the diagonal action of T on $\mathfrak{su}(n) \otimes \mathbb{C} = \mathfrak{sl}_n(\mathbb{C})$ where we have the same basis as above, the E_{jk}, so we have the same roots and root planes as above.

Since $\mathfrak{t} = \{\sum_i x_i e_i : \sum_i x_i = 0\} \subseteq \mathbb{R}^n$, $\mathfrak{t}^* = (\mathbb{R}^n)^*/\mathbb{R}(e_1^* + \cdots + e_n^*)$, the character lattice is the $\mathbb{Z}$-span of the images in $\mathfrak{t}^*$ of the e_i^*, and the roots in $\mathfrak{t}^*$ are $e_i^* - e_j^*$. Since $\mathfrak{su}(n)$ is an ideal in $\mathfrak{u}(n)$, $\kappa_{\mathfrak{su}(n)} = \kappa_{\mathfrak{u}(n)}|_{\mathfrak{su}(n)}$, i.e., $\kappa_{\mathfrak{su}(n)}(X,Y) = -2n\,\mathrm{Tr}(XY)$ since X and Y are themselves traceless.

Setting $n = 3$, we can draw a picture of $\mathfrak{t}^* = (\mathbb{R}^3)^*/\mathbb{R}(e_1^* + e_2^* + e_3^*)$. Let $\lambda = e_1^* + e_2^* + e_3^*$ and identify $\mathfrak{t}^*$ with $\lambda^\perp$, so the projection $(\mathbb{R}^3)^* \to \mathfrak{t}^*$ is identified with the orthogonal projection $\pi : (\mathbb{R}^3)^* \to \lambda^\perp$. Setting $v_i = \pi(e_i^*)$ we have

$$v_1 = \frac{1}{3}(2e_1^* - e_2^* - e_3^*) \qquad v_2 = \frac{1}{3}(-e_1^* + e_2^* - e_3^*) \qquad v_3 = \frac{1}{3}(-e_1^* - e_2^* + 2e_3^*)$$

so $v_1 + v_2 + v_3 = 0$ and the coefficients in each v_i sum to zero. The six vectors $\pm e_i^*$ project to the six vectors $\pm v_i^*$ which form the vertices of a regular hexagon:



The roots $e_i^* - e_j^*$ project to $v_i - v_j$ where the v_i generate a honeycomb lattice in $\lambda^\perp$. The character lattice modulo the lattice generated by the roots is $\mathbb{Z}/3$ in this instance.

Conjugacy Theorem

This apparent technical lemma turns out to have major implications in the theory of compact Lie groups; the consequences of this theorem will occupy the bulk of this section.

Theorem 4.4.1: Conjugacy Theorem

Let G be a compact connected Lie group, $T \subseteq G$ a maximal torus. Then

$$\bigcup_{g \in G} gTg^{-1} = G$$

Corollary 4.4.2

All maximal tori are conjugate.

Proof : Let T, T' be maximal tori in G. One can show that there exists $g \in T'$ a topological generator and $g = huh^{-1}$ for $u \in T$ by the theorem, so $\overline{\langle u \rangle} = h^{-1}\overline{\langle g \rangle}h = h^{-1}T'h \subseteq T$.

Similarly, $T' \subseteq hTh^{-1}$ and hTh^{-1}, so hTh^{-1} is a torus containing $hT'h^{-1} = T'$. ■

Corollary 4.4.3

The dimension of the maximal torus is an invariant of G, called its *rank*.

We computed (some cases of) the following ranks above:

$\operatorname{rank} \mathrm{U}(n)$ is n as we calculated above.

$\operatorname{rank} \mathrm{SU}(n)$ is $n-1$, also as above.

$\operatorname{rank} \mathrm{SO}(2n)$ is n. We didn't actually calculate this above in generality, but $\mathrm{SO}(2) \times \mathrm{SO}(2) \subseteq \mathrm{SO}(4)$ is a maximal torus and $SO(2)^n \subseteq \mathrm{SO}(2n)$ is similarly a maximal torus in general.

$\operatorname{rank} \mathrm{SO}(2n+1)$ is n. Similarly, we showed that the maximal torus in $\mathrm{SO}(3)$ is $\mathrm{SO}(2)$ given by rotations of the z-axis, and a similar recipe of block diagonal rotation matrices gives the maximal torus in $\mathrm{SO}(2n+1)$.

I don't think we've done any calculations of rank for non-matrix Lie groups, although perhaps we can expect some type of lifting result for e.g. $\mathrm{Spin}(n)$.

Corollary 4.4.4

If G is compact and connected then $\exp_G : \mathfrak{g} \to G$ is surjective.

PROOF : Let T be a maximal torus, $\mathfrak{t}$ its Lie algebra, then $\exp_G|_{\mathfrak{t}} = \exp_T : \mathfrak{t} \to T$ is surjective (we know explicitly what $\mathfrak{t}$ and T are by identifying $T = U(1)^n$ and it is easy to see that exp is surjective in this case). By naturality, $\exp(\operatorname{Ad} gt) = g \exp(t) g^{-1}$ where $t \in \mathfrak{t}$, so the image of exp contains gTg^{-1} for all $g \in G$, so, by the conjugacy theorem, $\exp_G$ is surjective. ■

The proof of the conjugacy theorem relies on the flag manifold, i.e., the left coset space G/T, so named since for $G = \mathrm{U}(n)$ with its standard torus (as above), the complete flags in $\mathbb{C}^n$ are parameterized by G/T. $\mathrm{U}(n)$ evidently acts transitively on the space of complete flags, so we need to understand the stabilizer of some flag F. Pick an ordered basis $v_1, \cdots, v_n$ for F so that F_i is the span of $v_1, \cdots, v_i$. The stabilizer of this flag is then evidently T, the set of diagonal matrices, so G/T parameterizes complete flags.

We mention the flag picture here but it doesn't really seem to be that important for the Lefschetz-theoretic proof below.

Similarly, with $G = \mathrm{SO}(n)$, G/T parameterizes the complete flags in $\mathbb{R}^n$. In what will follow, we will reduce the proof of the conjugacy theorem to the Lefschetz fixed point theorem applied to the map $G \times G/T \xrightarrow{\phi} G/T$ given by $(g, hT) \mapsto ghT$, but first, we need some basic results on homogeneous manifolds.

Homogeneous Manifolds

Lecture 28: November 11^{th}

A manifold M equipped with a transitive action of a Lie group G is called a homogeneous space; fixing a basepoint $x \in M$ and setting $H = \operatorname{stab}_G(x) \subseteq G$, we have a homeomorphism $G/H \to M$ given by $gH \mapsto g \cdot x$. The quotient G/H is a manifold, and in fact, this homeomorphism is a diffeomorphism. The key fact is the following:

Lemma 4.4.5

Let H be a closed subgroup of a Lie group G, then G/H is a manifold and the quotient map $\pi : G \to G/H$ is a submersion.

We omit the proof.

Corollary 4.4.6

A homogeneous manifold G/H with G a Lie group and H a connected embedded Lie subgroup is orientable.

We omit this proof as well, though note that connectedness of G is essential; for example, $\mathbb{RP}^2 = \mathrm{SO}(3)/\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(2))$ is not orientable but $\mathrm{O}(1)$ is disconnected.

Proof of the Conjugacy Theorem

G acts on the left on the closed manifold G/T by left multiplication; explicitly we have $G \times G/T \xrightarrow{\phi} G/T$ given by $\phi(g, hT) = ghT$. Write $\overline{x}$ to denote the coset xT in G/T, and let ϕ_g denote $\phi(g, -)$. A fixed point of ϕ_g is therefore a coset $\overline{x}$ such that $gxT = xT$, i.e., $x^{-1}gxT = T$, so $x^{-1}gx \in T$. Thus, it suffices to show that ϕ_g has a fixed point for all $g \in G$.

Among our lemmata above we showed that G/T is orientable, which is necessary to apply Lefschetz.

Recall the Lefschetz fixed point theorem in the following form: let M be a closed orientable manifold, $f : M \to M$ a smooth map with non-degenerate fixed points, i.e. the graph Γ_f is transverse to the diagonal Δ in $M \times M$. Points in this intersection are precisely the fixed points of f and transversality means that $Df_x : T_xM \to T_xM$ does not have 1 as an eigenvalue (i.e. $\det(Df_x - I) \neq 0$).

For (non)example, the fixed points of the identity map are all degenerate.

Define the *Lefschetz sign* $\epsilon(f;x) = \operatorname{sign}\det(Df_x - I)$; then the *Lefschetz number* is defined as the *signed* intersection number of Δ and Γ_f given by

$$\Delta \cdot \Gamma_f = \sum_{f(x)=x} \epsilon(f;x)$$

where there are finitely many such fixed points by compactness (since such fixed points are necessarily isolated, as otherwise, a tangent direction to an adjacent fixed point corresponds to 1 as an eigenvalue for Df_x).

For our purposes, the "weak" Lefschetz fixed point theorem, which states that $\Delta \cdot \Gamma_f$ depends only on the homotopy class of f, will suffice. Setting $f = \phi_g$, since $\Delta \cdot \Gamma_{\phi_g}$ depends only on the homotopy class of ϕ_g, it is independent of $g \in G$ (since a path from g to e in G connected corresponds to a homotopy of ϕ_g to ϕ_e). Thus, it suffices to show the following:

The logically necessary "strong" Lefschetz theorem is the one that I am more familiar with, which computes $\Lambda \cdot \Gamma_f$ as the alternating sum of the traces of f_* acting on H_* (generalizing the Euler characteristic, which is the Lefschetz number of the identity).

