Symplectic Topology **Fall 2023**

Chapter I: First Impressions

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Symplectic Vector Spaces

The chapters here will line up with the chapters in Tim's notes/ the syllabus.

Definition 1.1.1: Symplectic Vector Spaces

A *symplectic vector space* over a base field k is a pair (V, β) where V is the underlying vector space, and β is a *symplectic pairing* $\beta : V \times V \to k$ is a bilinear skew-symmetric ($\beta(v, v) = 0$) nondegenerate ($\beta^\sharp : V \to V^*$ given by $v \mapsto \beta(v, -)$ is an isomorphism) pairing.

Note that skew-symmetry of β implies that $\beta(u, v) = -\beta(v, u)$ (and is often defined this way), and this equality for all $u, v \in V$ in fact implies that β is skew-symmetric as long as the characteristic of the base field is not 2. If V is finite dimensional, we can pick a basis $e_1, \cdots, e_d$ for V and set $s_{ij} = \beta(e_i, e_j)$ and obtain a skew-symmetric matrix S representing $\beta^\sharp$, with non-degeneracy of β equivalent to $\det S \neq 0$. Non-degeneracy is an open condition with respect to the Zariski topology on k^{d^2}, i.e, $\det S \neq 0$ is an open condition since $\det S$ is a polynomial. Since the ordinary topology on $k = \mathbb{R}$ is finer than the Zariski topology, non-degeneracy is open with respect to it as well. For V finite dimensional, note also that $\beta^\sharp$ being an isomorphism implies that for all $0 \neq v \in V$, there exists u s.t $\beta(v, u) \neq 0$.

Definition 1.1.2: Symplectic Subspace

A linear subspace U of (V, β) symplectic is *symplectic* if $\beta|_U$ is non-degenerate.

We will denote the annihilator of U with U^β, i.e,

$$U^\beta = \{v \in V | \beta(v, U) = 0\}$$

One can show that $U \subseteq V$ is a symplectic subspace iff $V = U \oplus U^\beta$ iff U^β is a symplectic subspace.

As always with a proof about bilinear forms, we will induct on the dimension.

Lemma 1.1.3

Any finite dimensional symplectic vector space (V, β) has even dimension $2n$ and admits a symplectic basis $e_1, \cdots, e_n, f_1, \cdots, f_n$ s.t $\beta(e_i, e_j) = \beta(f_i, f_j) = 0$ and $\beta(e_i, f_j) = \delta_{ij}$.

PROOF : We will induct on the dimension. The base case is when $\dim V = 0$, in which case there is nothing to check. If $V \neq 0$, take $0 \neq e_1 \in V$ and find f_1 s.t $\beta(e_1, f_1) = 1$ (since β is non-degenerate), and let U be the span of e_1 and f_1. U^β is a symplectic vector space of dimension $\dim V - 2$ so we may apply the inductive hypothesis to obtain a symplectic basis for U^β, and adjoin e_1, f_1 to it. ■

The matrix of β with respect to a symplectic basis is $-J_0$ where

$$J_0 = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right)$$

The extra minus sign comes from identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via $e_i \mapsto e_i$, $f_j \mapsto ie_j$, and in this context, J_0 is the matrix of multiplication by i.

This still doesn't really explain the extra minus sign, which I suspect is probably due to historical reasons.

The vector space of skew pairings on V is $\bigwedge^2 V^*$ which consists of sums of the form $\sum_j \alpha_j \wedge \beta_j$ for $\alpha_j, \beta_j \in V^*$ and where

$$(\alpha \wedge \beta)(u, v) = \alpha(u)\beta(v) - \alpha(v)\beta(u)$$

for $u, v \in V$.

With respect to a symplectic basis $(e_1, \cdots, e_n, f_1, \cdots, f_n)$ for β, we can write

$$\beta = \sum_{j=1}^{n} e_j^* \wedge f_j^*$$

Since k^{2n} has a natural basis (written $(e_1, \cdots, e_n, f_1, \cdots, f_n)$), the *standard symplectic pairing* on k^{2n} is given by $\beta_0 = \sum_{i=0}^{n} e_i^* \wedge f_i^*$.

Definition 1.1.4: Isomorphisms

A *symplectic isomoprhism* $(V, \beta) \to (V', \beta')$ is a linear isomorphism $A : V \to V'$ s.t $\beta'(Au, Av) = \beta(u, v)$. Equivalently, $A^*\beta' = \beta$ as elements of $\bigwedge^2 V^*$.

Proposition 1.1.5

$\beta \in \bigwedge^2 V^*$ is symplectic iff $\beta^n = \beta \wedge \cdots \wedge \beta \neq 0 \in \det V^* = \bigwedge^{2n} V^*$.

PROOF : If β is symplectic, we can pick a symplectic basis so that it can be written as β_0, and then explicitly compute $\beta_0^n = n!(e_1 \wedge f_1) \wedge \cdots \wedge (e_n \wedge f_n) \neq 0$, and the converse is straightforward. ■

Tim pauses here to give the story of the word *symplectic*; Weyl liked the word complex, which is from Latin, so he pulled it back, translated it to Greek, and pushed it forward. Apparently mathematicians aren't the only clever ones to do this and biologists talk about symplectic bones, which some fish have.

Definition 1.1.6: Symplectic Linear Groups

The *symplectic linear group* of (V, β) is the group $\mathrm{Sp}(V, \beta)$ consisting of symplectic automorphisms of (V, β).

Note that for $A \in \mathrm{Sp}(V,\beta)$, $A^*\beta = \beta \in \bigwedge^2 V^*$ so $A^*(\beta^n) = (A^*\beta)^n = \beta^n \in \det V^*$, thus, A^* acts trivially on $\det V^*$, i.e $\det A = 1$. Thus, $\mathrm{Sp}(V,\beta) \subseteq \mathrm{SL}(V)$.

This inclusion is an equality in dimension 2 and a strict inclusion in higher dimensions.

$\mathrm{Sp}(V,\beta)$ is cut out from $\mathrm{GL}(V)$ by polynomial equations, i.e, it is an *algebraic group* over k (and in particular a Lie group), and has a Lie algebra denoted $\mathfrak{sp}(V,\beta)$, given by

$$\mathfrak{sp}(V,\beta) = \{\xi \in \mathrm{End}(V) | \beta(\xi u, v) = -\beta(u, \xi v)\}$$

Concretely, for (k^{2n}, β_0),

$$\mathrm{Sp}(k^{2n}) = \{A \in \mathrm{GL}(V) | A^T J_0 A = J_0\} \implies \mathfrak{sp}(k^{2n}) = \{\xi \in \mathfrak{gl}(V) | \xi^T J_0 + J_0 \xi = 0\}$$

by the regular yoga of calculating the Lie algebra by writing $A = I + t\xi$ and keeping only the first order terms in t. $\mathfrak{sp}(k^{2n})$ is equipped with the regular commutator Lie bracket $[-,-]$.

Note that $J_0^T = -J_0$ so $J_0\xi$ is symmetric by the condition $\xi^T J_0 + J_0\xi = 0$, so $\mathfrak{sp}(k^{2n})$ consists of matrices which are J_0 times the symmetric endomorphisms of k^{2n}, from which we can calculate $\dim \mathfrak{sp}(k^{2n}) = \dim \mathrm{Sp}(k^{2n}) = \frac{1}{2} 2n(2n+1) = n(2n+1)$.

Intro to Symplectic Manifolds

This section is largely just a list of results to come, with no proofs.

Definition 1.2.1: Symplectic Manifolds

A *symplectic manifold* (M,ω) is a C^∞ manifold M with a 2-form $\omega \in \Omega^2(M)$ that is non-degenerate (i.e, for all $x \in M$, $\omega_x \in \bigwedge^2(T_x^*M)$ is non-degenerate) and closed (i.e $d\omega = 0$).

In local coordinates $x_1, \cdots, x_{2n}$ on M, ω can be written as

$$\omega = \sum_{i<j} \omega_{ij}(x)\, dx_i \wedge dx_j$$

so $d\omega = 0$ is equivalent to the system of PDEs

$$\frac{\partial}{\partial x_i}\omega_{jk} + \frac{\partial}{\partial x_j}\omega_{ki} + \frac{\partial}{\partial x_k}\omega_{ij} = 0$$

for all i,j,k. Equivalently, $d\omega = 0$ implies that locally ω is exact, i.e, there exists a cover U_α s.t $\omega|_{U_\alpha} = d\eta_\alpha$.

This is the Poincaré lemma, and neither of the two equivalent conditions is specific to symplectic manifolds.

$d\omega = 0$ is a coordinate-independent (diffeomorphism-invariant) condition, since, given a smooth map $M \xrightarrow{\phi} N$, $\omega_N \in \Omega^2(N)$, then $d\omega_N = 0$ implies that $d(\phi^*\omega_N) = 0$ since the exterior derivative d commutes with pullbacks. Non-degeneracy is similarly coordinate independent.

Definition 1.2.2: Symplectomorphisms

A *symplectomorphism* between symplectic manifolds (M, ω_M) and (N, ω_N) is a diffeomorphism $\phi : M \to N$ s.t $\phi^*\omega_N = \omega_M$.

Definition 1.2.3: Symplectic Immersions

A *symplectic immersion* is a smooth map $\phi : M \to N$ s.t $\phi^*\omega_N = \omega_M$.

$\phi^*\omega_N = \omega_M$ implies that $D_x\phi$ is injective for all $x \in M$, as, otherwise, ω_M would be degenerate, so the maps that preserve our chosen structure in this subject are quite rigid.

Example 1.2.4

The basic example of a symplectic manifold is $(\mathbb{R}^{2n}, \omega_0)$, with coordinates $x_1, x_2, \cdots, x_n, y_1, \cdots, y_n$ and

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$$

The coordinate vector fields $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_n}$ form a symplectic basis of each tangent space. We can write

$$\omega_0 = -d\left(\sum_{j=1}^{n} y_j \, dx_j\right)$$

which immediately proves that ω_0 is closed.

We can also work in complex coordinates, with $z_j = x_i + iy_j$, so that

$$\omega_0 = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\overline{z}_j = \frac{i}{2}\partial\overline{\partial}\|z\|^2$$

where $\overline{\partial} f = \sum_j \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j$ and $\partial\alpha = \sum_j \frac{\partial \alpha}{\partial z_j} \wedge dz_j$.

Theorem 1.2.5: Darboux

Near any point p in a symplectic manifold (M^{2n}, ω), there exist coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ such that the coordinate vector fields $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$ form a symplectic basis for each tangent space, i.e, there exist open neighborhoods $p \in U \subseteq M$ and $0 \in U' \subseteq \mathbb{R}^{2n}$, and a symplectomorphism

$$\phi : (U, \omega|_U) \to (U', \omega_0|_{U'})$$

This theorem essentially shows that symplectic manifolds have no interesting local structure (the way that Riemannian manifolds do).

We will later give two proofs of this result, but first need to develop some more background.

Definition 1.2.6: Symplectic Volume

Given (M^{2n}, ω), U an open subset of M, then the *symplectic volume* of U is

$$\text{vol}_\omega(U) = \int_U \omega^n$$

Note that symplectomorphisms ϕ preserve volume, i.e, given $\phi : M \to M$, $\text{vol}_\omega(\phi(U)) = \text{vol}_\omega(U)$.

In $(\mathbb{R}^{2n}, \omega_0)$, $\text{vol}_{\omega_0}(U) = n!\,\text{Measure}(U)$, and there does not exist a symplectic automorphism of $\mathbb{R}^{2n}$ that maps the unit ball into a strictly smaller ball.

Example 1.2.7

Consider S^2 with any symplectic structure on it (we will assume that one exists). There is no symplectic automorphism of S^2 that moves the equator to anything other than a great circle, since, otherwise, the areas on either side of the putative image circle will be different than the areas of the hemispheres on either side of the equator.

I have heard this result also referred to as "Gromov's symplectic camel."

Theorem 1.2.8: Gromov's non-squeezing theorem

There does not exist a symplectic automorphism ϕ of $(\mathbb{R}^{2n}, \omega_0)$ carrying the open unit ball about the origin (denoted $B^{2n}(0; 1)$) into the cylinder $B^2(0; r) \times \mathbb{R}^{2n-2}$ for $r < 1$.

Theorem 1.2.9: Seidel

There exist symplectic automorphisms of certain compact symplectic 4-manifolds (M, ω) s.t ϕ is trivial in $\pi_0\text{Diff}^+(M)$ but nontrivial in $\pi_0\text{Aut}(M, \omega)$ (where Aut in this course always refers to the symplectic automorphism group).

Example 1.2.10: Stein Manifolds

If $X \subseteq \mathbb{C}^n$ is a closed complex submanifold, then $\omega_0|_X$ is a symplectic form on X.

Theorem 1.2.11: Seidel-Smith

For any even number $2k \geq 4$, there exists a smooth affine variety X of complex dimension $2k$ such that X is diffeomorphic to $\mathbb{R}^{4k}$ but not symplectomorphic to $(\mathbb{R}^{4k}, \omega_0)$.

Theorem 1.2.12: McLean

For all $k \geq 3$, there exists an infinite family of pairwise non-isomorphic symplectic (Stein) structures on $\mathbb{R}^{2k}$.

