Non-geometric veering triangulations

Ahmad Issa

University of Texas at Austin

Joint work with Craig Hodgson and Henry Segerman

- Today we consider **ideal** triangulations. An ideal triangulation of a 3-manifold M is a decomposition of M into tetrahedra with their vertices removed.

- Today we consider **ideal** triangulations. An ideal triangulation of a 3-manifold M is a decomposition of M into tetrahedra with their vertices removed.
- In 2010 Ian Agol gave an elegant construction of ideal triangulations of a subclass of hyperbolic 3-manifolds, characterised by a combinatorial condition which he called **veering**.

- Today we consider **ideal** triangulations. An ideal triangulation of a 3-manifold M is a decomposition of M into tetrahedra with their vertices removed.
- In 2010 Ian Agol gave an elegant construction of ideal triangulations of a subclass of hyperbolic 3-manifolds, characterised by a combinatorial condition which he called **veering**.
- He posed the question: is every veering triangulation *geometric*, i.e. realised by positive volume ideal hyperbolic tetrahedra in the complete hyperbolic structure?

- Today we consider **ideal** triangulations. An ideal triangulation of a 3-manifold M is a decomposition of M into tetrahedra with their vertices removed.
- In 2010 Ian Agol gave an elegant construction of ideal triangulations of a subclass of hyperbolic 3-manifolds, characterised by a combinatorial condition which he called **veering**.
- He posed the question: is every veering triangulation *geometric*, i.e. realised by positive volume ideal hyperbolic tetrahedra in the complete hyperbolic structure?
- Previously, this was verified for many examples.

- Today we consider **ideal** triangulations. An ideal triangulation of a 3-manifold M is a decomposition of M into tetrahedra with their vertices removed.
- In 2010 Ian Agol gave an elegant construction of ideal triangulations of a subclass of hyperbolic 3-manifolds, characterised by a combinatorial condition which he called **veering**.
- He posed the question: is every veering triangulation *geometric*, i.e. realised by positive volume ideal hyperbolic tetrahedra in the complete hyperbolic structure?
- Previously, this was verified for many examples.
- **Answer:** No! We found examples of non-geometric veering triangulations.

- Today we consider **ideal** triangulations. An ideal triangulation of a 3-manifold M is a decomposition of M into tetrahedra with their vertices removed.
- In 2010 Ian Agol gave an elegant construction of ideal triangulations of a subclass of hyperbolic 3-manifolds, characterised by a combinatorial condition which he called **veering**.
- He posed the question: is every veering triangulation *geometric*, i.e. realised by positive volume ideal hyperbolic tetrahedra in the complete hyperbolic structure?
- Previously, this was verified for many examples.
- **Answer:** No! We found examples of non-geometric veering triangulations.
- **Conjecture:** Every cusped hyperbolic 3-manifold admits a *geometric* ideal triangulation.