Lemma 4.4.7

There exists $g \in G$ with nonzero Lefschetz number.

The correct choice of g for ease of calculation turns out to be a topological generator for T since then we have

$$\phi_g(\overline{x}) = \overline{x} \iff x^{-1}gx \in T \iff x^{-1}Tx = T \iff x \in N(T)$$

where $N(T)$ is the normalizer defined as $N(T) = \{g \in G : gtg^{-1} \in T \text{ for all} t \in T\}$ and where we use the fact that g is a topological generator when we assert that $x^{-1}gx \in T \iff x^{-1}Tx = T$. Thus, the fixed points of ϕ_g correspond to points in $N(T)/T$, the *Weyl group* (which we will discuss in more detail below).

Note that since $g \in T$, $\phi_g(\overline{x}) = \overline{c_g(x)gxg^{-1}}$ where the conjugation map c_g descends to the quotient as a map $\overline{c_g}$ since T is abelian. Thus, ϕ_g is induced by the conjugation action of g on G.

Lecture 29: November 13th

The key calculation is that of the Lefschetz signs: note that $N(T)_0$ the connected component of the identity is T itself. To see this, note that $\operatorname{Aut}(T)$ is discrete since an automorphism of T is fully determined by its action on the co-character lattice $\operatorname{Hom}(\mathbb{R}^n/\mathbb{Z}^n, T)$, so the conjugation action $N(T) \to \operatorname{Aut}(T)$ is locally constant, and therefore constant on $N(T)_0$. This implies that $N(T)_0$ is abelian, and closed, and evidently contains T, so $N(T)_0 = T$.

We first check that $\overline{e} = T$ is non-degenerate as a fixed point of ϕ_g; note that $T_{\overline{e}}(G/T) = \mathfrak{g}/\mathfrak{t}$, and that $D_e c_g = \operatorname{Ad}(g)$, so $D_{\overline{e}}\phi_g = \operatorname{Ad}(g) \in \operatorname{End}(\mathfrak{g}/\mathfrak{t})$. De-composing $\mathfrak{g}/\mathfrak{t} = \bigoplus_j P_j$ into its root planes on which $\mathrm{Ad}(g)$ acts as rotation $R(\theta_j(g))$, we can compute the determinant of $I - D_e\phi_g \in \mathrm{End}\, T_{\overline{e}}(G/T)$:

$$\det(I - D_e\phi_g) = \prod_j \det(I - R(\theta_j(g)))$$

where $I - R(\theta_j(g)) \in \mathrm{End}\, P_j$. For a generic choice of $g \in T$, the rotations $R(\theta_j(g))$ are nontrivial for all j, and for a nontrivial rotation, $\det(I - R(\theta_j(g))) = 2(1 + \cos\theta_j(g)) > 0$, so $\overline{e}$ is a non-degenerate fixed point with Lefschetz sign $+1$.

Let $r_n : G/T \to G/T$ be the right multiplication map, with $n \in N(T)$, i.e. $r_n(xT) = xnT$. r_n is a diffeomorphism commuting with ϕ_g, and mapping $\overline{e}$ to $\overline{n}$, so, by the chain rule, $\epsilon(\phi_g; \overline{n}) = \epsilon(\phi_g; \overline{e}) = 1$, so all $\overline{n} \in N(T)$ are non-degenerate with Lefschetz sign $+1$.

Now, since G/T is compact, and non-degenerate fixed points are isolated, there are finitely many fixed points for any self map, so $N(T)/T$ is finite, and since all of the Lefschetz signs are $+1$, $\Delta \cdot \Gamma_{\phi_g} = |N(T)/T| \geq 1$, from which the lemma follows.

Another proof of the conjugacy theorem using algebraic topology comes from the *Bruhat decomposition*, which is a CW structure on G/T (and on some class of algebraic groups in general) with only even-dimensional cells, and the number of cells is $|N(T)/T|$. Thus, using cellular cohomology, we have that $\chi(G/T) = |N(T)/T|$, the Lefschetz number of the identity.

Corollary 4.4.8

$\chi(G/T) = |N(T)/T|$

PROOF : Apply the strong Lefschetz fixed point theorem. ■

The Weyl Group

The conjugacy theorem has the following additional consequence:

Proposition 4.5.1

In a compact connected Lie group G, a maximal torus T is also maximal among the abelian subgroups of G.

PROOF : Suppose $T \subseteq A$ with A abelian, and let $a \in A$. We want to show that there is a torus T' containing T and a, from which it follows by maximality that $T' = T$ and so $a \in T$.

Let $U = \overline{\langle a, T \rangle}$ which is a compact abelian Lie group, so its identity component U_0 must be a torus, with U/U_0 finite cyclic with generator a (which gives us cyclicity) and compactness gives us finiteness. Thus U is an extension of a finite cyclic group by a torus, and therefore admits a topological generator u, with $u \in T'$ by the conjugacy theorem, but then $T \cup \{a\} \subseteq U \subseteq T'$ and we can conclude that $T = T'$. ■

I skipped class and received moderately bad information on what we covered. As such, I skipped past what we actually covered (thinking that that material was just extra flavor in the notes) and wrote stuff up from two lectures ahead of where we were. If this section feels disorganized or repeats itself, that's why.

Note that there exist abelian subgroups not contained in a torus, e.g., the Klein 4-group embeds in SO(3) as the set $\{I, A, B, AB\}$ for the matrices

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

One can show that this copy of $\mathbb{Z}/2 \times \mathbb{Z}/2$ is not contained in any torus (A and B are rotations about different axes) or indeed any larger abelian subgroup, so is maximal abelian but not a torus.

Definition 4.5.2

The *Weyl group* of a compact connected Lie group G with respect to a chosen maximal torus T is $W(G,T) = N(T)/T$.

Meaningless pattern matching: the Weyl group is some sort of categorification of the fact that $\chi(G/T) = |W(G,T)| = |N(T)/T|$.

We write $\mathrm{Inn}(G)$ for the inner automorphisms of G (i.e. automorphisms of the form $x \mapsto gxg^{-1}$ indexed by elements of G). Note that the conjugacy theorem implies that, up to the induced action of inner automorphisms on cosets in $W(G,T)$, the Weyl group is independent of T.

Lecture 30: November 18th

Lemma 4.5.3

$$W(G,T) = \{\phi \in \mathrm{Inn}(G) : \phi(T) = T\}$$

PROOF : Let $\mathrm{Inn}_T(G) = \{\phi \in \mathrm{Inn}(G) : \phi(T) = T\}$, then any ϕ is of the form $\phi(t) = wtw^{-1}$ so $w \in N(T)$; this defines a homomorphism $c : N(T) \to \mathrm{Inn}_T(G)$ that is evidently surjective. $\ker c$ is the *centralizer* of T, i.e., the set $Z(T) = \{g \in G : gtg^{-1} = g \text{ for all } t \in T\}$ and $N(T)/Z(T) = \mathrm{Inn}_T(G)$. For any $g \in \ker c$, $\langle g, T\rangle$ is abelian and contains T, so is equal to T, so $\ker c = T$. ■

Example 4.5.4

Let $G = \mathrm{SU}(n)$ with its torus T of diagonal elements, $W = N(T)/T$. Pick $t \in T$ with distinct eigenvalues λ_i, then t has eigenspaces $\mathbb{C}e_1, \cdots, \mathbb{C}e_n$. If $w \in N(T)$ then t and wtw^{-1} share the same spectrum (since they are conjugate) and the same eigenvectors, so the action of $N(T)$ permutes the eigenspaces. This gives rise to a homomorphism $N(T) \to S_n$.

Permutation matrices lie in $\mathrm{U}(n)$ and have determinant ± 1, so, changing the sign of the first column matrix entry if necessary (i.e. for odd permutations), we can see that $N(T) \to S_n$ is surjective. The kernel of this map consists of diagonal matrices, i.e., T, so $W = N(T)/T = S_n$.

In particular, $W(\mathrm{SU}(2)) = S_2 = \mathbb{Z}/2$ (recall that this is also the character lattice modulo roots for SU(2)).

Example 4.5.5

Let $G = \mathrm{SO}(2n)$ with its torus T of block diagonal matrices, i.e., $T = \mathrm{SO}(2)^n$. Fix $R \in T$ which acts by rotations through $\theta_1, \cdots, \theta_n$

in its block factors, and assume that the θ_i are all distinct from each other and from $\pm I$. For $w \in N(T)$ acting on $\mathbb{R}^{2n}$, commuting with T implies that w must permute the planes $P_1 = \mathrm{span}(e_1, e_2), \cdots, P_n = \mathrm{span}(e_{2n-1}, e_{2n})$. As above, this gives a surjective homomorphism $f : W \to S_n$.

Suppose $N(T) \ni w \in \ker f$. Then w preserves the P_j, but not necessarily by an element of SO(2), as the determinant of each block of the matrix can be ± 1 (i.e. we can have rotations *and* reflections), i.e., $\ker f \ni w \in S(\mathrm{O}(2)^n)$.