Lagrangian Submanifolds

For (V, β) a symplectic vector space, $T \subseteq V$ (not necessarily a subspace), we write T^β for the annihilator of T (which is always a subspace).

Definition 1.2.13: Lagrangian Subspaces

$L \subseteq V$ is a *Lagrangian subspace* if $L^\beta = L$.

Definition 1.2.14: Isotropic and Coisotropic Subspaces

A subspace $U \subseteq V$ is *isotropic* if $U \subseteq U^\beta$ i.e $\beta = 0$ on $U \times U$, and *coisotropic* if $U \supseteq U^\beta$.

Therefore, a Lagrangian subspace is precisely one that is both isotropic and coisotropic.

Lemma 1.2.15

For a subspace $U \subseteq (V, \beta)$ (finite dimensional), U is isotropic iff there exists a symplectic basis $(e_1, \cdots, e_n; f_1, \cdots, f_n)$ and a $k \leq n$ s.t U is the span of $e_1, \cdots, e_k$.

The proof can be found in Tim's notes, but the reverse implication is immediate, and the forward implication is an easy induction on dimension.

Note that U is isotropic iff U^β is coisotropic since $(U^\beta)^\beta = U$, so we can dualize the above lemma in the obvious way (U is coisotropic iff there exists a symplectic basis $(e_1, \cdots, e_n; f_1, \cdots, f_n)$ and a $k \leq n$ s.t U is the span of $e_{k+1}, \cdots, e_n, f_1, \cdots, f_n$). Combining these two lemmas, we have:

Proposition 1.2.16

For $L \subseteq V$, the following are equivalent:

1. L is Lagrangian
2. L is isotropic and $\dim L = \frac{1}{2} \dim V$
3. L is maximal among isotropic subspaces
4. L is coisotropic and $\dim L = \frac{1}{2} \dim V$
5. L is minimal among coisotropic subspaces
6. There exists a symplectic basis $(e_1, \cdots, e_n, f_1, \cdots, f_n)$ s.t L is the span of the e_i.

The proofs are straightforward and left as an exercise. By the sixth equivalent condition above, we note that $\mathrm{Sp}(V, \beta)$ acts transitively on Lagrangian subspaces.

Definition 1.2.17: Lagrangian Submanifolds

Let (M, ω) be a symplectic manifold, L another manifold. A *Lagrangian immersion* of L in M is an immersion $i : L \to M$ s.t for all $x \in L$, $D_x(T_x L)$ is Lagrangian in $T_x M$ (note that this implies $\dim L = \frac{1}{2} \dim M$). In the case that i is also the inclusion of an embedded submanifold $L \subseteq M$, we call L a *Lagrangian submanifold*.

Example 1.2.18

Any circle embedded in any surface (not necessarily compact or orientable) is a Lagrangian submanifold, since in a 2-dimensional vector space, any one-dimensional subspace is isotropic since $\omega(v, \lambda v) = 0$ by skew-symmetry and bilinearity.

Any orientable surface has a symplectic structure, since by orientability, the surfaces in question have volume forms, which here are also symplectic forms (perhaps up to constant factors).

Example 1.2.19

Let $\phi : (M, \omega_M) \to (N, \omega_N)$ be a symplectomorphism, then its graph

$$\Gamma_\phi = \{(x, \phi(x)) : x \in M\} \subseteq (M \times N, (-\omega_M) \oplus \omega_N)$$

is a Lagrangian submanifold, as one can easily check.

Example 1.2.20: Clifford Tori

Let $C(a) \subseteq \mathbb{C}$ be a circle in $\mathbb{C}$ centered at 0 bounding a disk of *area* a, $C(a_1, \cdots, a_n) := C(a_1) \times \cdots \times C(a_n) \subseteq \mathbb{C}^n$, with $a_i > 0$. Such tori are Lagrangian with respect to ω_0.

One can ask whether two such Clifford Tori are symplectomorphic, and we will later give a necessary and sufficient condition for such a symplectomorphism to hold.

Example 1.2.21: Whitney Embedding Theorem

Let L be a smooth compact n-manifold, then there exists an embedding $L \hookrightarrow \mathbb{R}^{2n}$. One can ask which compact n-manifolds appear as either embedded or immersed Lagrangian submanifolds of $(\mathbb{R}^{2n}, \omega_0)$; in the immersed case, Gromov and Eliashberg gave a complete answer, that any such L is an immersed Lagrangian submanifold of $(\mathbb{R}^{2n}, \omega_0)$ (this is called "symplectic flexibility" and the proof uses a technique called the "h-principle"). Not all such L appear as embedded Lagrangian submanifolds, however; and in fact this fails even for some surfaces embedded in $\mathbb{R}^4$ as we will see below.

Proposition 1.2.22

Let $L \subseteq (\mathbb{R}^{2n}, \omega_0)$ be a closed embedded Lagrangian submanifold, with Euler characteristic $\chi(L)$. Then $\chi(L) \equiv 0 \pmod 2$, and if L is orientable, then $\chi(L) = 0$.

This can be refined when $\dim L = 2$, and there is a result that $\chi(L) \equiv 0 \pmod 4$ due to Audin.

This implies that the only compact orientable surface that can (and in fact does) admit a Lagrangian embedding in $\mathbb{R}^4$ is the torus T^2.

PROOF : Say L is orientable, so it has a self-intersection $L \cdot L \in \mathbb{Z}$ given by the signed count of the number of intersection points, so if L is closed, we can just push our perturbed copy of L far enough away that it doesn't intersect the original copy at all, so $\chi(L) = 0$ (one can also see this with Poincaré duality and cup products since $\mathbb{R}^{2n}$ has no cohomology in dimension $2n$). In the non-orientable case, $L \cdot L$ is only defined $\pmod 2$ and $\chi(L) \equiv 0 \pmod 2$ follows by the same reasoning. ■

Recall that J_0 represents the action of i on $T_x\mathbb{C}^n = \mathbb{C}^n$, so $\omega_0(v, J_0 v) = \|v\|^2 > 0$ for v nonzero. If $\Lambda \subseteq T_x\mathbb{C}^n$ is a Lagrangian subspace, then $\Lambda \cap J_0\Lambda = 0$ since, by the above, $J_0\Lambda$ is disjoint from the annihilator of Λ by ω_0 (which is Λ since Λ is Lagrangian), so $T_x\mathbb{C}^n = \Lambda \oplus J_0\Lambda$. So, an embedded Lagrangian submanifold $L \subseteq \mathbb{C}^n$ has normal bundle $N_{L/\mathbb{C}^n} = J_0(TL) \cong TL$, so, taking a section s of the normal bundle, $L \cdot L$ can be calculated as the number of zeros of s counted by sign, which in turn is equal to the Euler characteristic of L since s is a generic vector field on L, and we may apply Poincaré-Hopf.

Complex Projective Spaces

In this section we will give an infinite family of nontrivial symplectic manifolds, the complex projective spaces. Recall that $\mathbb{CP}^1 = S^2$; since a symplectic form must agree with a volume form on S^2, we must only write down the usual area measure on S^2. $\omega_{S^2} \in \Omega^2(S^2)$ ought to be $\mathrm{SO}(3)$-invariant, and $\int_{S^2} \omega_{S^2} = 4\pi$. We will construct ω_{S^2} by first considering $\mathrm{vol} = dx_1 \wedge dx_2 \wedge dx_3 \in \Omega^3(\mathbb{R}^3)$ which is manifestly rotationally invariant. This is a 3-form, and we want a 2-form, so we contract vol against the vector field

$$Z = \frac{1}{2}\nabla\|\cdot\|^2 = x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3}$$

which is also rotationally invariant (it points radially outwards at each point).

Recall that if X is a vector field on a manifold M, then $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ is the *contraction map* that sends a k-form ω to a $k-1$-form $\iota_X\omega$ via

$$\iota_X\omega(X_1, \cdots, X_{k-1}) = \omega(X, X_1, \cdots, X_{k-1})$$

One can calculate that

$$\iota_Z \,\mathrm{vol} = x_1\, dx_2 \wedge dx_3 + x_2\, dx_3 \wedge dx_1 + x_3\, dx_1 \wedge dx_2$$

and we set $\omega_{S^2} = (\iota_Z \operatorname{vol})|_{S^2}$. This satisfies our normalization condition:

$$\int_{S^2} \omega_{S^2} = \int_{S^2} (\iota_Z \operatorname{vol})|_{S^2} = \int_{B^3} d(\iota_Z \operatorname{vol}) = 3\int_{B^3} \operatorname{vol} = 3\frac{4}{3}\pi = 4\pi$$

Example 1.3.1

Consider S^2 sitting inside $\mathbb{R}^3$, together with its tangent cylinder (aligned with the z axis). Then $\Phi : S^2 \setminus \{\text{poles}\} \to S^1 \times (-1, 1)$ given by mapping (x_1, x_2, x_3) to $(\text{longitude}, x_3)$ is a symplectomorphism. This is due to Archimedes, since a symplectomorphism in this context is equivalent to area-preserving.

This map is what I understand as the Mercator projection, which is conformal but definitely not area-preserving, so I must be wrong, but I then don't know what kind of map projection this is then.

In higher dimensions, $\mathbb{CP}^n = (\mathbb{C}^{n+1} \backslash \{0\})/\mathbb{C}^\times = S(\mathbb{C}^{n+1})/\mathrm{U}(1) = S^{2n+1}/S^1$. We want to define the symplectic form $\tau = \tau_n$ on $\mathbb{CP}^n$ called the *Fubini-Study* form which is invariant under $\mathrm{PU}(n{+}1) = \mathrm{U}(n{+}1)/\text{diagonal matrices}$, satisfying $\tau_n|_{\mathbb{CP}^{n-1}} = \tau_{n-1}$, with normalization condition $\int_{\mathbb{CP}^1} \tau_n = 1$ for any projective line $\mathbb{CP}^1 \subset \mathbb{CP}^n$ (i.e, $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^n$ given by $[x : y] \mapsto [x : y : 0 : \cdots : 0]$). Our construction will be a prototype for symplectic reduction.

Consider the following maps:

$$\begin{array}{ccc} S^{2n+1} & \xhookrightarrow{\quad e \quad} & \mathbb{C}^{n+1} \\ \rho\big\downarrow & & \\ \mathbb{CP}^n = S^{2n+1}/S^1 & & \end{array}$$

where e is just the inclusion and ρ is given by $z \mapsto [z]$ (this map is called the *Hopf fibration*). The idea is to construct τ such that

$$\frac{1}{\pi} e^* \omega_0 = \rho^* \tau$$

where ω_0 is the symplectic form on $\mathbb{C}^{n+1}$.

By Weinstein's symplectic creed "everything is Lagrangian", we consider $(\rho, e) : S^{2n+1} \hookrightarrow \mathbb{CP}^n \times \mathbb{C}^{n+1}$ whose image is middle-dimensional; we want to show that it is Lagrangian with respect to the symplectic form $-\tau \oplus \frac{1}{\pi}\omega_0$.

Starting with some $z \in S^{2n+1}$, we consider its S^1-orbit which has a tangent direction ξ and an orthogonal direction $z^\perp$. $T_z S^{2n+1}$ is the real orthogonal complement to the vector z in $\mathbb{C}^{n+1}$ and therefore splits as $z^\perp \oplus \mathbb{R}\xi$ (where $z^\perp$ is the *Hermitian* orthogonal complement to z).

In the notes, it says an aside that this would make S^{2n+1} a *symplectic correspondence* between the symplectic manifolds $\left(\mathbb{C}^n, \frac{1}{\pi}\omega_0\right)$ and $(\mathbb{CP}^n, \tau_n)$.

$D_z\rho : T_z S^{2n+1} \to T_{[z]}\mathbb{CP}^n \cong z^\perp$ maps ξ to 0 by construction; setting $\eta = \omega_0|_{S^{2n+1}} = e^*\omega_0$, one can check that $\eta(\xi, -) = 0$, equivalently, $\iota_\xi \eta = 0$ (where ι is the contraction map by a vector field as above). Moreover, ω_0, hence η, is S^1-invariant. The above two properties imply that the Lie derivative vanishes, notationally $\mathcal{L}_\xi \eta = 0$.

The *Lie derivative* of a differential form ω by a vector field X is given by the formula

$$\mathcal{L}_X \omega = \iota_X(d\omega) + d(\iota_X \omega)$$

This is known as *Cartan's magic formula*. The Lie derivative has a more general definition in more general contexts, but morally is defined to give the rate of change of a general tensor field along the flow defined by a given vector field.

We want $\frac{1}{\pi}\eta = \rho^*\tau$. Explicitly, we want $\tau_{[z]}(u, v) = \frac{1}{\pi}\eta(u^\natural, v^\natural)$ where $u^\natural$ is the lift to $z^\perp$ of u and similarly for $v^\natural$, so we may as well define τ that way. The lifts are well-defined by the above splitting of $T_z S^{2n+1}$, and for z, z' both mapping to $[z]$, this definition is still well-defined since η is S^1-invariant. The last thing to check is that $d\tau = 0$, which we can see since

$$\rho^*(d\tau) = d(\rho^*\tau) = \frac{1}{\pi}d(\eta) = \frac{1}{\pi}e^*d(\omega_0) = 0$$

by repeatedly applying the commutativity of pullbacks with d.