Mapping torus of a homeomorphism

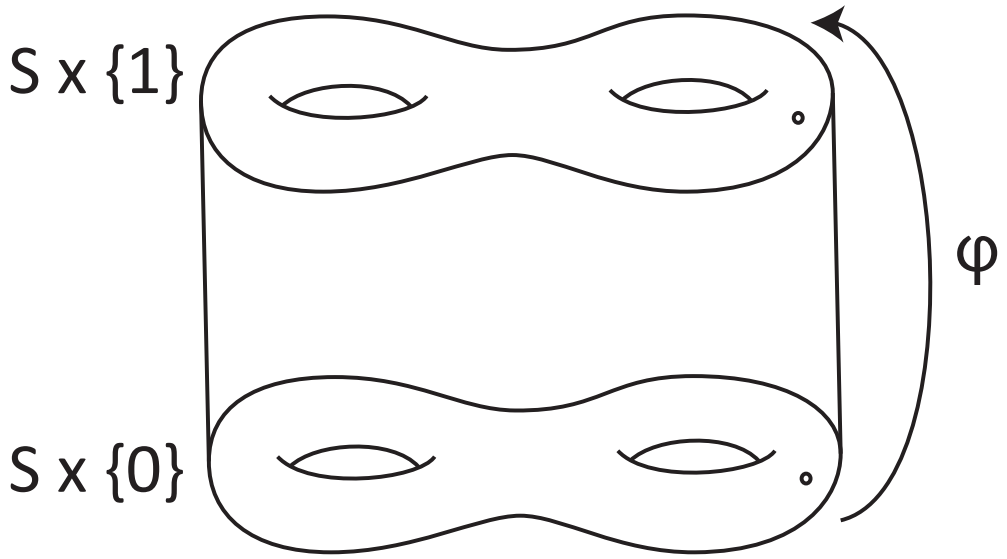
Mapping torus of a homeomorphism

- The veering triangulations we are interested in are triangulations of a subclass of mapping tori.

Mapping torus of a homeomorphism

- The veering triangulations we are interested in are triangulations of a subclass of mapping tori.
- The **mapping torus** of a homeomorphism $\varphi : S \to S$ of a surface S is the 3-manifold

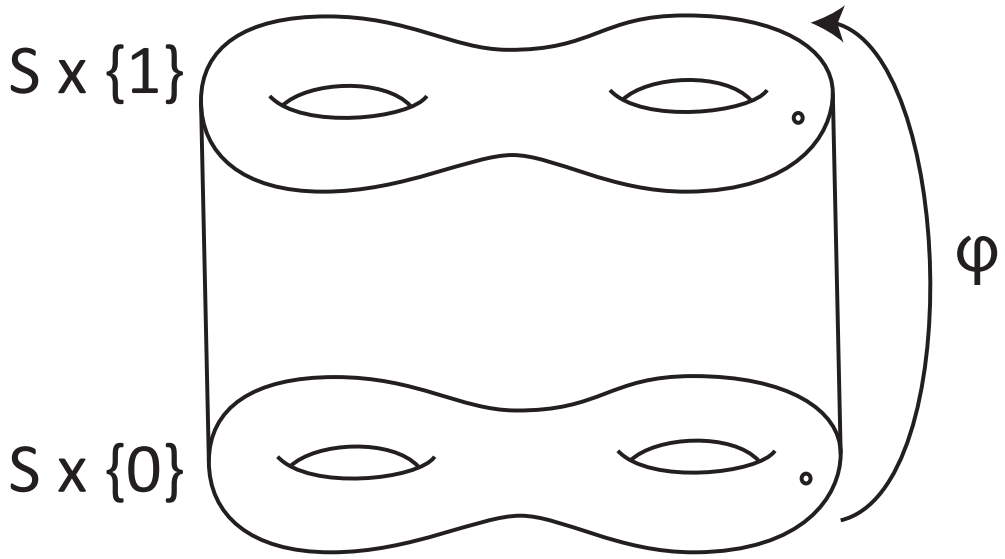
$$M_\varphi = S \times [0,1]/\{(x,0) \sim (\varphi(x),1)\}.$$



Mapping torus of a homeomorphism

- The veering triangulations we are interested in are triangulations of a subclass of mapping tori.
- The **mapping torus** of a homeomorphism $\varphi : S \to S$ of a surface S is the 3-manifold