Taking signs of the matrix blocks gives a map $\omega : \{1, \cdots, n\} \to \mathbb{Z}/2$ with the condition that $\sum_j \omega_j = 0$ (since the overall determinant is $+1$). The set of all such maps is an abelian group A of order 2^{n-1}, so we get a short exact sequence

$$1 \to A \to W(\mathrm{SO}(2n)) \to S_n \to 1$$

so $W(\mathrm{SO}(2n)) \cong A \rtimes S_n$. In particular, $|W| = n!2^{n-1}$ and the Weyl group is *morally* S_n, but augmented by some sign data.

A similar calculation for $\mathrm{SO}(2n+1)$ gives $W(\mathrm{SO}(2n+1)) = S_n \rtimes (\mathbb{Z}/2)^n$ so $|W(\mathrm{SO}(2n+1))| = n!2^n$.

Roots and SU(2) Representations

Theorem 4.5.6

Let G be compact, connected, and semisimple (i.e. $Z(G)$ is discrete $\iff Z(\mathfrak{g}) = 0 \iff \kappa_{\mathfrak{g}}$ is negative-definite), and let T be a maximal torus, α a root. Then there exists a homomorphism $r_\alpha : \mathrm{SU}(2) \to G$ such that the induced homomorphism $Dr_\alpha : \mathfrak{su}(2) \to \mathfrak{g}$ has image $P_\alpha \oplus \mathbb{R}t$ where P_α is the root plane corresponding to α and $t \in \mathfrak{t}$.

First, we want to fix some conventions. Our roots are complex, i.e. $\alpha \in (\mathfrak{t}_{\mathbb{C}})^*$, with $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \otimes \mathbb{C} \subseteq \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}_{\mathbb{C}}$, with $\mathfrak{g}_\alpha \subseteq \mathfrak{t}_{\mathbb{C}}$ given by $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}_{\mathbb{C}} : \mathrm{ad}(t)x = \alpha(t)x \text{ for all } t \in \mathfrak{t}_{\mathbb{C}}\}$ and $\Phi_{\mathbb{C}} = \{\alpha \in (\mathfrak{t}_{\mathbb{C}})^* : g_\alpha \neq 0\}$ is the *complex* root system.

Lemma 4.5.7

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$$

PROOF : The proof is essentially the Jacobi identity: let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, and $t \in \mathfrak{t}_{\mathbb{C}}$, then

$$\begin{aligned} \mathrm{ad}(t)[x,y] = [t,[x,y]] = -[x,[y,t]] - [y,[t,x]] = \\ -[x,-\beta(t)y] - [y,\alpha(t)x] = (\alpha(t)+\beta(t))[x,y] \end{aligned} \quad \blacksquare$$

So if $\alpha, \beta \in \Phi_{\mathbb{C}}$, either $\alpha + \beta \in \Phi_{\mathbb{C}}$ or $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ (note that this is true without the requirement that $\mathfrak{g}$ be semisimple).

Proposition 4.5.8

If $\alpha, \beta \in \Phi_{\mathbb{C}} \cup \{0\} \subseteq \mathfrak{t}_{\mathbb{C}}^*$, $\alpha + \beta \neq 0$, then $\kappa_{\mathfrak{g}}(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ i.e. the root spaces are orthogonal with respect to the Cartan-Killing form unless $\alpha = -\beta$.

PROOF : Let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, $t \in \mathfrak{t}_{\mathbb{C}}^*$, and note that $\mathfrak{g}_0 = \mathfrak{t}_{\mathbb{C}}$. By ad-invariance of κ,

$$0 = \kappa(\mathrm{ad}(t)x, y) + \kappa(x, \mathrm{ad}(t)y) = (\alpha(t) + \beta(t))\kappa(x, y)$$

Since t is arbitrary and $\alpha + \beta \neq 0$, the result follows. ■

Now, assuming that $\mathfrak{g}$ is semisimple (which was not required for the above), $\kappa_{\mathfrak{g}_{\mathbb{C}}}$ is non-degenerate.

Lemma 4.5.9

$\tilde{\kappa} : \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathfrak{g}_{\mathbb{C}}$ is a non-degenerate pairing.

PROOF : $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Phi_{\mathbb{C}}} \mathfrak{g}_\alpha$, so take $0 \neq x \in \mathfrak{g}_\alpha$. Since x is κ-orthogonal to $\mathfrak{t}_{\mathbb{C}}$ and all $\mathfrak{g}_\beta$ with $\beta \neq -\alpha$ by the above, and since κ is non-degenerate, there must exist $y \in \mathfrak{g}_{\mathbb{C}}$ s.t. $\kappa(x, y) \neq 0$, and the only place this y can therefore live is $\mathfrak{g}_{-\alpha}$. ■

This is essentially because there is nowhere else for the $\kappa_{\mathfrak{g}}$-dual of something in $\mathfrak{g}_\alpha$ to live.

Thus we have inclusions $\mathfrak{g}_\alpha \hookrightarrow (\mathfrak{g}_{-\alpha})^*$ and $\mathfrak{g}_{-\alpha} \hookrightarrow (\mathfrak{g}_\alpha)^*$, so $\mathfrak{g}_\alpha \xrightarrow{\sim} \mathfrak{g}_{-\alpha}^*$ since the two spaces have the same dimension.

I am meaninglessly alternating between Lemmas and Propositions to make the colors look nicer.

Proposition 4.5.10

Assume $\mathfrak{g}$ is negative-definite; then, if $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$,

$$[x, y] = \kappa_{\mathfrak{g}}(x, y) H_\alpha \in \mathfrak{t}_{\mathbb{C}}$$

where $H_\alpha \in \mathfrak{t}_{\mathbb{C}}$ is characterized by $\kappa(H_\alpha, t) = \alpha(t)$ for all $t \in \mathfrak{t}_{\mathbb{C}}$.

PROOF : By the above, $[x, y] \in \mathfrak{g}_{\alpha-\alpha} = \mathfrak{g}_0 = \mathfrak{t}_{\mathbb{C}}$, so we have

$$\kappa([x, y], t) = \kappa(\mathrm{ad}(x)y, t) = -\kappa(y, \mathrm{ad}(x)t) = \kappa(y, \mathrm{ad}(t)x) = \alpha(t)\kappa(x, y)$$

Since this holds for all $t \in \mathfrak{t}_{\mathbb{C}}$, the claim follows. ■

Now, to build our $\mathrm{SU}(2)$ representation, we begin by finding a copy of $\mathfrak{sl}_2(\mathbb{C}) \subseteq \mathfrak{g}_{\mathbb{C}}$. Recall that we have the following basis for $\mathfrak{sl}_2(\mathbb{C})$:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with the following relations:

$$[e, f] = h \qquad [h, e] = 2e \qquad [h, f] = -2f$$

So we want to find an $\mathfrak{sl}_2(\mathbb{C})$-triple $(e_\alpha, f_\alpha, h_\alpha)$ in $\mathfrak{g}_\mathbb{C}$ obeying the same relations. For what follows, we will denote by $(-,-)$ the pairing on $\mathfrak{t}^*$ induced by κ on $\mathfrak{t}$.

Somewhat noteworthy that Tim is giving a "basis"-dependent proof here as he is usually very opposed to that.

Pick any $0 \neq e_\alpha \in \mathfrak{g}_\alpha$, and $f_\alpha \in \mathfrak{g}_{-\alpha}$ with the normalization condition $\kappa(e_\alpha, f_\alpha = \frac{2}{(\alpha,\alpha)})$ so that

$$[e_\alpha, f_\alpha] = \kappa(e_\alpha, f_\alpha)H_\alpha = \frac{2}{(\alpha,\alpha)}H_\alpha =: h_\alpha \in \mathfrak{t}_\mathbb{C}$$

so the first relation is automatically satisfied. One can check that $\alpha(h_\alpha) = 2$:

$$[h_\alpha, e_\alpha] = \alpha(h_\alpha)e_\alpha = \frac{2}{(\alpha,\alpha)}\alpha(H_\alpha)e_\alpha = 2\frac{(\alpha,\alpha)}{(\alpha,\alpha)}e_\alpha = 2e_\alpha$$

Similarly, one can check that $[h_\alpha, f_\alpha] = -2f_\alpha$.

Thus, we may conclude that any $0 \neq e_\alpha \in \mathfrak{g}_\alpha$ extends to an $\mathfrak{sl}_2(\mathbb{C})$-triple $(e_\alpha, f_\alpha, g_\alpha)$. Now, we want to restrict to $\mathfrak{su}(2)$; recall the basis of Pauli matrices for $\mathfrak{su}(2)$:

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

which satisfy $\sigma_j^2 = -I$, $\sigma_1\sigma_2 = \sigma_3 = -\sigma_2\sigma_1$ and cyclic permutations thereof.

Clearly, $\sigma_1 = ih$, $\sigma_2 = e - f$, and $\sigma_3 = i(e+f)$, so our homomorphism $\theta : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}_\mathbb{C}$ given by $(e, f, h) \mapsto (e_\alpha, f_\alpha, h_\alpha)$ restricts to $\mathfrak{su}(2) \subseteq \mathfrak{sl}_2(\mathbb{C})$ via the Pauli matrices as above, mapping to $\mathfrak{g} \subseteq \mathfrak{g}_\mathbb{C}$.