τ_n is $\mathrm{PU}(n+1)$-invariant, which follows from the fact that η is $\mathrm{U}(n+1)$-invariant, and $\tau_n|_{\mathbb{CP}^{n-1}} = \tau_{n-1}$ (which follows by the analogous restriction statement for η_{2n+1}). For normalization, we want $\int_{\mathbb{CP}^1} \tau_n = 1$ but $\tau_n|_{\mathbb{CP}^1} = \tau_1$ so only have to check this once; our proof strategy will be related to the topology of fiber bundles.

Consider a fiber bundle given by the following data:

$$\begin{array}{ccc} F & \hookrightarrow & E \\ & & \downarrow{\scriptstyle p} \\ & & B \end{array}$$

Here, F is the fiber, B is the base, and E is the total space, with p smooth. E and B are both oriented, and we will take F compact. Then we have the following:

Proposition 1.3.2

There is a homomorphism $\partial : \pi_k(B, b) \to \pi_{k-1}(F_b)$.

Tim remarks that this is essentially the connecting homomorphism in the long exact sequence of homotopy groups.

PROOF : Given $\pi : (D^k, \partial D^k) \to (S^k, \bullet)$ a smooth map restricting to a diffeomorphism on the interior of D^k; given $f : (S^k, \bullet) \to (B, b)$, define $\tilde{f} = f \circ \pi : D^k \to B$. $\tilde{f}^*E$ is then a bundle over D^k which is automatically trivial since D^k is contractible, so $\tilde{f}$ lifts to a map $f^\natural : D^k \to E$ as follows:

$$\begin{array}{ccc} \tilde{f}^*E \cong F_b \times D^k & \longrightarrow & E \\ {\scriptstyle \mathrm{pr}_2}\downarrow & \overset{f^\natural}{\nearrow} & \downarrow{\scriptstyle p} \\ D^k & \xrightarrow[\tilde{f}]{} & B \end{array}$$

In general, if the pullback bundle is trivial then we can define a map $f^\natural$ as above by picking a point $b \in B$ (here we are given one by our setup), and regard our D^k as $\{b\} \times D^k$ and then travel along the map $\tilde{f}^*E \to E$. Then, finally, we can set $\partial f = f^\natural|_{\partial D^k}$. One has to show that this is well-defined i.e up to homotopy of f (we omit this verification). ■

What does this do for us? Suppose we have τ a closed k-form on B, and suppose $p^*\tau = d\gamma$ for some $\gamma \in \Omega^{k-1}(E)$, and $\delta = \gamma|_F$. Then, for $f : (S^k, \bullet) \to (B, b)$ as above, we have

$$\int_{S^k} f^*\tau = \int_{S^{k-1}} (\partial f)^*\delta$$

This follows by the following chain of equalities:

$$\int_{S^k} f^*\tau = \int_{D^k} \tilde{f}^*\tau = \int_{D^k} (f^\natural)^* p^*\tau = \int_{D^k} (f^\natural)^* d\gamma = \int_{D^k} d((f^\natural)^*\gamma) = \int_{S^{k-1}} (\partial f)^*\gamma$$

where the first equality is the pullback by the diffeomorphism π, the final equality is by Stokes' theorem (and the definition $\partial f = f^\natural|_{\partial D^k}$), and all intermediate equalities are by standard properties of integration on manifolds (i.e that pullbacks commute with the exterior derivative).

Using this we may finally verify that $\int_{\mathbb{CP}^1} \tau_1 = 1$ via the Hopf fibration; on $\mathbb{C}^2$, $\omega_0 = \frac{1}{2}\sum_{i=1}^2 x_i\, dy_i - y_i\, dx_i$, so on S^3, we have $\rho^*\tau = \frac{1}{\pi}\eta = d\gamma$ where

$$\gamma = \frac{1}{2\pi}(x_1\, dy_1 - y_1\, dx_1 + x_2\, dy_2 - y_2\, dx_2)|_{S^3}$$

and by the above lemma, $\int_{\mathbb{CP}^1 = S^2} \tau_1 = \int_F \delta$ where $F = S^1$ is some fiber of ρ and $\delta = \gamma|_F = \frac{1}{2\pi}(x_1\, dy_1 - y_1\, dx_1) = \frac{1}{2\pi} d\theta$. Hence $\int_F d\theta = 1$ and the result follows.

Complex Structures

Our basic examples of symplectic manifolds, $\mathbb{R}^{2n}$, have an obvious complex structure as $\mathbb{C}^n$, with hermitian inner product

$$h(z, w) = z \cdot w = \sum_j z_j \overline{w_j}$$

Specifically, we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via a basis for the former $(e_1, \ldots, e_n, f_1, \cdots, f_n)$ where $f_j = ie_j$, and we can then write

$$h_0 = g_0 - i\beta_0$$

where g_0 is the standard inner product on $\mathbb{R}^m$ and β_0 is the standard symplectic form on $\mathbb{R}^{2n}$.

Let J_0 denote the endomorphism of $\mathbb{R}^{2n}$ corresponding to multiplication by i, i.e

$$J_0 = \left(\begin{array}{c|c} 0 & -I_n \\ \hline I_n & 0 \end{array}\right)$$

Using the basis above, one can check that $g_0(u, v) = \beta_0(u, J_0 v)$, and since $h_0(J_0 u, J_0 v) = h_0(u, v)$ we have that $g_0(J_0 u, v) = \beta_0(J_0 u, J_0 v) = \beta_0(u, v)$. Thus, one can see that given any two of J_0, g_0, β_0, one can reconstruct the third one; the only remaining case is reconstructing J_0 from g_0 and β_0; choose a basis $v_1, \cdots, v_{2n}$, then

$$\sum_{k=1}^n \beta_0(u, v_k) v_k = \sum_{k=1}^n g_0(Ju, v_k) v_k = Ju$$

for u arbitrary (since $\sum_{k=1}^{n} g_0(-, v_k)v_k$ is the identity operator).

Given these relations of the standard structures on $\mathbb{C}^n$, we are motivated to define the following:

Definition 1.4.1: Compatible Complex Structures

Given a symplectic vector space (V, β), a *compatible complex structure* is an element $J \in \mathrm{End}(V)$ with $J^2 = -I$ (so J is a complex structure) such that if we set

$$g(u, v) := \beta(u, Jv)$$

then g is a real inner product.

Given (V, β, J, g), there are various symmetry groups inside $\mathrm{GL}(V)$ preserving the various structures on our vector space; we have already discussed $\mathrm{Sp}(V, \beta)$, the symmetries of β, and we also have $\mathrm{GL}(V, J)$, the complex automorphisms of V (symmetries of J), and $\mathrm{O}(V, g)$, the orthogonal automorphisms (symmetries of g). Considering the intersection of any two of these subgroups of $\mathrm{GL}(V)$, the "two out of three principle" implies that this must equal the triple intersection since any automorphism preserving two of the structures automatically preserves the third. In notation, we have

$$\mathrm{Sp}(V, \beta) \cap \mathrm{GL}(V, J) = \mathrm{Sp}(V, J) \cap \mathrm{O}(V, g) = \mathrm{O}(V, g) \cap \mathrm{GL}(V, J) = \mathrm{U}(V, J, h)$$

This seems like the vector space version of the definition of a Kähler manifold.

Polar Decomposition

Theorem 1.4.2: Polar Decomposition

Given a hermitian vector space $(V, J, h = g - i\beta)$, the unitary group $\mathrm{U}(V, J, h)$ is naturally a deformation retract of $\mathrm{Sp}(V, \beta)$. $\mathrm{U}(V, J, h)$ is a maximal compact subgroup of $\mathrm{Sp}(V, \beta)$, and any maximal compact subgroup of $\mathrm{Sp}(V, \beta)$ is conjugate to $\mathrm{U}(V, J, h)$.

Example 1.4.3

Consider $\mathrm{Sp}(\mathbb{R}^2) = \mathrm{SL}_2(\mathbb{R})$ which is homeomorphic (in fact, diffeomorphic) to a solid open torus $S^1 \times \mathring{D}^2$ (one can see this, for example, via Gram-Schmidt or the QR decomposition). $\mathrm{U}(1) \subseteq \mathrm{SL}_2(\mathbb{R})$ is the core longitudinal circle of the solid torus, and we can deformation retract to it in the natural way.

This is the general picture that polar decomposition generalizes, which is in turn an instance of a general structure theorem for non-compact semisimple Lie groups whose proof utilizes a so-called *Cartan involution*. In the base case, with V a real hermitian vector space, $\mathrm{GL}(V)$ contains $\mathrm{O}(V, g)$ as a maximal compact subgroup as a deformation retract; showing this and then restricting to $\mathrm{Sp}(V,\beta)$ will give us the desired result since $\mathrm{Sp}(V,\beta) \cap \mathrm{O}(V,\beta) = \mathrm{U}(V,J,h)$.

I don't understand why we can't obtain this deformation retract with Gram-Schmidt.

PROOF : Set $G = \mathrm{GL}(V)$; we define $\Theta : G \to G$ a Lie group homomorphism which will satisfy $\Theta^2 = \mathrm{id}$ (this is our Cartan involution) by $\Theta(Z) = (Z^*)^{-1}$ where Z^* denotes the g-adjoint, i.e, $g(Z^*u,v) = g(u,Zv)$ (in $\mathbb{R}^n$ with its standard inner product, $Z^* = Z^T$). Note that $(YZ)^* = Z^*Y^*$ and $(YZ)^{-1} = Z^{-1}Y^{-1}$ so the order flips two times, and Θ is in fact a homomorphism (that Θ is an involution is clear).

Let G^Θ denote the fixed point set of Θ; $Z = (Z^*)^{-1} \iff ZZ^* = I$, so $G^\Theta = \mathrm{O}(V,g)$. Passing to the Lie algebra $\mathfrak{g} = T_IG = \mathrm{End}(V)$, set $\theta = D_I\Theta : T_IG \to T_IG$; one can check that $\theta(\xi) = -\xi^*$ and thus $\theta^2 = \mathrm{id}_{\mathfrak{g}}$. Thus, $\mathfrak{g}$ splits into the ± 1 eigenspaces of θ (since finite order matrices are diagonalizable), which are $\mathfrak{o}(V,g)$ (skew-adjoint endomorphisms) and $\mathrm{symm}(V,g)$ (self-adjoint endomorphisms) respectively. One can show that $[\mathfrak{o}(V,g),\mathfrak{o}(V,g)] \subseteq \mathfrak{o}(V,g)$ and $[\mathfrak{o}(V,g),\mathrm{symm}(V,g)] \subseteq \mathrm{symm}(V,g)$ so $\mathrm{symm}(V,g)$ is a module for the Lie algebra $\mathfrak{o}(V,g)$.

We have the exponential map $\exp : \mathfrak{g} \to G$ given by $\xi \mapsto \exp(\xi) = \sum_{n\geq 0} \frac{\xi^n}{n!}$; look at $\exp|_{\mathrm{symm}(V,g)} \xrightarrow{\cong} S$ where S consists of positive definite symmetric automorphisms inside G, since any positive definite symmetric matrix P has a unique symmetric logarithm.

We can now define $\Phi : \mathrm{symm}(V,g) \times \mathrm{O}(V,g) \to G$ given by $\Phi(\xi,O) = \exp(\xi)\cdot O$. We claim that Φ is an $\mathrm{O}(V,g)$-equivariant diffeomorphism, and hence $\mathrm{O}(V,g)$ is a deformation retract of G.

That Φ is $\mathrm{O}(V,g)$-equivariant is immediate; moreover, if $Z = PO \in G$, with P positive-definite self-adjoint and O orthogonal, then

$$ZZ^T = POO^TP^T = PP^T = P^2$$

so P is a positive-definite square root of the self-adjoint positive-definite matrix ZZ^T, which must be unique, so Φ is injective. We can also recover P from Z as the square root of ZZ^T, and $O = P^{-1}Z$, so Φ is surjective. Φ is obviously smooth, and so is its inverse. ■

I don't know why it's clear that the square root is unique here. If you have a diagonal matrix then given any square root, you can get a different one by changing the signs of the diagonal entries; maybe the point is that there is a unique positive-definite square root of a positive-definite matrix? That tracks in the diagonal case.

Restricting to $\mathrm{Sp}(V,\beta)$, we have

$$\mathrm{Sp}(V,\beta)^\Theta = \mathrm{Sp}(V,\beta) \cap \mathrm{O}(V,g) = \mathrm{U}(V,h)$$

The derivative $\theta = D_I\Theta$ splits $\mathfrak{sp}(V,\beta) = \mathfrak{p} \oplus \mathfrak{u}$ with $\mathfrak{u} = \mathfrak{u}(V,h)$ the unitary Lie algebra of skew-adjoint endomorphisms and

$$\mathfrak{p} = \mathfrak{sp}(V,\beta) \cap J\mathfrak{sp}(V,\beta) = J\mathrm{symm}(V,g) \cap \mathrm{symm}(V,g)$$

We then have the following corollary:

Corollary 1.4.4

$\Psi : \mathfrak{p} \times \mathrm{U}(V,h) \to \mathrm{Sp}(V,\beta)$ given by $(\xi, U) \mapsto \exp(\xi) \cdot U$ is a $\mathrm{U}(V,h)$-equivariant diffeomorphism.