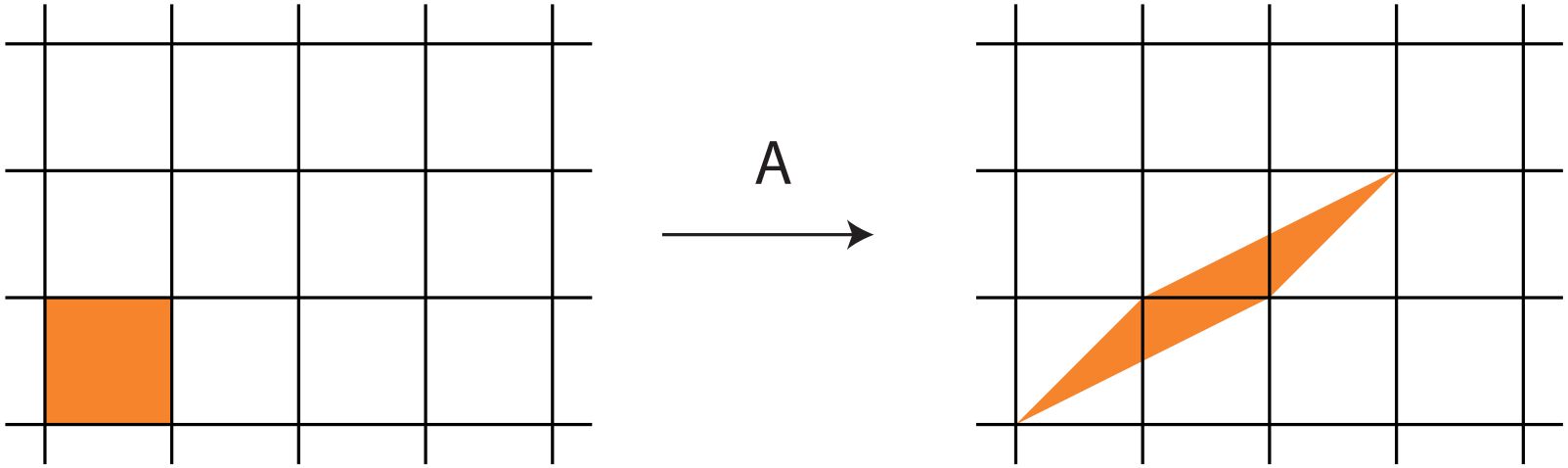
$$M_\varphi = S \times [0,1]/\{(x,0) \sim (\varphi(x),1)\}.$$



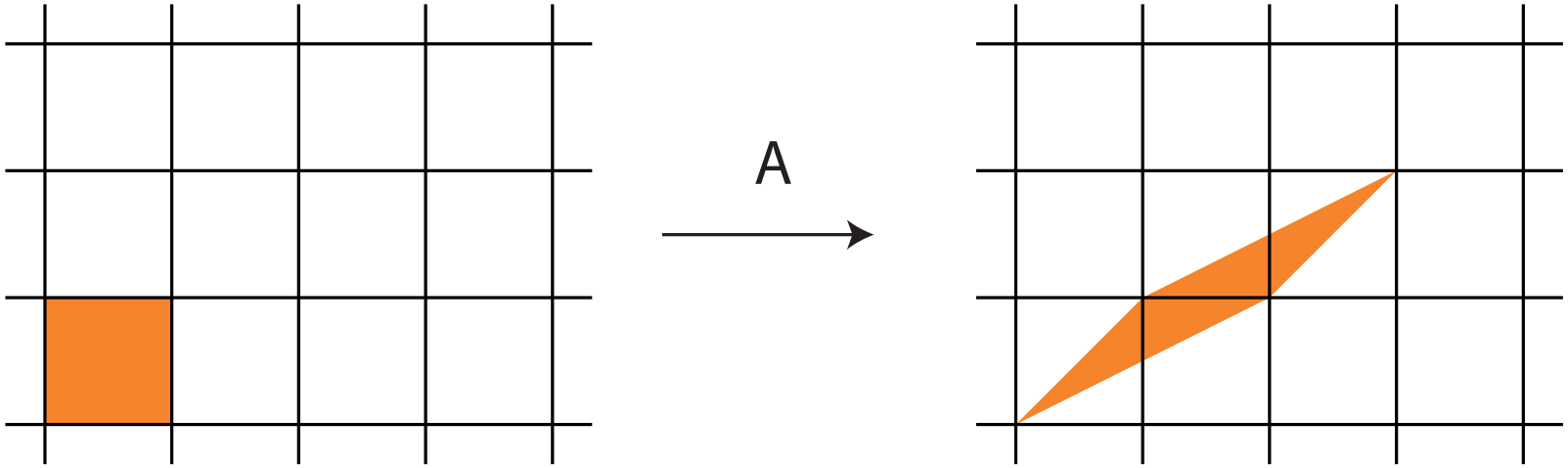
Hyperbolization theorem for mapping tori

The mapping torus M_φ admits a hyperbolic structure if and only if φ is (isotopic to) a pseudo-Anosov homeomorphism.