Since $\mathrm{SU}(2)$ is simply connected, we can integrate $\theta_\alpha|_{\mathfrak{su}(2)}$ to the promised homomorphism $r_\alpha : \mathrm{SU}(2) \to G$ via $r_\alpha(\exp t) = \exp_G \theta_\alpha(t)$. This r_α is unique, which will follow from the fact that the root spaces are one-dimensional, which we will see later on.

Lie Groups **Fall 2024**

The Geometry of Roots

PROFESSOR TIM PERUTZ ABHISHEK SHIVKUMAR

$\mathfrak{sl}_2(\mathbb{C})$ Representation Theory

Recall that a Lie algebra representation is a map $\theta : \mathfrak{g} \to \operatorname{End} V$ that respects brackets.

By the final construction from the last chapter, for all roots α of some compact, connected, semisimple Lie groups G (with discrete center), we have a copy of $\mathfrak{sl}_2(\mathbb{C})_\alpha \subseteq \mathfrak{g}_\mathbb{C}$ generated by $e_\alpha, f_\alpha, h_\alpha$. The adjoint representation $\operatorname{ad} : \mathfrak{g}_\mathbb{C} \to \operatorname{End} \mathfrak{g}_\mathbb{C}$ then gives us a representation of $\mathfrak{sl}_2(\mathbb{C})$ into $\operatorname{End} \mathfrak{g}_\mathbb{C}$. To exploit this structure towards an understanding of the representation theory of $\mathfrak{g}_\mathbb{C}$, it behooves us to understand the representation theory of $\mathfrak{sl}_2(\mathbb{C})$.

We won't prove much here, this is just an accounting of the salient facts (sometimes with an indication of how they are proved).

Let V be a representation of $\mathfrak{sl}_2(\mathbb{C})$, i.e., $\mathfrak{sl}_2(\mathbb{C})$ acts on a finite dimensional complex vector space V via a homomorphism $\mathfrak{sl}_2(\mathbb{C}) \to \operatorname{End} V$, and let V_λ be the *weight space* of weight λ, which is the λ-eigenspace of $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. By the commutation relations, $e \cdot V_\lambda \subseteq V_{\lambda+2}$ and $f \cdot V_\lambda \subseteq V_{\lambda-2}$. A nonzero vector $v \in V_\lambda$ (for some λ) is *primitive* if $e \cdot v = 0$, and it is easy to see that a primitive vector exists since h has finitely many eigenvalues and e raises the weight by 2.

For $v \in V_\lambda$ primitive, set $v_n = \frac{1}{n!} f^n \cdot v \in V_{\lambda-2n}$ for $n \geq 0$, so that $v_n \in V_{\lambda-2n}$ and $f \cdot v_n = (n+1)v)_{n+1}$, $e \cdot v_n = (\lambda - n + 1)v_{n-1}$. It follows by the commutation relations that the weight λ of a primitive vector is a non-negative integer m, and that $v_0, \cdots, v_m$ are linearly independent and $v_k = 0$ for $k > m$.

One can show that, if W_m is the vector space with basis $(v_0, \cdots, v_m)$ as above, then W_m is an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ of dimension $m+1$ with weights $m, m-2, \cdots, -m+2, -m$. It also follows that any irreducible representation of dimension $m+1$ is isomorphic to W_m.

Not sure where the tensoring by $P_m(V)$ comes from, it's a bit less clear than the rest of it.

There is a corresponding decomposition theorem for our arbitrary representation V:

$$V \cong \bigoplus_{m \geq 0} P_m(V) \otimes_\mathbb{C} W_m$$

where $P_m(V) = V_m \cap \ker e$ is the space of primitive elements of weight m.

$\mathfrak{sl}_2(\mathbb{C})$ acts on $\mathfrak{g}_{\mathbb{C}}$

Lecture 31: November 22nd

Returning to the point of this discussion (to construct from a root of a compact connected semisimple Lie group G a homomorphism $\mathrm{SU}(2) \to G$), let $\Phi_{\mathbb{C}}$ denote as above the set of complex roots and let $\alpha \in \Phi_{\mathbb{C}}$. We have a copy $\mathfrak{sl}_2(\mathbb{C})_\alpha$ of $\mathfrak{sl}_2(\mathbb{C})$ inside $\mathfrak{g}_{\mathbb{C}}$ generated by $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$, $h_\alpha \in \mathfrak{t}_{\mathbb{C}}$. The claim is now that ad restricted to $\mathfrak{sl}_2(\mathbb{C})_\alpha$ makes $\mathfrak{g}_{\mathbb{C}}$ a representation of $\mathfrak{sl}_2(\mathbb{C})$. But, first, we need the following:

Lemma 5.2.1

$\dim \mathfrak{g}_\alpha = 1$.

PROOF : We have $e_\alpha \in \mathfrak{g}_\alpha$, and we know by the above discussion that $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are dually paired via κ (Lemma 4.5.9), with $[x, y] = \kappa(x, y)H_\alpha$ for $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, and $H_\alpha \in \mathfrak{t}_{\mathbb{C}}$ corresponding to α under $\mathfrak{t}_{\mathbb{C}} \cong \mathfrak{t}_{\mathbb{C}}^*$ (Proposition 4.5.10).

If $\dim \mathfrak{g}_\alpha > 1$, take $0 \neq y \in \mathfrak{g}_{-\alpha}$ κ-orthogonal to e_α, i.e., $[e_\alpha, y] = 0$. Then

$$\mathrm{ad}(h_\alpha, y) = -\alpha(h_\alpha)y = -2y$$

so the weight is -2.

So y is primitive (as $e_\alpha \cdot y = 0$) and has weight -2, but this is a contradiction since primitives have non-negative integer weight. ■

Now suppose $\alpha, \beta \in \Phi_{\mathbb{C}}$ are linearly independent roots and take $\bigoplus_{m \in \mathbb{Z}} \mathfrak{g}_{\beta + m\alpha} \subseteq \mathfrak{g}_{\mathbb{C}}$ which is a sub-representation of $\mathfrak{sl}_2(\mathbb{C})_\alpha$ in $\mathfrak{g}_{\mathbb{C}}$ (this takes some checking). Then we have the following:

The proof is omitted but is straightforward.

Proposition 5.2.2: Serre Relations

With α, β as above with $\mathfrak{g}_{\alpha - \beta} = 0$, then $V = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}_{\beta + m\alpha}$ is an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})_\alpha$. The indices where $\mathfrak{g}_{\beta + m\alpha}$ is nonzero are those with $m \in \{0, \cdots, N\}$ where $N = 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}$. Thus V has a basis $(e_\beta, \mathrm{ad}(e_\alpha)e_\beta, \cdots, \mathrm{ad}(e_\alpha)^N e_\beta)$ and

$$\mathrm{ad}(e_\alpha)^{N+1} e_\beta = \mathrm{ad}(f_\alpha)^{N+1} f_\beta = 0$$

The above are called the *Serre relations*.

The Serre relations are named for their discoverer, Chevalley.

The Weyl Group as Isometries

The Weyl group $W(G, T)$ can be thought of by Lemma 4.5.3 as the group of automorphisms of T arising from an inner automorphism of G. One can check that Ad embeds $N(T)$ into $\mathrm{O}(\mathfrak{t})$ using the naturality of the Cartan-Killing form, hence W into $\mathrm{O}(\mathfrak{t})$. Using the isomorphism $\mathfrak{t} \cong \mathfrak{t}^*$, we also have $W \hookrightarrow \mathrm{O}(\mathfrak{t}^*)$ via $wT \mapsto (\mathrm{Ad}(w)|_{\mathfrak{t}})^*$. The set of real infinitesimal roots Φ lives in $\mathfrak{t}^*$, and one can check that the action of W preserves Φ.

Lecture 32: December 2nd

For any $0 \neq \alpha \in \mathfrak{t}^*$, we have the reflection s_α in $\alpha^\perp$ which leaves the α direction unchanged and rotates $\alpha^\perp$. On $\mathfrak{t}$ we have the inner product $-\kappa(-,-)$ which gives rise to an inner product $(-,-)$ on $\mathfrak{t}^*$, and the formula for our reflection is

$$s_\alpha(v) = v - 2\frac{(\alpha, v)}{(\alpha, \alpha)}\alpha$$

Theorem 5.3.1

$W \subseteq \mathrm{O}(\mathfrak{t}^*)$ contains the reflection s_α for all $\alpha \in \Phi$.

More precisely, one can show that W is generated by the s_α, but we don't yet have the technology for that. The proof follows using *coroots*, which are sometimes easier to work with.

PROOF : Recall that for any complex root α, we introduced an element $h_\alpha \in \mathfrak{t}$ characterized by $\theta(h_\alpha) = 2\frac{(\alpha,\theta)}{(\alpha,\alpha)}$ for all $\theta \in (\mathfrak{t}_{\mathbb{C}})^*$, called a *coroot*.

Now, for α a *real* root, we may repeat this definition with a change of notation: the coroot $\alpha^\vee$ is the element of $\mathfrak{t}$ such that

$$\theta(\alpha^\vee) = 2\frac{(\alpha, \theta)}{(\alpha, \alpha)}$$

for all $\theta \in \mathfrak{t}^*$. Note that $\alpha(\alpha^\vee) = 2$ so $s_\alpha(v) = v - v(\alpha^\vee)\alpha$.