Symplectic and Complex Vector Bundles

Let (V,β) be a symplectic vector space, and let $\mathcal{J}(V,\beta) \subset \mathrm{End}(V)$ be the space of compatible complex structures, i.e,

$$\mathcal{J}(V,\beta) = \{J \in \mathrm{End}(V) : J^2 = -I, \beta(Ju,v) = -\beta(u,Jv) = 0 \text{ and } \beta(v,Jv) > 0 \text{ for all } v \neq 0\}$$

By the above discussion, these criteria are equivalent to the requirement that $g(u,v) := \beta(u,Jv)$ defines a real inner product.

$\mathrm{Sp}(V,\beta)$ acts transitively on $\mathcal{J}(V,\beta)$ by conjugation, and the stabilizer of some fixed complex structure J is $\mathrm{U}(V,J,g-i\beta)$ so, fixing $J \in \mathcal{J}(V,\beta)$, we have

$$\mathcal{J}(V,\beta) \cong \mathrm{Sp}(V,\beta)/\mathrm{U}(V,J,g-i\beta)$$

Note that this is not a group quotient, but a space of cosets (a homogeneous space).

Neither assertion in this paragraph is obvious to me.

From our discussion of polar decomposition above, $\mathrm{Sp}(V,\beta) = \exp(\mathfrak{p})$ (where $\mathfrak{p} = \mathfrak{sp}(V,\beta) \cap J\mathfrak{sp}(V,\beta) = J\mathrm{symm}(V,g) \cap \mathrm{symm}(V,g)$ as above) hence $\mathrm{Sp}(V,\beta)/\mathrm{U}(V,J,g-i\beta)$ is homeomorphic to $\mathfrak{p}$ which is a vector space, and therefore contractible (note that we need a choice of $J \in \mathcal{J}(V,\beta)$ for this contraction).

Tim uses some commutative diagrams here to define what we mean by a vector space structure on the fiber that I've omitted.

Definition 1.4.5: Symplectic Vector Bundles

Recall that a *smooth vector bundle* is a surjection of smooth manifolds $V \xrightarrow{p} B$ where B is the base manifold, such that each fiber V_b of p has the structure of a vector space and such that for all $b \in B$ there exists an open neighborhood $U \ni b$ such that $p^{-1}(U) \cong U \times V_b$ i.e V is locally diffeomorphic to a product of an open set of B with the fiber.

A *symplectic vector bundle* is a smooth vector bundle together with β a smoothly varying family of symplectic pairings β_b on the fibers V_b, such that the local trivializations are symplectomorphisms i.e $V_b \times \{x\} \xrightarrow{\sim} V_x$ is a symplectomorphism (where x is a nearby point).

Example 1.4.6

If η is a non-degenerate 2-form on a manifold M, then (TM, η) is a symplectic vector bundle. Note that we do not require that η is closed, so M may not have a symplectic structure.

We can similarly define a compatible complex structure on a symplectic vector bundle as a smoothly varying family of complex structures on the fibers such that $g_b := \beta_b(-, J_b-)$ is an inner product on V_b. As before, we will denote the set of such structures $\mathcal{J}(V, \beta)$.

Recall that $C^\infty(B; E)$ denotes the vector space of smooth sections of our vector bundle $E \xrightarrow{p} B$; for any compact $K \subseteq B$, any $r \in \mathbb{N}$, there exists a seminorm on $C^\infty(B; E)$ controlling the suprema of sections s and of its derivatives through order r over K. Taking an exhaustion by compacta of the base (i.e $K_1 \subseteq K_2 \subseteq \cdots$ all compact such that $B = \bigcup_i K_i$) and $r \to \infty$ we get a countable set of seminorms $\|\cdot\|_k$, for $k \in \mathbb{N}$, which gives a complete metric on $C^\infty(B; E)$:

$$d(s_1, s_2) = \sum_k \frac{\|s_1 - s_2\|_k}{2^k(1 + \|s_1 - s_2\|_k)}$$

which makes $C^\infty(B; E)$ into a *Fréchet space*.

I think this should be equivalent to just taking the compact-open topology, i.e, when considering $C(X, Y)$, take the topology generated by the subbase consisting of $V(K, U)$ defined as

$$V(K, U) = \{f \in C(X, Y) : f(K) \subseteq U\}$$

with K compact in X and U open in Y. Also, recall that a subbase just means that you are allowed to take arbitrary unions and finite intersections of the opens in question (or, equivalently, that the set of all finite intersections of elements in the subbase forms a base).

Then we topologize $\mathcal{J}(V, \beta)$ with the subspace topology of $C^\infty(B, \underline{\mathrm{End}}(V))$ (where $\underline{\mathrm{End}}(V)$ is the vector bundle of vector bundle endomorphisms).

Theorem 1.4.7

$\mathcal{J}(V, \beta)$ is contractible.

$\mathcal{J}(V, \beta)$ is the space of sections of a fiber bundle over B with fibers $\mathcal{J}(V_b, \beta_b) \cong \mathfrak{p}_{J_b}$ given $J_b \in \mathcal{J}(V_b, \beta_b)$, so once we've exhibited one symplectic structure on the vector bundle V, i.e, a $J \in \mathcal{J}(V, \beta)$, $\mathcal{J}(V, \beta)$ is isomorphic to the C^∞ sections of a vector bundle, $\mathfrak{p}_J \to B$, and this space of sections is a metric vector space, hence contractible. The trouble then is exhibiting a single symplectic structure on the vector bundle in question.

One approach is via obstruction theory, where one can construct J inductively over k-skeleta of the base B with respect to a CW decomposition of B, and then deform this *a priori* continuous function to a smooth function. However, a more direct approach is possible: concretely, given (V, β) a symplectic vector space, we may take any inner product g on V (not necessarily compatible with β), and define $K \in \mathrm{End}(V)$ by $g(u, v) = \beta(u, Kv)$, so K is skew-adjoint.

If $K \in \mathrm{O}(V, g)$, then $K \in \mathcal{J}(V, \beta)$ since $g(K^2u, v) = -g(Ku, Kv) = -g(u, v)$ so $K^2 = -I$, and since K is defined by $g(u, v) = \beta(u, Kv)$, this implies that $K \in \mathcal{J}(V, \beta)$.

In general, K has a polar decomposition $K = PO$ with O orthogonal, and $P = \exp(\xi)$ with ξ symmetric (via polar decomposition); we claim that O

is a compatible complex structure, and omit the verification of this. Note that a partition of unity argument would not work here because, unlike with a Riemannian metric, a convex combination of symplectic forms is not guaranteed to be symplectic.

The Canonical Line Bundle and the Adjunction Formula

A complex structure on the tangent bundle (as a vector bundle) is called an *almost complex structure* on the underlying manifold M. On a symplectic manifold (M, ω), we refer to the set of compatible almost complex structures by $J \in \mathcal{J}(TM, \omega)$. If $J_0, J_1 \in \mathcal{J}(TM, \omega)$, then $(TM, J_0) \cong (TM, J_1)$ as complex vector bundles via an isomorphism that is canonical up to homotopy.

Definition 1.4.8: Canonical Line Bundle

Given (M, ω), the *canonical line bundle* on M is $K_M := \det_{\mathbb{C}}(T^*M, J^*)$ where the determinant of a vector bundle is simply its top exterior power.

This line bundle does not depend on the choice of J; note that, last time, we showed that $\mathcal{J}(TM, \omega)$ is contractible, hence path-connected (and nonempty), so for any J_0, J_1 there exists a path between them J_t which gives a complex line bundle over $M \times I$ whose restriction to $M \times \{t\}$ is $\det_{\mathbb{C}}(T^*M, J_t^*)$, so $\det_{\mathbb{C}}(T^*M, J_0^*) \cong \det_{\mathbb{C}}(T^*M, J_1^*)$; moreover, by contractibility, any two paths between J_0 and J_1 are homotopic (since $\mathcal{J}(TM, \omega)$ is simply connected) so the latter isomorphisms are canonical up to homotopy.

Note that we only need to specify that K_M is independent of J because we are taking a complex determinant; the argument given here also does not seem to use any special properties of the determinant line bundle, and seems to gives a recipe for proving that the isomorphism type of a complex vector bundle does not depend on the specific almost complex structure in question, although I'm not completely sure that this is true.

K_M is a symplectic invariant in that $K_M \cong \phi^* K_N$ if ϕ is a symplectomorphism between (M, ω_M) and (N, ω_N). Note that for any manifold X, the isomorphism classes of complex line bundles $L \to X$ form an abelian group $\text{Pic}(X)$ under the tensor product, with inverses given by duals. The first Chern class gives an isomorphism $c_1 : \text{Pic}(X) \xrightarrow{\sim} H^2(X; \mathbb{Z})$; there are many ways to calculate $c_1(L)$. For example, in algebraic topology, one can identify Pic(X) with $[X, \text{BU}(1)]$ since all line bundles pull back from the universal line bundle over BU(1) which turns out to be equal to $\mathbb{CP}^\infty$, which is in turn equal to $K(\mathbb{Z}, 2)$, so $[X, \text{BU}(1)] = H^2(X; \mathbb{Z})$ by Brown representability.

More concretely, $H^2(X; \mathbb{Z})$ consists of singular 2-cochains, which you can evaluate on 2-chains in some CW structure on X (and you can demand that the 2-chains are smooth), so given a line bundle L over X, pick some generic section s and look at its intersection Z with the zero section (generically these intersections are transverse); so Z is some codimension 2 oriented submanifold of X since $\dim_{\mathbb{R}} L_x = 2$, which we can think of as a 2-cochain by taking a 2-chain u to the intersection number $u \cdot z$, i.e, $c_1(L) = \delta_Z$,

$Z = s^{-1}(0)$. c_1 is also natural in the sense that it commutes with pullbacks, i.e, $c_1(f^*L) = f^*c_1(L)$ for $f : Y \to X$. Thus, $c_1(K_M)$ is an important symplectic invariant of a symplectic manifold M.

In fact, one can define the higher Chern classes in a similar way, with c_i an obstruction class to the existence of $n-i+1$ linearly independent sections.

Example 1.4.9

Let (Σ, ω) be a symplectic surface, closed, orientable, connected, and genus g. Then K_Σ is $(T^*\Sigma, J)$, and $c_1(K_\Sigma) = c_1(T^*\Sigma) = -c_1(T\Sigma)$, and by Poincaré-Hopf, the signed count of zeros of a generic vector field (i.e section of $T\Sigma$) is the Euler characteristic, so $\langle c_1(K_\Sigma), [\Sigma]\rangle = 2g-2$. Since $H^2(\Sigma) = \mathbb{Z}$, this fully characterizes $c_1(K_\Sigma)$, i.e, no information is lost when evaluating against the fundamental class.

For Σ non-orientable, $H^2(\Sigma) = 0$ so $c_1(K_\Sigma) = 0$ trivially.

Example 1.4.10

Consider $(S^2 \times S^2, \omega = \mathrm{pr}_1^*\zeta_1 \oplus \mathrm{pr}_2^*\zeta_2)$ where ζ_1, ζ_2 are symplectic forms on S^2, with areas $a_i = \int_{S^2} \zeta_i > 0$. $K_{S^2\times S^2} = \mathrm{pr}_1^* K_{S^2} \otimes \mathrm{pr}_2^* K_{S^2}$ so

$$c_1(K_{S^2\times S^2}) = c_1(\mathrm{pr}_1^* K_{S^2}) + c_1(\mathrm{pr}_2^* K_{S^2}) = -2e_1 - 2e_2$$

where $H^2(S^2 \times S^2) = \mathbb{Z}\cdot e_1 \oplus \mathbb{Z}\cdot e_2$.

Proposition 1.4.11

Suppose $[\omega] = a_1e_1 + a_2e_2 \in H^2(S^2\times S^2; \mathbb{R})$ with $a_1 \neq a_2$, and $\phi \in \mathrm{Aut}(S^2\times S^2, \omega)$, then $\phi^* : H^*(S^2\times S^2) \to H^*(S^2\times S^2)$ is the identity on cohomology.

Proof : Recall that

$$H^*(S^2\times S^2) = \underset{0}{\mathbb{Z}} \oplus (\mathbb{Z} \underset{2}{\oplus} \mathbb{Z}) \oplus \underset{4}{\mathbb{Z}}$$

ϕ^* is the identity on H^0 since this is true for any self-homeomorphism of a path-connected space, and on H^4 since ϕ is orientation preserving (as a symplectomorphism). On H^2, since $[\omega] = a_1e_1 + a_2e_2 \in H^2(S^2\times S^2;\mathbb{R}) = H^2(S^2\times S^2)\otimes_{\mathbb{Z}}\mathbb{R}$, and we know that $c_1(K) = -2e_1 - 2e_2$, since $a_1 \neq a_2$ by assumption, $[\omega]$ and $c_1(K)$ form a basis for $H^2(S^2\times S^2;\mathbb{R})$, and both are symplectic invariants and hence fixed by ϕ^*, so ϕ^* must be be the identity on H^2 since it fixes two linearly independent vectors. ■

If $a_1 = a_2$, then the map $(x, y) \mapsto (y, x)$ is a self-symplectomorphism that is nontrivial on cohomology.