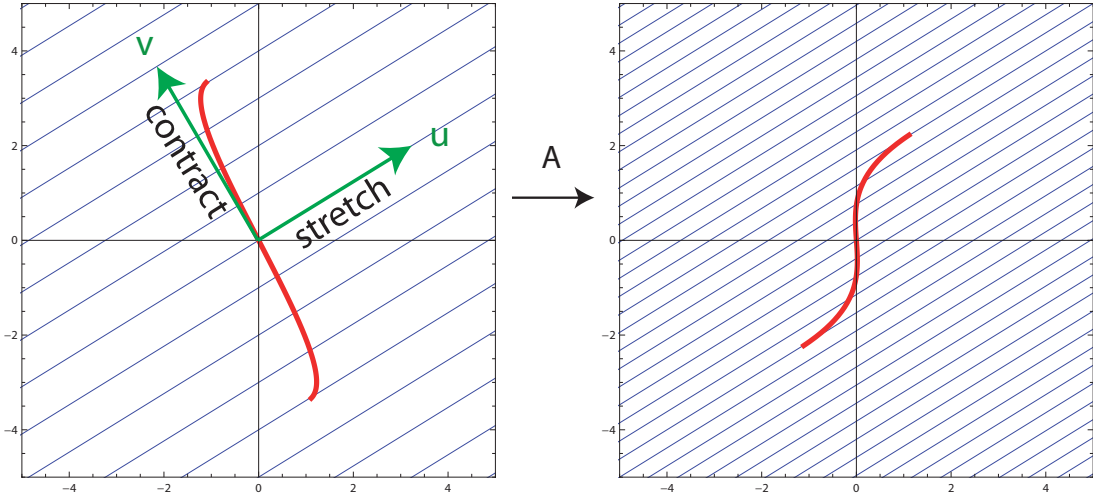
- Consider $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$.



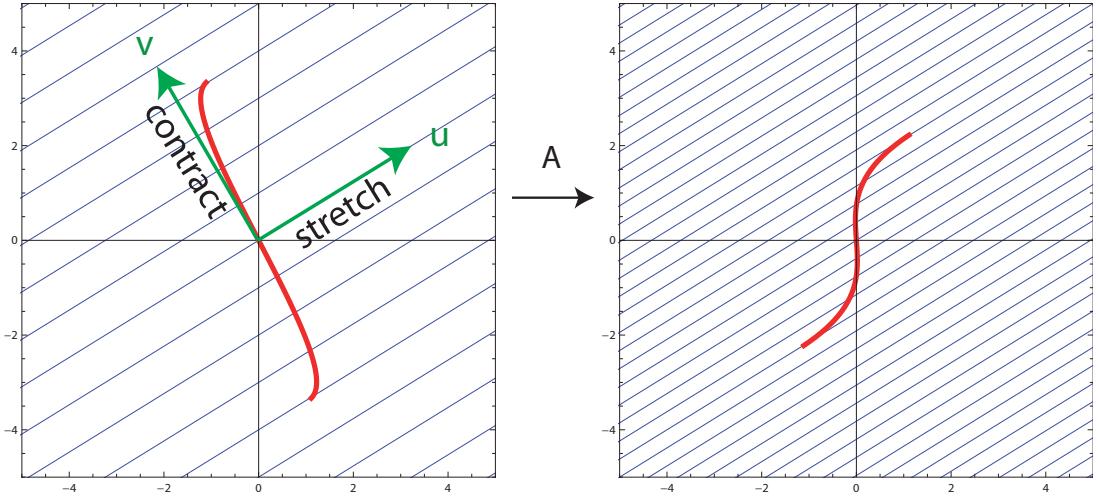
- Consider $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) : \mathbb{R}^2 \to \mathbb{R}^2$.
- Descends to a homeomorphism $\varphi : S \to S$ where $S = (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$ is the once-punctured torus.