We want to find an element $w_\alpha \in W$ whose action on $\mathfrak{t}^*$ is s_α. To that end, note that we have a homomorphism $r_\alpha : \mathrm{SU}(2) \to G$ such that $\mathrm{Im}(D_e r_\alpha) = \mathbb{R}\alpha^\vee \oplus P_\alpha$; in particular, $D_e r_\alpha(\sigma_1) = \alpha^\vee$ where σ_1 is the first Pauli matrix. The Weyl group of $\mathrm{SU}(2)$ is $S_2 = \mathbb{Z}/2$ and its unique nontrivial element is represented by $w = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right) = \exp\left(\frac{\pi}{2}\sigma_3\right)$. Note that $w^2 = -I \in T$ and $w^4 = I = \exp(2\pi\sigma_3) = I$ so $2\pi\sigma_3$ is a cocharacter of $\mathrm{SU}(2)$. Set $w_\alpha = r_\alpha(w)$.

Lemma 5.3.2

w_α normalizes T and so represents an element of $W(G)$ of order 2. Its action on $\mathfrak{t}$ is

$$\mathrm{Ad}(w_\alpha)(t) = t - \alpha(t)\alpha^\vee$$

and its action on $\mathfrak{t}^*$ is by s_α.

The proof of this lemma follows by the naturality of exp and some tedious (hence omitted) calculation, which completes the proof of the claim. ■

Abstract Root Systems

Definition 5.4.1: Abstract Root Systems

An *(abstract) root system* (V, Φ) is a finite dimensional inner-product space $(V, (-,-))$ together with a finite subset $\Phi \subseteq V \setminus \{0\}$ whose elements are called *roots*, such that:

1. $\operatorname{span} \Phi = V$
2. For all $\alpha \in \Phi$, $s_\alpha(\Phi) = \Phi$
3. For all $\alpha, \beta \in \Phi$, the number $n_{\alpha\beta} = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer (called the *Cartan integers* of Φ)
4. If $\alpha \in \Phi$ and $c \in \mathbb{R}$, then $c\alpha \in \Phi \iff c = \pm 1$

Note that the second axiom implies that Φ is closed under antipodal reflection $\alpha \mapsto -\alpha$, so in the fourth axiom, one of the implications in $\iff$ is redundant. Moreover, omitting the fourth axiom entirely, if α maximizes length among vectors in $\Phi \cap \mathbb{R}\alpha$, one can show that either

$$\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\} \text{ or } \{-\alpha, -\frac{1}{2}\alpha, \frac{1}{2}\alpha, \alpha\}$$

The significance of an abstract root system, whose axioms may seem arbitrary, is the following:

Theorem 5.4.2

The (real, infinitesimal) roots $\Phi \subseteq \mathfrak{t}^*$ coming from a compact connected semisimple Lie group form a root system. The set of coroots $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$ is a root system in $\mathfrak{t}$.

The proof of this theorem is omitted; it is somewhat calculation heavy and intuition light, and can be found in all of the standard texts. Due to the acceleration of content towards the end of the semester, an increasing proportion of proofs will be omitted.

Although they capture equivalent information, it is useful to have both notions (roots and coroots) as proofs of various lemmata are often more natural in one setting than the other.

Let the *rank* of Φ be defined as the dimension of the vector space in which it lives, so the rank of $\Phi \subseteq \mathfrak{t}^*$ as above is equal to the rank of G. With the axioms of an abstract root system in place, it is possible to classify them:

Lecture 33: December 4th

Rank 1

A rank 1 root system is generated by a single element $\alpha \neq 0$, and must also contain $-\alpha$, and no other elements by the fourth axiom. All such root systems are evidently equivalent and denoted A_1.

A_1 is the root system for $\mathrm{SU}(2)$.

Rank 2

TikZ for the root systems as always stolen from Tim.

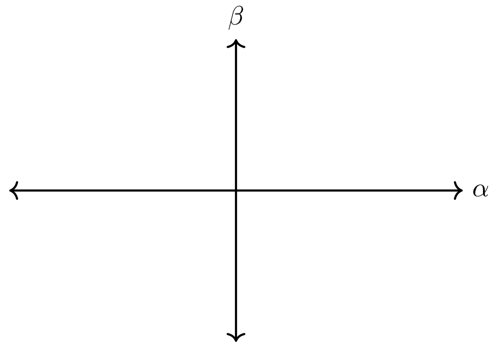
Remark 5.4.3

If (V, Φ) and (V', Φ') are root systems, so is their sum $(V \oplus V', \Phi \times \{0\} \cup \{0\} \times \Phi')$.

Thus, we get our first and most obvious rank 2 root system, $A_1 \times A_1$ (somewhat idiosyncratically this is denoted as a product):

$A_1 \times A_1 = D_2$ is the root system for $\mathrm{SU}(2) \times \mathrm{SU}(2) = \mathrm{Spin}(4)$.

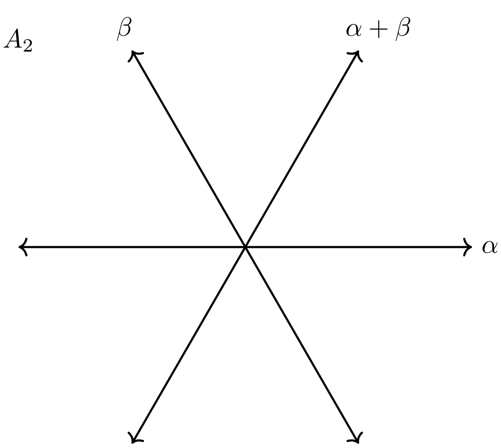
$A_1 \times A_1$



when $|\alpha| = |\beta|$, this root system is also known as D_2

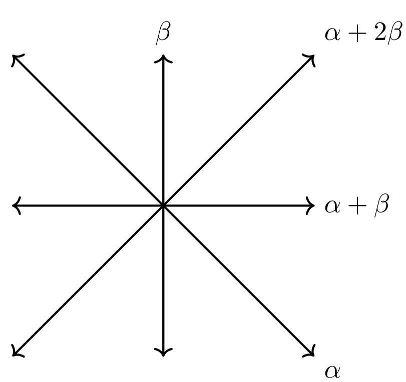
The remaining root systems are the following:

A_2

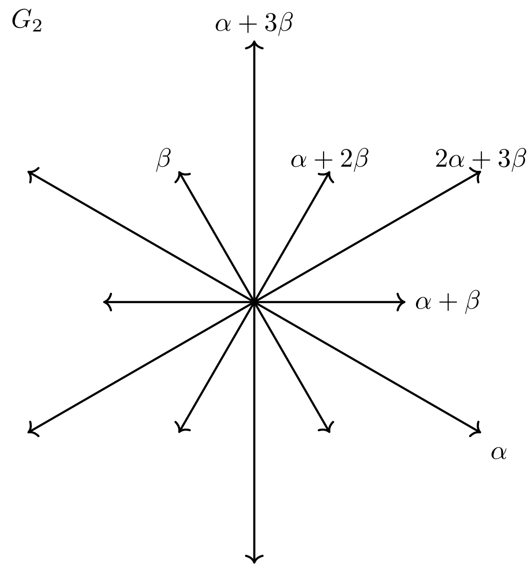


A_2 is the root system for SU(3).

$B_2 = C_2$



B_2 is the root system for Spin(5) = Sp(2).



G_2 is the root system for G_2, an exceptional Lie group constructed from this root system.

G_2 is our first *exceptional* root system, corresponding to an exceptional Lie algebra lying outside of the infinite families A_n, B_n, C_n, D_n (which have corresponding infinite families in the root system/Dynkin diagram setting). We can arrive at this short list of rank 2 root systems systematically after first developing some basic properties of abstract root systems.

Reducibility and Isomorphism

Towards the notion of equivalence for root systems (which we have implicitly used above), we have the following observations:

- If Φ is a root system in V, then so is $cA\Phi$ with $c \neq 0$, $A \in \mathrm{O}(V)$. These ought to be isomorphic.
- We noted above that there is a natural notion of sum or product of root systems. If we can decompose (V, Φ) as a sum, then it is *reducible* and *irreducible* otherwise.
- Given (V_1, Φ_1) and (V_2, Φ_2), we can form their sum with arbitrary scalars: $(V_1 \times V_2, c_1\Phi_1 \times \{0\} \cup \{0\} \times c_2\Phi_2)$ with $c_1c_2 \neq 0$ and we would like all of these root systems in $V_1 \times V_2$ to be considered isomorphic.
- Given (V, Φ), we may define an equivalence relation on Φ generated by $\alpha \sim \beta$ if $(\alpha, \beta) \neq 0$. This gives rise to equivalence classes Φ_i, and it turns out that $V = \bigoplus_i V_i$ where $V_i = \operatorname{span} \Phi_i$ is an orthogonal decomposition. This decomposes (V, Φ) as a product of sub-root systems (V_i, Φ_i) each of which is irreducible.

Then, we have the following definition:

Definition 5.4.4: Isomorphism of (Abstract) Root Systems

If (V, Φ) and (V', Φ') are root systems, an isomorphism between them is an invertible linear map $\theta : V \to V'$ with $\theta(\Phi) = \Phi'$ and if (V, Φ) has irreducible components (V_i, Φ_i), (V', Φ') has irreducible components (V_j', Φ_j'), with $\theta(V_i) = V_j'$ and $\theta|_{V_i} : V_i \to V_j'$ is a homothety i.e. of the form cA where $c \neq 0$ and B is an isometry.