Proposition 1.4.12: The Adjunction Formula

Let (X, ω) be a symplectic manifold, D a closed, codimension 2 submanifold of X. Then, there is an isomorphism of complex line bundles (which is an isomorphism of *holomorphic* line bundles when X is a complex manifold and D a complex submanifold)

$$K_D \cong N_{D/X} \otimes_{\mathbb{C}} K_X|_D$$

We never actually got to this in class, and I'm just including the result without proof for some measure of completion.

Maslov Indices and Classes of Lagrangians

Definition 1.5.1: Maslov Class

Given (M^{2n}, ω) and a Lagrangian submanifold $L^n \subseteq M^{2n}$, there is a *Maslov class* $\mu_L \in H^2(M, L)$ whose image in $H^2(M)$ is $-2c_1(K_M)$.

Here we develop a generalization of $c_1(K_M)$ to the data of (M, ω) together with a Lagrangian submanifold L. Why we do this is totally opaque to me.

The Lagrangian Grassmanian

Let (V, β) be a symplectic vector space, then we write $\mathcal{L}(V, \beta)$ to denote the set of Lagrangian subspaces of V, and $\mathcal{L}(n) := \mathcal{L}(\mathbb{R}^{2n}, \beta_0)$. Evidently, $\mathcal{L}(V, \beta) \subseteq \mathrm{Gr}_n(V)$ where $\dim V = 2n$, from which $\mathcal{L}(V, \beta)$ obtains a submanifold structure.

Recall that if $V = \mathbb{R}^{2n}$ with its standard inner product g, given $L \in \mathrm{Gr}_n(V)$, $L^\perp \in \mathrm{Gr}_n(V)$ as well. Then, we obtain the standard charts on $\mathrm{Gr}_n(V)$ via maps $\phi_L : \mathrm{Hom}_{\mathbb{R}}(L, L^\perp) \to \mathrm{Gr}_n(V)$ by taking graphs, i.e,

$$\phi_L(\alpha) = \Gamma_\alpha = \{l + \alpha l : l \in L\}$$

The ϕ_L give charts for the Grassmanian.

The charts for other Grassmanians (i.e, not middle dimensional) are defined similarly.

Then, for (V, β) symplectic, g a compatible Riemannian metric (for example, arising from a compatible complex structure J), and given $L \in \mathcal{L}(V)$, one can ask for which $\alpha \in \mathrm{Hom}_{\mathbb{R}}(L, L^\perp)$ is $\Gamma_\alpha \in \mathcal{L}(V)$? Here, β defines a linear isomorphism $L^\perp \cong L^*$, so we can think of α as a map $L \to L^*$ i.e $l \mapsto \alpha(l)(-) := \beta(\alpha(l), -|_{L^\perp})$, so for Γ_α to be a Lagrangian subspace, it is necessary and sufficient that $\alpha(l)(l') = \alpha(l')(l)$, i.e, Γ_α is Lagrangian iff α is symmetric. This defines a linear subspace $\mathrm{symm}(L, L^*) \subseteq \mathrm{Hom}(L, L^*)$ so the ϕ_L work as a submanifold chart for $\mathcal{L}(V) \subseteq \mathrm{Gr}_n(V)$ giving $\mathcal{L}(V)$ the structure o a submanifold.

Alternatively, we can give the manifold structure by noting that $\mathrm{U}(V)$ acts transitively on $\mathcal{L}(V)$, so, fixing $L \in \mathcal{L}(V)$, we can set $\mathcal{L}(V, \beta) \cong \mathrm{U}(V)/\mathrm{O}(L)$ where $\mathrm{O}(L)$ stabilizes L.

Lemma 1.5.2

$\pi_1(\mathcal{L}(V, \beta)) \cong \mathbb{Z}$ for $V \neq 0$.

We won't prove this here, but the idea is to use the long exact sequence of homotopy groups arising from the following diagram:

I guess implicitly we're picking some embedding $\mathrm{O}(n) \hookrightarrow \mathrm{U}(n)$ corresponding to the stabilizer of a Lagrangian L but what is the "canonical" Lagrangian submanifold of $\mathbb{R}^{2n}$? The span of $x_1, \cdots, x_n$?

$$\begin{array}{ccccc} \mathrm{O}(n) & \longrightarrow & \mathrm{U}(n) & \longrightarrow & \mathcal{L}(n) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{O}(n+1) & \longrightarrow & \mathrm{U}(n+1) & \longrightarrow & \mathcal{L}(n+1) \end{array}$$

The maps $\mathcal{L}(n) \hookrightarrow \mathcal{L}(n+1)$ will induce an isomorphism on π_1 for all $n \geq 1$.

The Maslov Index

Consider the complex line $\det_{\mathbb{C}}(V,J)^{\otimes 2}$. A Lagrangian $L \in \mathcal{L}(V)$ defines a half-ray in $\det_{\mathbb{C}}(V,J)^{\otimes 2}$, i.e, a point denoted in $\det^2(L) \in S_V$ which is the circle of half-rays in $\det_{\mathbb{C}}(V,J)^{\otimes 2}$. $\det L = \bigwedge^n_{\mathbb{R}} L \hookrightarrow \bigwedge^n_{\mathbb{C}} V$, and $0 \neq \eta \in \det L$ gives rise to $\eta \otimes \eta \in \det_{\mathbb{C}}(V,J)^{\otimes 2}$ which is invariant under $\eta \mapsto -\eta$, so $\det^2(L) := \mathbb{R}_+(\eta \otimes \eta) \in S_V$. This gives us a more canonical identification of $\pi_1(\mathcal{L}(V,\beta))$ with $\mathbb{Z}$; in particular $\det^2_* : \pi_1\mathcal{L}(V,\beta) \xrightarrow{\sim} \pi_1(S_V)$ is an isomorphism, and since S_V is an oriented circle, the right hand side is canonically equivalent to $\mathbb{Z}$ (i.e no sign indeterminacy). We omit the proof is this here. This isomorphism $m = \det^2 : \pi_1\mathcal{L}(V,\beta) \xrightarrow{\sim} \mathbb{Z}$ is called the *Maslov index*.

The Maslov Class

Now, passing to the case of Lagrangian submanifolds $L^n \subseteq (M^{2n}, \omega)$, we have a bundle $\mathcal{L}(TM,\omega) \to M$ of Lagrangian Grassmanians $\mathcal{L}(T_xM, \omega_x)$ above $x \in M$. Then, L defines a section τ_L of $\mathcal{L}(TM,\omega)|_L \to L$ since the tangent space of a Lagrangian submanifold is Lagrangian (this is simply the definition of a Lagrangian submanifold).

Fixing $J \in \mathcal{J}(M,\omega)$, we can form a complex line bundle $\det^{\otimes 2}_{\mathbb{C}}(TM,J) \to M$ with circle bundle $S_{TM} \to M$ and over L there is a section of $S_{TM}|_L \to L$ given by $\det^2(\tau_L)$. τ_L is a trivialization of $\det^{\otimes 2}(TM) = (K_M^*)^{\otimes 2}$ over L, and is one formulation of the *Maslov class*.

More concretely, a line bundle $E \to X$ has a Chern class $c_1(E) \in H^2(X)$, and similarly a line bundle $E \to X$ with a trivialization t over $A \subseteq X$ has a *relative Chern class* $c_1^{\mathrm{rel}}(E,t) \in H^2(X,A)$. We want to consider $\det^2(\tau_L) \in H^2(M,L)$ as the relative Chern class of $S_{TM} \to M$ relative to the trivialization $\det^2_L$.

Maybe this comes later, but why exactly are we interested in the Maslov class? What information does it give us? What does it obstruct?

To define it, pick a generic section σ of $(K_M^*)^{\otimes 2} \to M$ that extends the section $\det^2 \tau_L$ on L, then $\sigma^{-1}(0)$ is a closed, oriented, codimension two submanifold of $M \setminus L$ and so has a fundamental class $[Z] \in H_{2n-2}(M \setminus L) \cong H^2(M,L)$ (where the identification of homology with cohomology is via Lefschetz duality) which is precisely the relative Chern class (here, specifically, the Maslov class) μ_L.

Using the map to absolute cohomology $H^2(M,L) \to H^2(M)$ we have

$$\mu_L \mapsto c_1(\det^{\otimes 2}_{\mathbb{C}}(TM)) = 2c_1(K_M^*) = -2c_1(K_M)$$

so μ_L has the promised image.

I missed a lecture here with more on the Maslov class; for now I'm omitting the content of this lecture as I don't think it really further motivated the Maslov class, just kind of elaborated on it. The one result was that if the Clifford torus $C(a_1, \cdots, a_n)$ is symplectomorphic to $C(1, \cdots, 1)$ (as Lagrangian submanifolds of $\mathbb{C}^n$), then $(a_1, \cdots, a_n) = (1, \cdots, 1)$ which used the Maslov class.

Symplectic Topology **Fall 2023**

Differential Geometry of Symplectic Manifolds

Professor Tim Perutz Abhishek Shivkumar

Hamiltonian Mechanics

Flows and Lie Derivatives

Definition 2.1.1: Flows

Let M^n be a manifold, and X_t a smooth time-dependent vector field on M (with $t \in I$ an interval in I containing 0). Then the *flow* of $\{X_t : t \in I\}$ is a smooth map $\phi : I \times M \to M$ (often rewritten $\phi_t : M \to M$) such that $\phi_0 = \mathrm{id}_M$ and $\frac{d\phi_t}{dt} = X_t \circ \phi_t$.

In saying ϕ_t is *the* flow of X_t, we are implicitly assuming the fact that ϕ_t is unique. Moreover, one can always show that ϕ_t exists on some neighborhood $t \in (-\epsilon, \epsilon)$ of 0. If M is compact, then a flow exists for $t \in \mathbb{R}$.

We previously defined the Lie derivative of forms in the margins above via Cartan's magic formula, which is in fact a theorem.

Definition 2.1.2: Lie Derivatives

Given a vector field X on a manifold M, the *Lie derivative* is a map $\mathcal{L}_X : \Omega^k(M) \to \Omega^k(M)$ given by

$$\mathcal{L}_X \alpha = \frac{d}{dt}\bigg|_{t=0} (\phi_t^* \alpha)$$

Note that since d commutes with pullbacks, $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$. One can also check that there is a Leibniz rule, $\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X(\alpha) \wedge \beta + (-1)^{|\alpha|} \wedge \mathcal{L}_X(\beta)$.

Finally, we have Cartan's magic formula, $\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d$. To prove this, one works in $\mathbb{R}^n$ since both sides of the equation are local operators. Both sides are $\mathbb{R}$-linear and obey the same Leibniz rule, so it suffices to check on 0 and 1-forms (since all other forms are sums of wedges of these), i.e to check $\alpha = f\, dg$ where f and g are functions. In fact, it suffices to check $\alpha = f$ and $\alpha = dg$ separately by using the Leibniz rule; we can reduce this even further by noting that the right hand side commutes with d, so if the formula holds for g, it holds for dg, so finally we may check the formula only for functions f in which case the formula simply states $\mathcal{L}_X(f) = df(X)$ which holds.

This is philosophically an important reduction technique for proving identities between forms.

Lemma 2.1.3

Let X_t be a time-dependent vector field on M with flow ϕ_t, $\alpha \in \Omega^k(M)$. Then

$$\frac{d}{dt}(\phi_t^*\alpha) = \phi_t^*\mathcal{L}_{X_t}(\alpha)$$

We will omit the proof but note that the strategy is similar to proving Cartan's magic formula, i.e, reducing to the case of functions where it is easy.

Proposition 2.1.4

Let (M, ω) be a symplectic manifold, I an interval containing the origin, $\{\phi_t\}_{t\in I}$ a family of diffeomorphisms $M \to M$ with $\phi_0 = \mathrm{id}_M$. Then, the following are equivalent:

1. $\phi_t^*\omega = \omega$ for all t (i.e ϕ_t is a symplectomorphism at all times)
2. $\mathcal{L}_{X_t}\omega = 0$ for all t where X_t generates the flow, i.e, $\frac{d\phi_t}{dt} = X_t \circ \phi_t$ (which can be taken as a definition since ϕ_t are diffeomorphisms)
3. $da_t = 0$ for all t where $a_t \in \Omega^1(M)$, $a_t = \iota_{X_t}\omega = \omega(X_t, -)$ i.e a_t is always closed

Proof : We have the following cases:

$1 \iff 2$ We use the above lemma, that

$$\frac{d}{dt}(\phi_t^*\omega) = \phi_t^*\mathcal{L}_{X_t}(\omega)$$

which immediately gives us that the first and second assertions are equivalent.

$2 \iff 3$ Cartan's magic formula tells us that

$$\mathcal{L}_{X_t}\omega = d \circ \iota_{X_t}\omega + \iota_{X_t} \circ d\omega = d \circ \iota_{X_t}\omega = da_t$$

since ω is closed which immediately tells us that the second and third assertions are equivalent. ■

Definition 2.1.5: Symplectic Vector Fields

A time-dependent vector field is called *symplectic* if $\mathcal{L}_{X_t}\omega = 0$ for all t, i.e, it generates a flow by symplectic automorphisms.