This definition is a little verbose and can be simplified by ignoring the irreducible sub-root systems entirely; the following are equivalent to the above definition as demands on $\theta : V \to V'$:

1. $\theta(\Phi) = \Phi'$ and θ preserves the Cartan integers: $n_{A\beta, A_\alpha} = n_{\beta,\alpha}$ for all roots α, β.
2. $\theta(\Phi) = \Phi'$ and $A \circ s_\alpha = s_{A\alpha}$ for all $\alpha \in \Phi$ i.e. A intertwines the reflections.

We can now explicitly check that the above list of rank 2 root systems is complete by casework on the nontrivial Cartan integer, which will tell us the angle θ between α and β with α, β the generators for some rank 2 root system Φ. Note that $n_{\alpha,\alpha} = 2$, $n_{\alpha,-\alpha} = -2$ (similarly for β), and $\cos\theta = \frac{(\alpha,\beta)}{|\alpha||\beta|}$; then $n_{\beta\alpha} = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} = 2\cos\theta\frac{|\beta|}{|\alpha|}$ so

$$n_{\beta\alpha} n_{\alpha\beta} = 4\cos^2\theta \in \{0, 1, 2, 3, 4\}$$

For this to be an integer then gives us a very short list of possible values for θ:

$n_{\beta\alpha} n_{\alpha\beta}$	θ and length constraints	Root System
4	$\theta = 0, \pi$, $\alpha = \pm\beta$	degenerate configuration
3	$\cos\theta = \pm\frac{\sqrt{3}}{2}$, so $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ and $\lvert\beta\rvert^2 = 3\lvert\alpha\rvert^2$	G_2
2	$\cos\theta = \pm\frac{1}{\sqrt{2}}$, so $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ and $\lvert\beta\rvert^2 = 2\lvert\alpha\rvert^2$	$B_2 = C_2$
1	$\cos\theta = \pm\frac{1}{2}$, so $\theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$ and $\lvert\beta\rvert^2 = \lvert\alpha\rvert^2$	A_2
0	$\theta = \frac{\pi}{2}$ so $\alpha \perp \beta$ and there is no length constraint	$A_1 \times A_1$

We call an irreducible root system *simply-laced* if all its roots have the same length (e.g. A_2).

Root Lattices

It is possible to extend some of the above examples of root systems to infinite families of root systems indexed by dimension. Our technique to do this is to consider the root system associated to a lattice. Suppose $L \subseteq \mathbb{R}^n$ is an integer lattice, i.e., a discrete subgroup of full rank n s.t. $\lambda_1, \lambda_2 \in L$ implies that $(\lambda_1, \lambda_2) \in \mathbb{Z}$; L is called a *root lattice* if the set $\Phi_L = \{\lambda \in L : (\lambda, \lambda) = 2\}$ spans L. In this case, Φ_L is a root system in $\mathbb{R}^n$. When L is an *even* root lattice, i.e., $(\lambda, \lambda) \in 2\mathbb{Z}$ for all $\lambda \in L$, Φ_L is simply-laced. This follows by the reflection formula, which becomes

$$s_\alpha(v) = v - 2\frac{(\alpha, v)}{(\alpha, \alpha)} = v - (\alpha, v)\alpha \in L$$

for α of length-squared 2 (i.e., in Φ_L), since $(\alpha, v) \in \mathbb{Z}$ and lattices are $\mathbb{Z}$-linearly closed.

One can obtain non-simply-laced root systems via a modification of this construction by setting

$$\Phi_L = \{\lambda \in L : (\lambda, \lambda) = 1 \text{ or } 2\}$$

Example 5.4.5: A_n

To construct A_n, the root system for the Lie algebra $\mathfrak{su}(n+1)$, consider $v = (1, \cdots, 1) \in \mathbb{R}^{n+1}$, and in particular, the subspace $V = v^\perp$ with its induced inner product arising from the standard inner product on $\mathbb{R}^{n+1}$. Set

$$\Phi := \{e_i - e_j : i \neq j\}$$

and let $\alpha_{ij} = e_i - e_j$. It is routine to check that Φ forms a simply-laced root system with all lengths equal to 2. Φ spans

$$L_{A_n} = V \cap \mathbb{Z}^{n+1} = \{\sum_i n_i e_i : n_i \in \mathbb{Z}, \sum_i n_i = 0\}$$

Example 5.4.6: D_n

D_n is the root system of the Lie algebra $\mathfrak{so}(2n)$ and is given by $\Phi = \{e_i \pm e_j : i \neq j\} \subseteq \mathbb{Z}^n$ which is simply-laced with length-square 2 in the D_n lattice

$$L_{D_n} = \{\sum_i n_i e_i : n_i \in \mathbb{Z}, \sum_i n_i \in 2\mathbb{Z}\}$$

Since L_{D_n} is an even lattice spanned by Φ, Φ is a simply-laced root system.

Example 5.4.7: B_n and C_n

B_n and C_n are the root systems for $\mathfrak{so}(2n+1)$ and $\mathfrak{sp}(2n)$ respectively, and are given by

$$\Phi_{B_n} = \{e_i \pm e_j : i \neq j\} \cup \{\pm e_i\} \subseteq \mathbb{Z}^n$$

and

$$\Phi_{C_n} = \{e_i \pm e_j : i \neq j\} \cup \{\pm 2e_i\} \subseteq \mathbb{Z}^n$$

The corresponding lattices

$$L_{B_n} = \{\sum_i n_i e_i : n_i \in \mathbb{Z}\} = \mathbb{Z}^n$$

and

$$L_{C_n} = \{\sum_i n_i e_i : n_i \in \mathbb{Z}, \sum_i n_i \in 2\mathbb{Z}\}$$

are spanned by their respective *short roots* (it turns out that (λ, λ) can only take on at most two values for a given root system, and B_n and C_n are evidently not simply-laced).

Starting from even lattices (finding these is itself a non-trivial task), one can also find three of the five exceptional root systems E_6, E_7, E_8; the remaining two, F_4 and G_2 are not simply-laced.

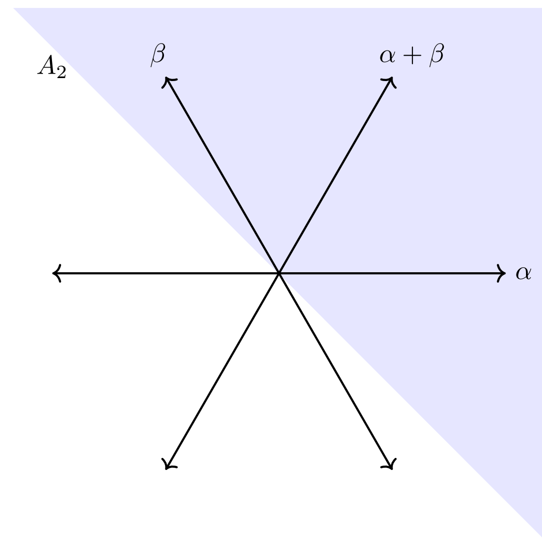
Positive and Simple Roots

Lecture 34: December 6th

One important notion in the study of root systems is that of *positive roots*, which involves a choice of separating hyperplane in V so that half of the roots are on one side and their negatives are on the other. Explicitly, pick $\lambda \in V^*$ such that $\lambda(\alpha) \neq 0$ for all $\alpha \in \Phi$. Then the positive roots $\Phi^+ \subseteq \Phi$ are precisely those for which $\lambda(\alpha) > 0$ and $\Phi = \Phi^+ \sqcup (-\Phi^+)$. In the above pictures of rank 2 root systems, the labeled roots form a set of positive roots.

Example 5.4.8

A half plane $\lambda > 0$ is shaded in blue showing a choice of positive roots for A_2:



It is instructive to think about rotating this shaded region and at what points in this rotation the shaded half plane does not give us a valid subset of positive roots.

Definition 5.4.9: Simple Roots

With respect to a chosen set of positive roots Φ^+, the set of *simple roots* in Φ^+ are those which are not the sum of two positive roots.

In the above example, α and β are simple while $\alpha + \beta$ is not.

Proposition 5.4.10

Any positive root is a sum of simple roots.

PROOF : Let Φ^+_λ be given by $\lambda > 0$ for some $\lambda \in V^*$. Order the positive roots $\alpha_1, \cdots, \alpha_n$ where $0 < \lambda(\alpha_1) \leq \cdots \leq \lambda(\alpha_n)$. If some α_k is not simple, then $\alpha_k = \alpha_i + \alpha_j$ with $i, j < k$ by definition, and we can recursively expand α_i, α_j into sums of other positive roots, eventually terminating in a sum of simple roots. This process terminates since $\lambda(\alpha_i)$ is bounded below by 0 and attains finitely many values, and decreases at each step of this expansion. ■

Thus for any root α, either α or $-\alpha$ (but never both) is a sum of simple roots (with some choice of positive roots fixed).

Proposition 5.4.11

If $\alpha \neq \beta \in \Phi$ with $(\alpha, \beta) > 0$ (i.e. they meet at an acute angle), then $\alpha - \beta$ is a root.

PROOF : $n_{\beta\alpha} > 0$ by assumption so $n_{\beta\alpha} n_{\alpha\beta} \in \{1, 2, 3\}$, so one of $n_{\beta\alpha}, n_{\alpha\beta}$ is equal to 1 (we may assume $n_{\beta\alpha} = 1$). But then $\alpha - \beta = s_\beta \alpha$. ■

Proposition 5.4.12

Distinct simple roots α, β meet at a right or obtuse angle (i.e. $(\alpha, \beta) \leq 0$).

PROOF : If $(\alpha, \beta) > 0$ then $\alpha - \beta$ would be a root, as would $\beta - \alpha$. One of these must be positive (WLOG $\alpha - \beta$), but $\alpha = (\alpha - \beta) + \beta$ is a sum of positive roots, which contradicts simplicity of α. ■

Putting these together, we have the following:

Lemma 5.4.13

The set Δ of simple roots forms a basis for the ambient vector space V.

PROOF : The simple roots $\mathbb{N}$-linearly span the positive roots, therefore they $\mathbb{Z}$-linearly span all of the roots (and therefore $\mathbb{R}$-linearly span V itself). Linear independence follows from an analysis of the angles between the simple roots: if there is a non-trivial linear dependence relation, write it as

$$\varphi = \sum_i x_i \beta_i = \sum_j y_j \gamma_j$$

where $\beta_i \in \Delta_1$, $\gamma_j \in \Delta_2$ and $\Delta = \Delta_1 \sqcup \Delta_2$, and where the coefficients are all positive. Then

$$0 \leq (\varphi, \varphi) = \sum_{i,j} x_i y_j (\beta_i, \gamma_j) \leq 0$$

since simple roots met at a right or obtuse angle, hence $\varphi = 0$, contradicting the non-triviality of the relation. ■

Lecture 35: December 9th

Definition 5.4.14: A Base

A subset Δ of Φ is called a *base* if it is linearly independent and $\mathbb{N}$-linearly spans Φ^+.

The simple roots form a base by the above, and it turns out that every base arises as the set of simple roots for some selection of positive roots, so "base" is just an occasionally convenient way to say "simple roots with respect to some $\lambda \in V^*$."

The Weyl Group Redux

Definition 5.5.1: The Weyl Group of a Root System

The *Weyl group* $W = W(\Phi)$ of a root system Φ is the subgroup of $\mathrm{O}(V)$ generated by the reflections s_α, $\alpha \in \Phi$.

Recall that we have shown that a compact connected semisimple Lie group G with maximal torus T gives rise to a Weyl group $W(G, T) = N(T)/T \subseteq$

O(t*) and we have shown that this Weyl group contains the Weyl group of reflections (Theorem 5.3.1) i.e. $W(G,T) \supseteq W(\Phi)$. As suggested by our otherwise incoherent nomenclature, these groups will end up being equal, though this has yet to be shown.

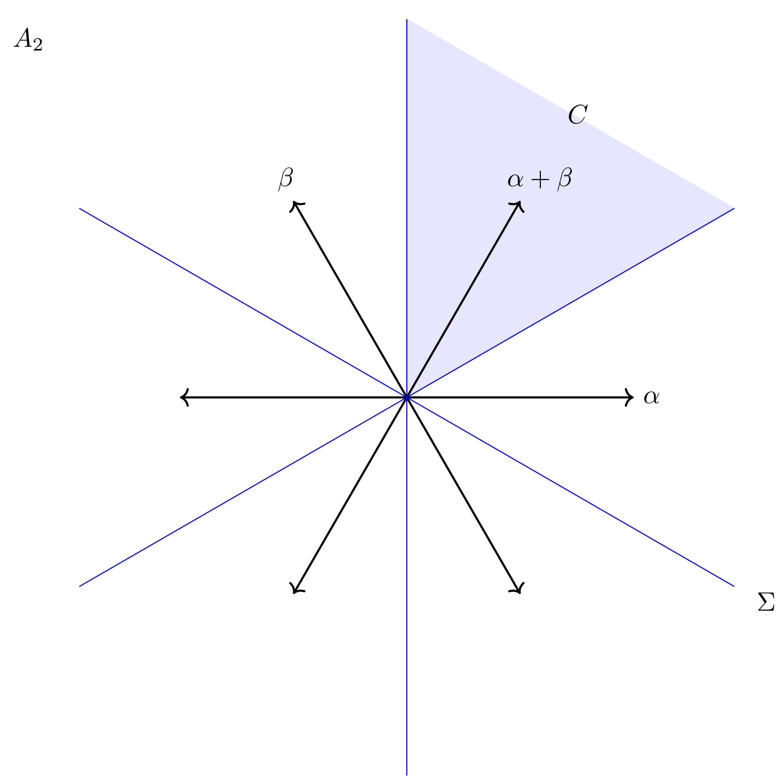
Definition 5.5.2: Weyl Chambers

Let $\Sigma = \cup_{\alpha\in\Phi}\alpha^\perp \subseteq V$; the connected components C of $V \setminus \Sigma$ are called the *Weyl chambers* of Φ.

Note that the Weyl chambers are in bijection with choices of subsets of positive roots Φ^+ since the only invalid choices for v in $\lambda = (v, -)$ as above are precisely the roots α themselves, and passage through the hyperplanes $\alpha^\perp$ switches a given root from positive to negative. Varying λ in some fixed chamber C evidently does not change the set of positive roots.

Example 5.5.3

Here is the chamber structure on A_2, with a Weyl chamber C shaded (this is the choice of Weyl chamber that gives us the choice of positive roots shown in our previous depiction of A_2):



More than a simple correspondence, we in fact have a W-equivariant bijection between Weyl chambers and choices of positive roots defined explicitly as follows: given a Weyl chamber C, fix any $v \in C$ and set $\lambda = (v, -)$, and define Φ^+_λ as usual. This bijection intertwines the action of W on chambers and the action on sets of positive roots.

The action of $W(\Phi)$ on the Weyl chambers is transitive since reflection through any of the hyperplanes $\alpha^\perp$ exchanges adjacent chambers on opposite sides of $\alpha^\perp$, so we can take a chamber C to any chamber adjacent to it, and, repeating this, to any other chamber. Thus, W acts transitively on the set of bases for Φ.

Theorem 5.5.4

$W(\Delta) = \Phi$ and W is generated by the reflections through simple roots.

The proof is omitted but straightforward.

Theorem 5.5.5

Our two definitions for the Weyl group are equivalent for G compact connected and semisimple. Moreover, $W(G,T)$ acts freely and transitively on the set of Weyl chambers.

The proof is omitted. What remains to be shown is that $w \in W(G,T)$ is a reflection through some root α; the proof proceeds by studying the action of w on the Weyl chambers. The proof is not too complicated, but is not particularly interesting.

Cartan Matrices and Dynkin Diagrams

We have seen in our discussion above that the data of a root system is completely captured by the coefficients $n_{\alpha\beta}$; we can further organize this data as follows:

Definition 5.6.1: Cartan Matrix

Let (V,Φ) be a root system, Φ^+ a choice of positive roots, $\Delta \subseteq \Phi^+$ the corresponding simple roots. The *Cartan matrix* associated to this data is the function $A : \Delta \times \Delta \to \mathbb{Z}$ given by

$$A(\alpha,\beta) = n_{\alpha\beta}$$

If one chooses an ordering of Δ, then A can be given by an actual matrix.

Example 5.6.2

The Cartan matrices for the rank 2 root systems with the positive roots as above (ordered lexicographically) are as follows:

$$A(A_1 \times A_1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad A(A_2) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad A(B_2) = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad A(G_2) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Proposition 5.6.3

The Cartan matrix A for an ordered base fully determines the root system.

Proof : As noted above, the reflections through simple roots generate the Weyl group W, and $W(\Delta) = \Phi$. Moreover, $s_{\alpha_i}(\alpha_j) = \alpha_j - A_{ji}\alpha_i$ and the simple roots span V, so from A we may determine the reflections through simple roots, hence W, hence $W(\Delta) = \Phi$. ∎

Dynkin Diagrams

On the final class day (which I missed) or perhaps in its unrealized coda, we have finally gotten to the main reason I took this class.

We are now (at last) ready to introduce Dynkin diagrams, which are the key combinatorial gadget used to classify Lie algebras and their Lie groups subject to adequate adjectives.

Definition 5.6.4: Dynkin Diagrams

A *Dynkin diagram* is decorated graph of the following kind: there is a vertex set, and for each pair of distinct vertices (v, v') a multiplicity $m(v, v') \in \mathbb{N}$. If $m(v, v') > 1$ there is also an arrow pointing from v to v' or vice versa — i.e., a distinguished element of the pair $\{v, v'\}$.

Definition 5.6.5: Dynkin Diagram of a Root System

The Dynkin diagram associated to a root system Φ (with a specified system of simple roots Δ) is a Dynkin diagram whose vertex set is Δ, and multiplicities $m(\alpha, \beta) = n_{\alpha\beta}n_{\beta\alpha} \in \{0, 1, 2, 3\}$. If the multiplicity is 2 or 3, the arrow points from the longer to the shorter root.

We will see shortly how these Dynkin diagrams correspond to the Lie algebras and root systems whose names they share.