Example 2.1.6

Let (M,ω) be $(T^2 = \mathbb{R}^2/\mathbb{Z}^2, \omega = dx \wedge dy)$ with vector field $X = \frac{\partial}{\partial x}$ which generates the translation flow $\phi_t(x,y) = (x+t, y)$. The corresponding one-form $a = \omega(X, -) = dy$ is clearly closed (but not exact since y is not a well-defined function on the torus for the same reason that θ is not a well-defined function on S^1).

Definition 2.1.7: Hamiltonian Flows

A time-dependent vector field X_t on (M,ω) is called *Hamiltonian* if $a_t = \iota_{X_t}\omega$ is *exact*, say $a_t = dH_t$. H_t, which is well-defined up to the addition of locally constant functions, is called the *Hamiltonian*. A flow ϕ_t is also called Hamiltonian if its generating vector field is Hamiltonian.

Moving backwards, $H_t \in C^\infty(M)$ (varying smoothly in t) for M symplectic defines a vector field X_t characterized by $dH_t = \omega(X_t, -)$ and thereby a flow. We will often write the flow associated to a Hamiltonian as X_t^H. With respect to a compatible complex structure J, we can explicitly write

$$X_t^H = -J \circ \nabla_g H_t$$

where ∇_g is the gradient with respect to the metric g defined by J and ω. Note that the flow is locally orthogonal to the gradient of the Hamiltonian.

Proposition 2.1.8

Let $H \in C^\infty(M)$, independent of t (called an *autonomous* Hamiltonian), then $\phi_t^* H = H$.

PROOF : Since H is just a function, the pullback is just $\phi_t^* H = H \circ \phi_t$ so

$$\frac{d}{dt}\phi_t^* H = \phi_t^*(\mathcal{L}_{X^H} H) = \phi_t^*(dH(X^H)) = \phi_t^*(\omega(X^H, X^H)) = 0$$ ■

This result has a generalization to the non-autonomous case.

Example 2.1.9

Consider (S^2, ω) where ω is the SO(3)-invariant volume form with total area 4π, and let $H(x,y,z) = z$ (with coordinates coming from the standard embedding into $\mathbb{R}^3$). Since the flows preserve H, the contour lines are isolatitudinal, and the flow is just rotation about the z-axis.

Hamiltonian Dynamics

Given a Hamiltonian $H : I \times M \to \mathbb{R}$ for (M,ω) symplectic, denote by $\phi_t^H \in \mathrm{Aut}(M,\omega)$ its flow, which satisfies $\frac{d\phi_t^H}{dt} = X_{H_t} \circ \phi_t^H$ with $\phi_0 = \mathrm{id}$. These flows are called *Hamiltonian diffeomorphisms*. The $t = 1$ maps ϕ_1^H generate the subgroup $\mathrm{Ham}(M,\omega) \subseteq \mathrm{Aut}(M,\omega)$ of Hamiltonian diffeomorphisms.

Example 2.1.10

Consider (V, β) a symplectic vector space regarded as a symplectic manifold, with $\xi \in \mathfrak{sp}(V, \beta)$, and consider the flow $t \mapsto \exp(t\xi) \in \text{Diff}(V)$. This flow is Hamiltonian. To see this, note that the given vector field is time-independent, given by $x \mapsto \xi x$ by differentiating.

Example 2.1.11

Consider $(\mathbb{C}^n, \omega_0)$ with Hamiltonian $H(z) = -\frac{1}{2}\|z\|^2$, which generates the flow $\phi_t(z) = \exp(tJ_0)z = (\cos t)z + i(\sin t)z$ which is just the S^1 action on $\mathbb{C}^n$ $(z \mapsto e^{2\pi i t}z)$.

Poisson Brackets

Our discussion of Lie group structures on various automorphism groups here will necessarily be somewhat imprecise and impressionistic to avoid spending the time it would take to fully flesh out all the details.

We want to think of $\text{Diff}(M)$ as an infinite dimensional Lie group, and consider its Lie algebra $T_{\text{id}_M}\text{Diff}(M)$. Elements of the tangent space arise from paths passing through the origin, i.e, $\phi_t : I \times M \to M$ with $\phi_0 = \text{id}_M$, which are the same as flows, so we may identify $T_{\text{id}_M}\text{Diff}(M)$ with the set of vector fields on M.

We would also want an exponential map $\exp : T_{\text{id}_M}\text{Diff}(M) \to \text{Diff}(M)$, which has a natural candidate in the $t = 1$ flow ϕ_1 (we will assume M is compact so that this flow always exists). The final desiderata is a Lie algebra structure, i.e, a bracket of vector fields, which is given by

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

regarding vector fields as derivations.

Recall that in local coordinates, we can write the vector field X as

$$X = \sum_i X^i \frac{\partial}{\partial x_i} \implies X(f) = \sum_i X^i \frac{\partial}{\partial x_i}(f)$$

which gives the action of X on functions.

So far, we have been ignoring the symplectic structure; we can think of $\text{Aut}(M, \omega)$ as a Lie subgroup of $\text{Diff}(M)$, and think of $T_{\text{id}_M}\text{Aut}(M, \omega)$ as a subspace of $T_{\text{id}_M}\text{Diff}(M)$, which naturally consists of the symplectic vector fields ξ s.t $\mathcal{L}_\xi \omega = 0$ (so that the corresponding flow is symplectic).

Given two symplectic vector fields X, Y, is the bracket $[X, Y]$ also symplectic? Yes, since we have

$$\mathcal{L}_{[X,Y]}\omega = (\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X)\omega = 0$$

since $\mathcal{L}_X\omega = \mathcal{L}_Y\omega = 0$, so the set of symplectic vector fields indeed forms a Lie algebra, $\text{svect}(M, \omega)$. The Hamiltonian automorphisms should form a Lie subgroup of $\text{Aut}(M, \omega)$ with corresponding Lie subalgebra consisting of the Hamiltonian vector fields, but Hamiltonian vector fields correspond to functions in $C^\infty(M)$ upto overall additive constants, so we should obtain a bracket of functions themselves giving a Lie algebra structure to $C^\infty(M)/\mathbb{R}$.

This is all in the time-independent (autonomous) case. Also, it's not clear to me why $\mathcal{L}_{[X,Y]}$ is equal to $\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X$.

Definition 2.1.12: Poisson Brackets

The *Poisson bracket* on $C^\infty(M)$ (not on $C^\infty(M)/\mathbb{R}$) is defined by

$$H, K = \omega(X_H, X_K)$$

It is easy to check that $\{-,-\}$ is $\mathbb{R}$-linear, skew-symmetric, and satisfies the Jacobi identity (so it is a Lie bracket). Moreover, $\{\text{const}, -\} = 0$, so it descends to a bracket on $C^\infty(M)/\mathbb{R} = \text{hvect}(M, \omega)$ (the Lie algebra of the Lie group of Hamiltonian diffeomorphisms). Note that we can also write

$$\{H, K\} = \iota_{X_K} \circ \iota_{X_H} \omega = \iota_{X_K} dH = dH \circ X_K = \mathcal{L}_{X_K} H$$

so we can think of the Poisson bracket as measuring the variation of H in the direction of X_K.

Example 2.1.13

On $(\mathbb{R}^{2n}, \omega_0)$, we have

$$X_H = -J_0 \nabla H = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial y_j} \frac{\partial}{\partial x_j} - \frac{\partial H}{\partial x_j} \frac{\partial}{\partial y_j} \right)$$

so we can write

$$\{H, K\} = \omega_0(X_H, X_K) = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial y_j} \frac{\partial K}{\partial x_j} - \frac{\partial H}{\partial x_j} \frac{\partial K}{\partial y_j} \right)$$

Proposition 2.1.14

$X_{\{H,K\}} = [X_H, X_K]$.

This result tells us that $\{-,-\}$ is the "correct" bracket since it gives us what we want on vector fields.

PROOF : We want to show that $d\{H, K\} = \omega([X_H, X_K], -)$. Using $\{H, K\} = \mathcal{L}_{X_K} H$, we may calculate

$$d\{H, K\} = d(\mathcal{L}_{X_K} H) = \mathcal{L}_{X_K} dH$$

On the other side, we have

$$\iota_{[X_H, X_K]} \omega = \iota_{\mathcal{L}_{X_K} H} \omega$$

which we may transform into $\mathcal{L}_{X_K} dH$ via application of the identity

$$\mathcal{L}_U(\iota_V \alpha) = \iota_{\mathcal{L}_U V} \alpha + \iota_V \mathcal{L}_U \alpha$$

for vector fields U, V, and forms α. ■

The Poisson bracket satisfies one additional identity aside from the general axioms of a Lie bracket, the Leibniz rule.

Lemma 2.1.15: Leibniz Rule

$$\{f \cdot g, h\} = f\{g, h\} + g\{f, h\}$$

The proof is omitted but this follows from the ordinary Leibniz rule for the exterior derivative i.e $d(fg) = f\,dg + g\,df$.

Definition 2.1.16: Poisson Algebras

A commutative algebra over a field equipped with a Lie bracket that satisfies the Leibniz rule as above is a *Poisson algebra*.

Commuting Hamiltonians

Consider an autonomous Hamiltonian $H \in C^\infty(M)$ with accompanying flow ϕ_t^H. If $K \in C^\infty(M)$, we can ask how K evolves along the flow associated to H, i.e, we may evaluate

$$\frac{d}{dt} K \circ \phi_t^H = \frac{d}{dt}(\phi_t^H)^* K = (\phi_t^H)^*(\mathcal{L}_{X_H} K) = (\phi_t^H)^*\{H, K\}$$

If H and K *Poisson commute*, i.e, $\{H, K\} = 0$, then K is constant along the flow of H, equivalently it is a conserved quantity of the flow. In fact, the above calculation shows that K is a conserved quantity iff K Poisson commutes with the Hamiltonian.

For the physics of symplectic geometry, Tim recommends Arnold's Mathematical Methods of Classical Mechanics.

Lemma 2.1.17

If $(H_1, \cdots, H_m)$ are pairwise Poisson-commuting functions on (M^{2n}, ω), then

$$I_x := \text{span}(X_{H_1}(x), \cdots, X_{H_m}(x)) \subseteq T_x M$$

is *isotropic*, i.e, $\omega|_{I_X} = 0$.

PROOF : This is immediate from the fact that $\omega(X_{H_i}, X_{H_j}) = \{H_i, H_j\} = 0$. ■

This simple observation tells us that $\dim I_X \leq n$, or, equivalently,

$$\text{span}(dH_1(x), \cdots, dH_m(x)) \subseteq T_x^* M$$

has dimension at most n. We call the H_i *independent* at x if the corresponding vectors or covectors are linearly independent at x, so this shows that there are at most n independent Hamiltonians at each point.

Example 2.1.18

On $(\mathbb{R}^{2n}, \omega_0)$, $(x_1, \cdots, x_n)$ Poisson-commute and are independent everywhere. If $H \in C^\infty(\mathbb{R}^{2n})$ Poisson commutes with all the x_i, it must be a function of the x_i independent of the y_i. Note that $dx_1, \cdots, dx_n$ here span a Lagrangian submanifold of $\mathbb{R}^{2n}$.

This situation generalizes:

Proposition 2.1.19

Let $(F_1, \cdots, F_n)$ be Poisson commuting functions on (M^{2n}, ω) that are independent at (and therefore near) $p \in M$. Say $H \in C^\infty(M)$, $\{H, F_i\} = 0$ for all i. Then, near p, H is dependent on the F_i, i.e, there exists a function G such that

$$H(x) = G(F_1(x), \cdots, F_n(x))$$

for all x near p.

PROOF : We want to show that, near p, H is constant on the level sets of $F = (F_1, \cdots, F_n) : M \to \mathbb{R}^n$ to encode functional dependence. Set $v_i = X_{F_i}$, so that at p, $(v_1, \cdots, v_n)$ spans the tangent space to the level set of F (by the independence of the F_i). If p is a regular point of F, the same is true near p, and since $\{H, F_i\} = 0$, $dH(v_i) = 0$, so dH vanishes on tangent spaces to the level set, so H is a constant. ■

If $F_1, \cdots, F_m$ are independent at almost all $p \in M$, then we call the F_i *functionally independent*. A Hamiltonian H on (M^{2n}, ω) such that there exist n functionally independent F_i such that $\{F_i, H\} = 0$ and $\{F_i, F_j\} = 0$ is called a *totally integrable system*.

Totally integrable systems are the setting for the celebrated Liouville-Arnold theorem, which gives a canonical transformation to "action-angle" coordinates, in which the Hamiltonian depends only on the action coordinates, and the angle coordinates evolve linearly through time.

Symplectic Origins

Hamiltonian mechanics is where symplectic geometry comes from. To define Hamiltonian mechanics beyond what we have already discussed, it is helpful to first define *Lagrangian mechanics*; consider some dynamical system in which "points" are parameterized by a manifold Q. For example, we could have $Q = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ to parameterize the positions of the Sun, Earth, and Moon, or we could consider a spinning top whose point of contact is fixed, in which case $Q = \mathrm{SO}(3)$.

At each time t in our abstract dynamical system, we have position coordinates $q(t) \in Q$, and a velocity vector $\dot{q}(t) \in T_{q(t)}Q$. In Lagrangian mechanics, we take a function $L = L(q, \dot{q}) \in C^\infty(TQ)$; in classical mechanics, $L = T - V$ where T is the kinetic and V is the potential energy. In the planetary setup, the kinetic energy is translational and the potential energy is from mutual gravitational attraction; for the top, the potential energy is again from gravity, and the kinetic energy is rotational.