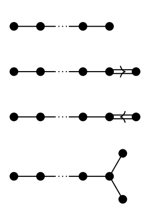
Example 5.6.6

A_n :

B_n :

C_n :

D_n :



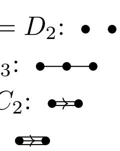
The coefficient n refers to the number of vertices. We may also (as particular instantiations of the above) give more explicit Dynkin diagrams for the rank 2 root systems which we explicitly computed above:

$A_1 \times A_1 = D_2$:

$A_3 = D_3$:

$B_2 = C_2$:

G_2:



Lemma 5.6.7

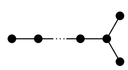
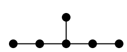
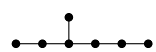
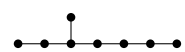
The connected components of the Dynkin diagram correspond to the irreducible components of the root system.

This follows directly from definitions.

The angles between the simple roots can be read off directly from the Dynkin diagram from the multiplicities for $i \neq j$ (setting $(e_i, e_i) = 1$):

$$(e_i, e_j) = \begin{cases} 0, & m = 0, \\ \cos(2\pi/3) = -1/2, & m = 1, \\ \cos(3\pi/4) = -\sqrt{2}/2, & m = 2, \\ \cos(5\pi/6) = -\sqrt{3}/2, & m = 3. \end{cases}$$

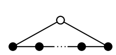
From these restrictions, we may progressively whittle down the possible simply-laced (i.e. all multiplicity 1) Dynkin diagrams and produce a complete classification. Our strategy will be to start from the valid simply-laced Dynkin diagrams:

A_n: D_n:  E_6:  E_7:  E_8: 

whose existence we accept as an axiom from e.g. the above calculations and lattice/quadratic form theory (in the case of E_6, E_7, and E_8). From a valid diagram, we add extra vertices and then demonstrate that this leads to a contradiction.

Lemma 5.6.8

The diagram



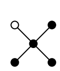
does not arise as a sub-Dynkin diagram of any root system (where by sub-Dynkin diagram we mean merely a subgraph).

Proof : Suppose $e_0, \cdots, e_n$ is the basis of $\mathbb{R}^{n+1}$ coming from this diagram, and set $v = e_0 + \cdots + e_n$. Then

$$(v, v) = \sum_{i=0}^{n} e_i^2 + 2\sum_{i<j}(e_i, e_j) = n + 1 - \frac{1}{2}2(n+1) = 0$$

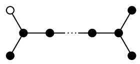
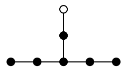
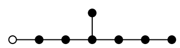
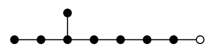
■

We may also rule out a vertex of valency ≥ 4:



Set v_0 to the central vertex, $v_1, \cdots, v_4$ the outer vertices, then $v = 2e_0 + (e_1 + \cdots + e_4)$ has $(v, v) = 0$. We omit the following verifications which are substantially similar to the two above:

- A connected simply-laced sub-Dynkin diagram cannot have two distinct vertices of valency at least 3.
- The following sub-Dynkin diagrams are "illegal" (i.e. give rise to a contradiction):

(i)  (ii)  (iii)  (iv) 

It then follows that the only connected simply-laced Dynkin diagrams are the ADE ones listed above. One can extend this combinatorial playtime to show that the only non-simply-laced Dynkin diagrams arising from irreducible root systems are B_n, C_n, G_2 and F_4 .

The ADE Dynkin diagrams show up everywhere in math, and a desire to understand them was the main reason I took this class. Here are a few flavors of DynkinADE:

- As alluded to throughout this impressionistic discussion is that simply laced Dynkin diagrams are intimately related to lattices; a theorem of Witt states that any integral lattice having a basis consisting of vectors of norm-square 2 is a direct sum of ADE lattices. Moreover, an integral lattice with a basis of vectors of norm-square 1 or 2 is a sum of ADEs and copies of $\mathbb{Z}$.
- A quiver is simply a directed graph (where loops and multiple arrows are allowed) and a quiver representation is a functor from this graph (regarded as a category) to the category of finite-dimensional (complex) vector spaces. There is an evident category of representations $\mathrm{Rep}(Q)$ of any quiver Q; Gabriel's theorem states that a quiver Q has *finite type* (i.e. has finitely many isomorphism classes of indecomposable objects) iff the underlying undirected graph of Q is an ADE Dynkin diagram. Moreover, in this case, the indecomposable representations of Q are in bijection with the positive roots of the Dynkin diagram. This can be extended to all quivers using Kac-Moody algebras.
- There is a correspondence, due to McKay, between the ADE Dynkin diagrams and the finite subgroups of $SO(3)$ (or $SU(2)$).

It remains to realize E_6, E_7, E_8, G_2, and F_4 as lattices (with corresponding root systems) giving rise to these Dynkin diagrams; as these constructions can only appear *ad hoc* without the development of much further theory, we omit them here. E_8 is particularly important and appears ubiquitously in lattice theory.

Theorem 5.6.9

Any irreducible root system is isomorphic to A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 4)$, or E_6, E_7, E_8, G_2, or F_4.

Sketch : The above list represents all connected Dynkin diagrams, and any irreducible root system determines such a diagram. Since a given Dynkin diagram determines the simple roots Δ, and hence reflections through them (which generate the Weyl group), and $W(\Delta) = \Phi$, the Dynkin diagram determines Φ. ■

To complete our classification of Lie algebras from Dynkin diagrams, we need now to be able to reconstruct a Lie algebra $\mathfrak{g}$ from its root system $\Phi \subseteq \mathfrak{t}^*$; we also need to show that every root system arises from a compact connected semisimple Lie group.

Lie Algebras from Root Systems

Let Φ be a root system, $\Delta = \{\alpha_1, \cdots, \alpha_r\}$ a base for Φ with associated Cartan matrix A_{ij}. We can now build a complex Lie algebra associated to these data by hand with generators and relations: Let X_i, Y_i, H_i for $i = 1, \cdots, r$ be our generators, and let $\tilde{\mathfrak{g}}$ be the complex Lie algebra with these generators and the following *Weyl relations*:

$$[H_i, H_j] = 0 \qquad [X_i, Y_j] = \delta_{ij} H_i \qquad [H_i, X_j] = A_{ij} X_j \qquad [H_i, Y_j] = -A_{ij} Y_j$$

$\tilde{\mathfrak{g}}$ is infinite-dimensional (since, e.g., $[X_i, X_j]$ is not *a priori* determined in terms of our existing generators), so is not our target. Let $\tilde{\mathfrak{n}}_+$ and $\tilde{\mathfrak{n}}_-$ denote the Lie subalgebras generated by the X_i and Y_i respectively, and $\tilde{\mathfrak{h}}$ the Lie subalgebra generated by the H_i. The H_i commute, so $\tilde{\mathfrak{h}} = \mathbb{C}\{H_1, \cdots, H_n\}$. For any $0 \neq \alpha \in \tilde{\mathfrak{h}}^*$, set $\tilde{\mathfrak{g}}_\alpha := \{x \in \tilde{\mathfrak{g}} : \mathrm{ad}(x)(H_i) = \alpha(H_i)x \text{ for all } i\}$ to be the "root space," and set $Q = \mathbb{Z}\Phi$, $Q_\pm \subseteq \mathbb{Q}$ to be the monoids generated by $\pm\Delta$ (so elements of Q_+ are positive sums of simple roots).

Theorem 5.7.1: Chevalley, Harish-Chandra

We have that $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_+$ as vector spaces. Moreover, $\tilde{\mathfrak{n}}_+$ (resp. $\tilde{\mathfrak{n}}_-$) is the free Lie algebra on the X_i (resp. the Y_i) and $\tilde{\mathfrak{n}}_\pm = \oplus_{\alpha \in Q_\pm \setminus \{0\}} \tilde{\mathfrak{g}}_\alpha$. tk reenumerate

By free Lie algebra on a set of generators, we simply mean the vector space spanned by those generators together with all possible bracketings thereof.

Finally, we define $\mathfrak{g}$ as the quotient of $\tilde{\mathfrak{g}}$ by the following additional relations:

$$\mathrm{ad}(X_i)^{-A_{ij}+1} X_j = 0 \qquad \mathrm{ad}(Y_i)^{-A_{ij}+1} Y_j \text{ for all } i \neq j$$

These relations give what is called the *Serre presentation* of $\mathfrak{g}$.

The Serre presentation is named for its discoverers, Chevalley and Harish-Chandra.

Theorem 5.7.2: Serre

The quotient $\mathfrak{g} = \tilde{\mathfrak{g}}/\mathfrak{r}$ where $\mathfrak{r}$ is the ideal generated by the above two relations, is a semisimple complex lie algebra of finite dimension $|\Phi| + r$ with abelian subalgebra $\mathfrak{h} = \mathbb{C}\{H_1, \cdots, H_r\}$ acting on $\mathfrak{g}$ via the root system Φ.

Although this is the punchline of what we have done so far, the proof is omitted because it is long and also relies on (in the notes) an unproven technical lemma.

Corollary 5.7.3

Every root system Φ arises from a compact, simply connected, semisimple Lie group K with maximal torus T; moreover, the pair (K, T) is uniquely determined up to isomorphism by Φ.

There's some material in Tim's notes beyond the end of the course on universal enveloping algebras (another main reason I took this class) but I will end my notes here.