The time evolution of our dynamical system is then given by the *Euler-Lagrange* equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

The total energy $E = T + V$ is constant along the time-evolution, i.e,

$$\frac{\partial L}{\partial \dot{q}}\dot{q} - L(q, \dot{q})$$

is constant. In Hamiltonian mechanics, we convert $L \in C^\infty(TQ)$ to a function $H = E \in C^\infty(T^*Q)$ by introducing *conjugate momenta* $p_i = \frac{\partial L}{\partial \dot{q}_i}$ which is a function on the cotangent fiber T_q^*Q.

The fact that momentum lives on the cotangent bundle is still somewhat obscure to me.

In these new coordinates, we may rewrite the Euler-Lagrange equations as

$$\frac{\partial H}{\partial p} = \dot{q} \text{ and } \frac{\partial H}{\partial q} = -\dot{p}$$

This kind of resembles the trick of introducing extra variables to turn a higher order ODE into a system of first order ODEs.

These are *Hamilton's equations*, with $H(p, q) = T + U = p \cdot q - L$. If $f \in C^\infty(T^*Q)$ is some other function, perhaps representing a physical quantity, we may check that f evolves correctly along a solution $(q(t), p(t))$ of Hamilton's equations by checking that

$$\frac{df}{dt} = \{H, f\}$$

If f Poisson commutes with H, then f is a conserved quantity of our dynamical system, as stated somewhat more abstractly above. Before we continue our analysis of a generic dynamical system, we must observe a fundamental fact: the cotangent bundle T^*Q comes with a canonical symplectic structure.

To write down the symplectic form on T^*Q, first we will define the *tautological one-form* $\theta_{\text{taut}} \in \Omega^1(T^*Q)$ and set $\omega = -d\theta_{\text{taut}}$. Let $\pi : T^*Q \to Q$ be the natural projection map, and let $p \in T_q^*Q$, $v \in T_p(T_q^*Q)$ so that $\pi(p) = q$ and $D\pi_p(v) \in T_qQ$. Then, set

$$\langle \theta_{\text{taut}}, v\rangle = \langle p, D\pi_p(v)\rangle$$

This is quite abstract, and we can rephrase this more concretely using local coordinates $q_1, \cdots, q_n$ on an open set $U \subseteq Q$ with corresponding forms $p_i = dq_i$ giving a local trivialization of T^*Q. Thus, since $\pi(q, p) = q$, $D\pi(\partial_{q_i}) = \partial_{q_i}$ and $D\pi(\partial_{p_i}) = 0$, and we get the following formula for θ_{taut} valid on the open set $T^*U \subseteq T^*Q$:

$$\theta_{\text{taut}} = \sum_i p_i \, dq_i$$

In these coordinates, we have

$$\omega_{T^*Q} = \sum_i dq_i \wedge dp_i = \omega_0$$

There's some mental yoga here as a symplectic form on the cotangent bundle lives on the double cotangent bundle, making some of the notation kind of clunky.

More generally, we have the following definition:

Definition 2.1.20

An *exact symplectic manifold* (M, θ, ω) is a manifold M with a one-form θ such that $\omega = d\theta$ is non-degenerate (and evidently closed).

What are the interesting examples of these that are not cotangent bundles?

Thus, we may see that T^*Q comes as an exact symplectic manifold. Note that a diffeomorphism $\phi : (M_1, \theta_1, \omega_1) \to (M_2, \theta_2, \omega_2)$ is a symplectomorphism iff

$$\phi^*\omega_2 = \omega_1 \iff \phi^*(d\theta_2) = d\theta_1 \iff d(\phi^*\theta_2) = d\theta_1 \iff d(\phi^*\theta_2 - \theta_1) = 0$$

So ϕ defines a class $\phi^*\theta_2 - \theta_1 \in H^1(M;\mathbb{R})$. We say ϕ is *exact* if $\phi^*\theta_2 - \theta_1 = df$. A diffeomorphism $Q_1 \xrightarrow{f} Q_2$ induces a diffeomorphism $T^*Q_1 \xrightarrow{\tilde{f}} T^*Q_2$ covering f in the sense that the following diagram commutes:

$$\begin{array}{ccc} T^*Q_1 & \xrightarrow{\tilde{f}} & T^*Q_2 \\ \downarrow & & \downarrow \\ Q_1 & \xrightarrow{f} & Q_2 \end{array}$$

$\tilde{f}$ is defined as $\tilde{f}|_{T_q^*Q_1} = (D_{f(q)}f^{-1})^*$. The fact that θ_{taut} is constructed in a coordinate-independent manner makes it clear that $\tilde{f}^*(\theta_{\text{taut}}) = \theta_{\text{taut}}$ so $\tilde{f}$ is an exact symplectomorphism.

Given $\alpha \in \Omega^1(Q)$, α can be regarded as a map $\tilde{\alpha} : Q \to T^*Q$ (this is just the definition of a global section) which has a graph $\Gamma(\alpha) \subseteq T^*Q$ which should be a middle dimensional submanifold since $\pi|_{\Gamma(\alpha)} : \Gamma(\alpha) \to Q$ is a diffeomorphism with inverse $\tilde{\alpha}$. Moreover, one can check that $\tilde{\alpha}^*(\theta_{\text{taut}})|_{\Gamma(\alpha)} = \alpha$.

A submanifold $L \subseteq Q$ is Lagrangian iff it is middle-dimensional and $\omega|_L = 0$; if Q is exact, then this is equivalent to $\theta_{\text{taut}}|_L$ being closed, so we may associate to a Lagrangian submanifold a class $[\theta|_L] \in H^1(L, \mathbb{R})$. A Lagrangian submanifold is *exact* if $\theta|_L = df$ for some $f \in C^\infty(L)$.

Lemma 2.1.21

In $(T^*Q, \omega_{T^*Q}, \theta_{\text{taut}})$, the fibers T_q^*Q are exact Lagrangian submanifolds. Moreover, the graph $\Gamma(\alpha)$ of a one-form α is Lagrangian (resp. exact Lagrangian) iff α is closed (resp. exact).

What, if any, Lagrangian submanifolds do not arise as the graph of a one-form? Why is this significant?

PROOF : It is straightforward from the $p\,dq$ formula for θ_{taut} above that θ_{taut} vanishes on the fibers T_q^*Q, and the formula $\tilde{\alpha}^*\theta_{\text{taut}} = \alpha$ gives the second claim. ∎

An Interlude on the Lagrange Top

As a basic but instructive example of an integrable system, consider a spinning top (a solid body) fixed at its tip in a frictionless system. With no further assumptions, this is not explicitly solvable in full generality. The Lagrange case is where the top is rotationally symmetric, i.e, its moment of inertia tensor has two equal eigenvalues.

There are two coordinate systems in which we can consider this problem: the stationary frame, with unit vectors (e_x, e_y, e_z) and gravity pointing in the negative e_z direction, and the dynamical frame, where we have moving basis vectors $(e_1(t), e_2(t), e_3(t)) = (R_t e_x, R_t e_y, R_t e_z)$ where $R_t \in \mathrm{SO}(3)$ is a rotation operator describing the position of the top. e_3 points up through the axis of the top.

The position of our top is then described by points of $Q = \mathrm{SO}(3)$. The angular velocity at time t will be essentially given by $\dot{R}_t \in T_{R_t}\mathrm{SO}(3)$. For $R \in \mathrm{SO}(3)$, $T_R\mathrm{SO}(3)$ is identified with $\mathfrak{so}(3)$ which consists of 3×3 skew-symmetric matrices. As a Lie group, SO(3) is parallelizable (also, all 3-manifolds are parallelizable), so we can write $T\mathrm{SO}(3) \cong \mathrm{SO}(3) \times \mathfrak{so}(3)$ given by $(R, \alpha) \mapsto (R, \alpha R^{-1})$. We may also identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$ as Lie algebras (with the bracket on $\mathbb{R}^3$ the standard cross product), via $\mathbb{R}^3 \to \mathfrak{so}(3)$ given by $\xi(u)(v) = [u \times v]$.

Thus, we have

$$\xi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$$

One can show that $\xi(Rv) = R\xi(v)R^{-1}$ where $R \in \mathrm{SO}(3)$.

Concretely, then, the angular velocity is given by

$$\omega(t) = \xi^{-1}(\dot{R}_t R_t^{-1}) \in \mathbb{R}^3$$

The kinetic energy of the top can be written as

$$E_{\text{kinetic}}(t) = \frac{1}{2} I_1 \omega_1(t)^2 + \frac{1}{2} I_2 \omega_2(t)^2 + \frac{1}{2} I_3 \omega_3(t)^2$$

where $\omega_j(t) = \omega(t) \cdot e_j(t)$ and the I_j are the moments of inertia, with $I_1 = I_2$ as we required at the beginning. The potential energy is

$$E_{\text{potential}}(t) = mgl(e_3(t) \cdot e_2)$$

where m is the mass, g is the acceleration due to gravity, and l is the distance of the center of gravity from the ground, and $e_3(t) \cdot e_2$ is a correction factor that just picks out the vertical component of l.

We can define the kinetic energy function on $T\mathrm{SO}(3)$:

$$T(R, \omega) = \frac{1}{2} I_1 (\omega \cdot Re_x)^2 + \frac{1}{2} I_2 (\omega \cdot Re_y)^2 + \frac{1}{2} I_3 (\omega \cdot Re_z)^2$$

where $\omega \in \mathfrak{so}(3)$ and a potential energy function

$$U(R, \omega) = mgl(Re_z \cdot e_z)$$

Together, this gives a Lagrangian $\mathcal{L}(R,\omega) = T - U \in C^\infty(T\mathrm{SO}(3))$. We can now write down a Hamiltonian in terms of the conjugate momenta $L_j := \frac{\partial L}{\partial \omega_j} = I_j\omega_j$ understood as angular momenta about dynamical axes, and $H = T + U$ now regarded as a function in $C^\infty(T^*\mathrm{SO}(3))$, where $T^*\mathrm{SO}(3) \cong \mathrm{SO}(3) \times \mathbb{R}^3$ as above, and $\mathbb{R}^3$ has coordinates given by the L_j. We have

$$H(R,L) = \frac{1}{2I_1}L_1^2 + \frac{1}{2I_2}L_2^2 + \frac{1}{2I_3}L_3^2 + mglu(R)$$

where $u(R) = Re_z \cdot e_z$ is just the height of the center of mass.

There are two circle actions on our system: the first is given by rotation of the plane that the top is sitting on, i.e, around the e_z-axis, which does not affect our system at all. The second action depends on the fact that our top is symmetric, so we have an action around the axis of the top, the e_1-axis. The former action can be written as the matrix

$$\rho(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SO}(3)$$

with the action by right translation: $\phi \cdot A = A\rho(\phi)$ for $A \in \mathrm{SO}(3)$. This induces an action of S^1 on $T^*\mathrm{SO}(3)$.

I don't think there's anything special here about the symplectic structure giving us an induced action; if we have a diffeomorphism $f : M \to M$ (which our group action is) then $d(f^{-1})^* : T^*M \to T^*M$ should be the induced action on the cotangent bundle.

We have

$$\left.\frac{d}{d\phi}\right|_{\phi=0} R \cdot \rho(\phi) = R\xi(e_z) = (R\xi(e_z)R^{-1})R = \xi(Re_z)R$$

Thus, the generating vector field on $\mathrm{SO}(3)$ is $X = \xi(Re_z)R$ which lifts to a Hamiltonian vector field $\tilde{X}$ on $T^*\mathrm{SO}(3)$.

Since H doesn't change along S^1-orbits, $\{J,H\} = 0$ where J is the generating function in $C^\infty(T^*\mathrm{SO}(3))$ for $\tilde{X}$.

The second circle action is given by left translation, by rotations around the axis of the top. Since this is a left action, it commutes with our other action which was a right action, and repeating similar calculations as above, we find that $\{K,H\} = \{K,J\} = 0$ where K is a generating function for the Hamiltonian vector field on $T^*\mathrm{SO}(3)$ corresponding to our second circle action, so our top is a completely integrable system.

Kind of just filling in details from the notes now, but just in broad strokes because we didn't cover this in class.

Recall the term $u(R) = Re_z \cdot e_z$ in our Hamiltonian above; one can show that

$$\dot{u}(R_t) = \gamma_1\omega_2 - \gamma_2\omega_1$$

Using the other equations that we have already setup, one can show that there exists a relation of the form $(\dot{u}(R_t))^3 = f(u(R_t))$ where f is a cubic polynomial; such a relation is solved precisely by the Weierstraß $\wp$ function, from the theory of elliptic curves, which satisfies the relation

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3$$

where the g_i are multiples of the Eisenstein series from the theory of modular forms. This is somewhat far afield from our original discussion of integrable systems, but perhaps the punchline is this: we exhibited a commuting full rank set of Hamiltonians, and, as promised, found a somewhat explicit solution to our system.

We had an in class vote as to whether we would continue discussing the Lagrangian top for a second lecture or just move on; I really wish we hadn't moved on. There is some commentary in the notes about how this relates to symplectic reduction, but they're fairly sketchy and I'll wait until we actually cover that topic to include them.

Moser's Method and Darboux's Theorem

Note that when considering de Rham cohomology, we only care about closed forms modulo exact forms, but so far we have been discussing closed forms explicitly (i.e not as classes). In fact, we can show that deforming a symplectic form ω by an exact form will not change the symplectomorphism class of the underlying symplectic manifold, which will enable us to prove the celebrated Darboux theorem, which states that all symplectic manifolds are locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$.

First, we must discuss how we may control the primitives of exact forms (where the *primitive* of $d\alpha$ is α). Consider $U \subseteq \mathbb{R}^n$ an open, star-shaped set (i.e not convex) with a marked point $\bullet$; by the Poincaré lemma, for $k > 0$, for $\eta \in \Omega^k(U)$, if η is closed, then it is exact. More precisely, there is a homotopy operator $K : \Omega^k(U) \to \Omega^{k-1}(U)$ s.t $d \circ K + K \circ d = \mathrm{id} - p^* \circ i^*$ where $p : U \to \{\bullet\}$ and $i : \{\bullet\} \to U$.

Why is it reasonable to expect such an operator K to exist? Is there some geometric intuition underlying it?

K can be taken such that if $t \mapsto \alpha_t$ is a smooth family of forms, then so is $t \mapsto K\alpha_t$. Moreover, if $\alpha \in \Omega^k(U)$, and α vanishes on $\bullet$, then we can demand that $K\alpha$ vanishes there as well.

We can work in greater generality using vector bundles: let $p : V \to M$ be a real vector bundle, $i : M \to V$ its zero-section, $U \subseteq V$ an open neighborhood of $i(M)$. Let $r_t : V \to V$ be the function $v \mapsto tv$; we want that $r_t U \subseteq U$ for all $t \in [0, 1]$ (where $r_t U$ is evidently a thickened copy of the zero-section). U deformation retracts to $i(M)$ so in particular the de Rham cohomology of U agrees with that of M. We claim that there exists $K : \Omega^k(U) \to \Omega^{k-1}(U)$ s.t $d \circ K - K \circ d = \mathrm{id} - p^* \circ i^*$ as above, satisfying that if α_t is a smooth family of forms, then so is $K\alpha_t$ and if α is zero along $i(M)$, then so is $K\alpha$.

We can write an explicit formula for K:

$$K\alpha = \int_0^1 r_t^*(\iota_e \alpha) \frac{dt}{t}$$

where e denotes the radial Euler vector field that points outwards, given by $\frac{d}{dt}|_{t=1} r_t$. We will omit the verification that this operator satisfies the given properties.

Lemma 2.2.1

For any smooth path of exact forms $t \mapsto \beta_t \in \Omega^k(Q)$, there exists a smooth path $t \mapsto \alpha_t$ of primitives such that $d\alpha_t = \beta_t$.

We omit the verification, but one may prove this by using the quasi-isomorphism of the de Rham complex with the Čech-de Rham complex along with the Poincaré homotopy operator K (to reduce from global to local).

Now, we may discuss Moser's method.

Definition 2.2.2: Symplectic Isotopies

A path $t \mapsto \omega_t$ of symplectic forms on M is called a *symplectic isotopy* if there exists a flow ϕ_t with $\phi_0 = \mathrm{id}$ such that $\phi_t^* \omega_t = \omega_0$ so that all the (M, ω_t) are symplectomorphic.

Lemma 2.2.3: Moser

Take $t \mapsto \omega_t$ a path of symplectic forms on M, $t \mapsto \alpha_t$ a path of one-forms such that $\frac{d\omega_t}{dt} = d\alpha_t$, X_t a vector field such that $\omega_t(X_t, -) = -\alpha_t$, and $U \subseteq M$ an open set on which the flow ϕ_t of X_t is well-defined as a map $[0,1] \times U \to M$.

Then $\phi_t^* \omega_t = \omega_0$ on $\phi_t^{-1}(U)$.

Of course, if M is compact, then we may take $U = M$.

PROOF : It is clear that $\phi_0^* \omega_0 = \mathrm{id}^* \omega_0 = \omega_0$. In general, we want to show $\frac{d}{dt}(\phi_t^* \omega_t) = 0$ which will allow us to conclude that $\phi_t^* \omega_t = \omega_0$ for all t:

$$\frac{d}{dt} \phi_t^* \omega_t = \phi_t^*(\mathcal{L}_{X_t} \omega_t) + \phi_t^*(\dot{\omega}_t) = \phi_t^*(d(\iota_{X_t} \omega)) + \iota_{X_t}(d\omega) + \phi_t^* d\alpha_t$$

The middle term vanishes since $d\omega = 0$ and since $\iota_{X_t} \omega = -\alpha_t$ by construction, the first and third terms cancel, and the derivative is equal to zero as claimed. ■

Theorem 2.2.4: Moser's Isotopy Theorem

Let M^{2n} be compact, ω_t a smooth path of symplectic forms with $[\dot{\omega}_t] = 0 \in H^2(M; \mathbb{R})$ for all t. Then ω_t is a symplectic isotopy.

PROOF : There exists a smooth path α_t such that $\dot{\omega}_t = d\alpha_t$ by assumption, so we may apply Moser's lemma on $U = M$. ■

Corollary 2.2.5

Let Σ be a compact connected oriented surface, ω_0, ω_1 area forms such that $\int_\Sigma \omega_0 = \int_\Sigma \omega_1$. Then $(\Sigma, \omega_0) \cong (\Sigma, \omega_1)$.

PROOF : To see this, note that $[\omega_0] = [\omega_1] \in H^2(\Sigma; \mathbb{R})$ so we may take $\omega_t = (1 - t)\omega_0 + t\omega_1$; clearly $[\dot{\omega}_t] = 0 \in H^2(M; \mathbb{R})$, so we may apply Moser's isotopy theorem. ■

Now, let (M, J) be a complex manifold arising from a holomorphic atlas. A symplectic form ω compatible with J is called a *Kähler form* (we have seen several examples of these, for example, $\mathbb{CP}^n$ with its Fubini-Study form). If M is compact, you can linearly interpolate between Kähler forms ω_0, ω_1 on (M, J) as above and get a family of Kähler forms and show that a Kähler form, up to symplectomorphism, only depends on its cohomology class.

I don't know if this bit about Kähler classes was fully explained; I think it might be related to the $\partial\overline{\partial}$ lemma , but I don't think it's obvious that it's true specifically for Kähler forms and not for symplectic forms in general.

We are now ready to prove Darboux's theorem:

Theorem 2.2.6

For any $p \in (M^{2n}, \omega)$, there exists an open neighborhood $U \ni p$ and a symplectomorphism ϕ to $U' \subseteq (\mathbb{R}^{2n}, \omega_0)$ taking p to 0.

PROOF : We may apply a manifold chart to map p to 0 and U to an open neighborhood of 0 in $(\mathbb{R}^{2n}, \omega)$ (with ω *a priori* not equal to ω_0), and by choosing a symplectic basis for $T_0\mathbb{R}^{2n}$, may assume that $\omega(0) = \omega_0(0)$. Our idea then is to interpolate, using Moser's lemma: let $\omega_t = (1-t)\omega_0 + t\omega$ be a path of closed 2-forms, with $\omega_t(0) = \omega_0(0)$, hence ω_t is non-degenerate on some open neighborhood of 0. $\dot{\omega}_t = \omega_1 - \omega_0 = d\alpha$ since we are in $\mathbb{R}^{2n}$, and since $(\omega_1 - \omega_0)(0) = 0$, we may assume $\alpha(0) = 0$ (this is true up to a constant we can subtract).

Let X_t be a vector field such that $\omega_t(X_t, -) = -\alpha$ as in Moser's lemma, ϕ_t the corresponding flow, and note that $X_t(0) = 0$ by assumption, so $\phi_t(0) = 0$ for all t, so there exists an open neighborhood $0 \in V \subseteq U$ such that $\phi_t(V) \subseteq U$ for all $t \in [0, 1]$ (with $\overline{U}$ compact).

Then, Moser's lemma tells us that $\phi_t^*\omega_t = \omega_0$ on $\phi_t^{-1}(V)$, so at $t = 1$, $\omega_t = \omega$, and we have our result. ■

This result tells us that there are no interesting local invariants of symplectic manifolds, as any two points in two equal-dimensional symplectic manifolds have symplectomorphic neighborhoods.

This result allows us to define the following invariant:

Definition 2.2.7: Gromov Width

The *Gromov width* of a symplectic manifold (M, ω) is given as

$$w(M, \omega) = \{\sup \pi_r^2 : (B^{2n}(0; r), \omega_0) \hookrightarrow (M, \omega)\}$$

This is an example of a "symplectic capacity", i.e, an invariant $c(M^{2n}, \omega) \in [0, \infty]$ such that if $(M, \omega) \hookrightarrow (M', \omega')$ then $c(M, \omega) \le c(M', \omega')$. c should also satisfy $c(M, a\omega) = |a| \cdot c(M, \omega)$ and normalized such that $c(B^{2n}(0; r), \omega_0) = \pi r^2$.

Note that in a finite volume symplectic manifold, the volume determines an upper bound for the Gromov width since symplectic embeddings are volume preserving. Gromov's celebrated non-squeezing theorem amounts to the claim that $w(B^2(0; r) \times \mathbb{R}^{2n-2}) = \pi r^2$.

There also exist neighborhood theorems for the important symplectic submanifolds, i.e, coisotropic, isotropic, and Lagrangian submanifolds.

Theorem 2.2.8: Symplectic Neighborhood Theorem

Given (M^{2n}, ω), (M'^{2n}, ω'), and (S^{2n-2k}, ζ) which is equipped with symplectic embeddings ι, ι' into M and M' respectively. Then there are normal bundles, N, N' respectively over S which we may regard as $N = (TS)^{\omega}) \subseteq \iota^* TM$, $N' = (TS)^{\omega'} \subseteq \iota'^* TM'$ (recall that in a symplectic vector bundle (V, β), T a linear subspace, then $T^{\omega} = \{v \in V : \beta(v, T) = 0\}$; we define $(TS)^{\omega}$ similarly).

Assume that there exists $\theta : N \to N'$ an isomorphism of symplectic vector bundles over S. Then there exist open neighborhoods $\iota(S) \subseteq U \subseteq M$ and $\iota'(S) \subseteq U' \subseteq M'$ and a symplectomorphism $\phi : U \to U'$ such that $\phi|_{\iota(S)} = \iota' \circ \iota^{-1}$. Moreover, $D\phi|_{\iota(S)} : N \to N'$ is equal to θ.

Before we prove this theorem, we give an application:

Corollary 2.2.9

Say (M^4, ω), (M'^4, ω') are symplectic 4-manifolds, $S \subseteq M$, $S' \subseteq M'$ symplectic 2-manifolds, compact, connected, and of the same genus g, satisfying $\int_S \omega = \int_{S'} \omega'$ (i.e they have equal area), and that the self-intersections $S \cdot S$ and $S' \cdot S'$ are equal. Then, S and S' have symplectomorphic neighborhoods.

Note that any area form on a surface is a symplectic form, and are classified as symplectic forms by their total areas.

Proof : There exists a diffeomorphism $S \cong S'$ by the classification of surfaces, so by the Moser isotopy theorem, it follows that $(S, \omega|_S)$ is symplectomorphic to $(S', \omega'|_{S'})$. Moreover, $S \cdot S = e(N_S)[S] = c_1(N_S)[S]$ so $S \cdot S$ fully determines $c_1(N_S) \in H^2(S) = \mathbb{Z}$ and therefore uniquely determines the normal bundle (since in this dimension it is determined as a complex line bundle (and hence a symplectic vector bundle) by the first Chern class), so $N_S \cong N_{S'}$ and the theorem applies. ∎

Now we may prove the theorem, whose proof is modeled on the proof of Darboux:

Proof : We start by exhibiting ϕ as a diffeomorphism by the tubular neighborhood theorem, which states that for $Y \subseteq X$ a submanifold, there exist neighborhoods $Y \subseteq V \subseteq X$ and $0_Y \subseteq V' \subseteq N_{Y/X}$ (where 0_Y is the zero section) and a diffeomorphism $\phi : V \to V'$ mapping Y to the image of the zero-section such that $D\phi|_Y$ is essentially the identity. Hence, we need only show that we may take ϕ to be a symplectomorphism.

Consider symplectic forms ω_0, ω_1 near $S \subseteq M$, both non-degenerate when restricted to TS. We may assume that $\omega_0 = \omega_1$ along S. Set $\omega_t = (1-t)\omega_0 + t\omega_1$ which is symplectic near S for all $t \in [0, 1]$, and $\dot{\omega}_t = \omega_1 - \omega_0 = d\alpha$

is zero along S, so we may take α to be zero along S as above. Take X_t s.t $\omega(X_t, -) = -\alpha$ as above, with $X_t = 0$ along S, so there exists a neighborhood U' of S such that the flow ϕ_t carries U' into the neighborhood U where ω_t is symplectic, and ϕ_t is the identity along S. Now, Moser's lemma tells us that $\phi_1^* \omega_1 = \omega_0$, from which the result follows. ■