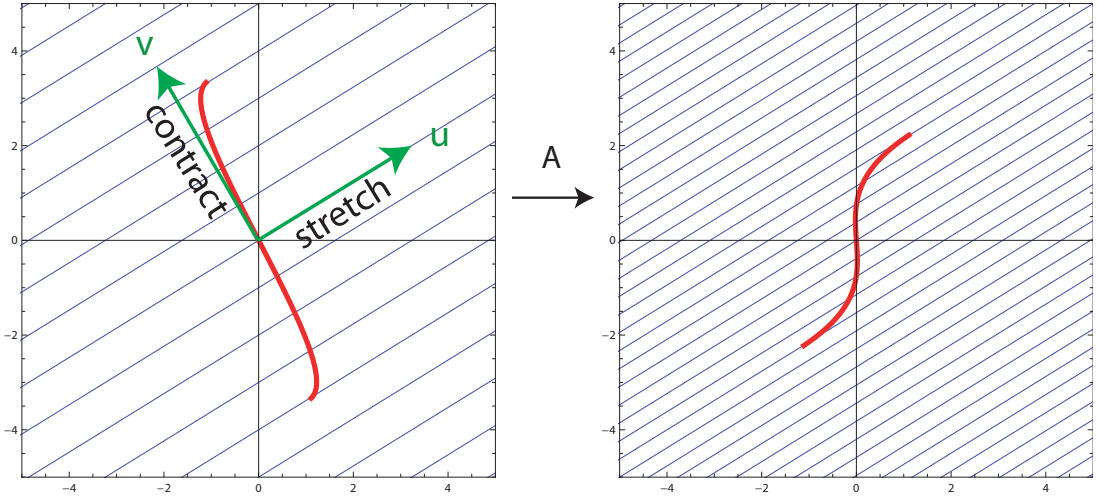
- Consider $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) : \mathbb{R}^2 \to \mathbb{R}^2$.
- Descends to a homeomorphism $\varphi : S \to S$ where $S = (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$ is the once-punctured torus.
- A has eigenvectors u and v with eigenvalues $\lambda > 1$ and $\frac{1}{\lambda} < 1$.



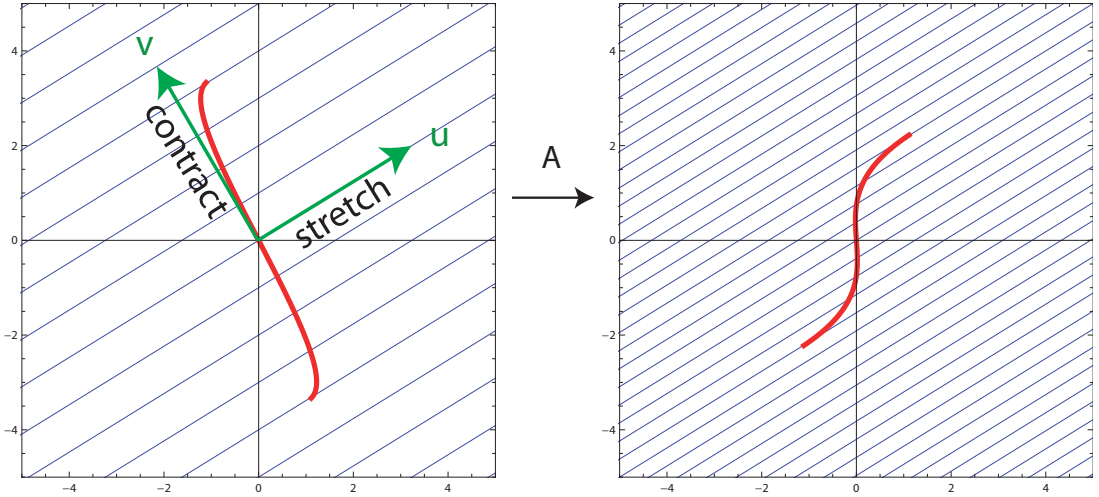
- Consider $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) : \mathbb{R}^2 \to \mathbb{R}^2$.
- Descends to a homeomorphism $\varphi : S \to S$ where $S = (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$ is the once-punctured torus.
- A has eigenvectors u and v with eigenvalues $\lambda > 1$ and $\frac{1}{\lambda} < 1$.
- Lines parallel to u **foliate** (partition) $\mathbb{R}^2 - \mathbb{Z}^2$. Projecting these lines onto S gives a foliation $\mathcal{F}^u$ of S into **leaves** (curves).



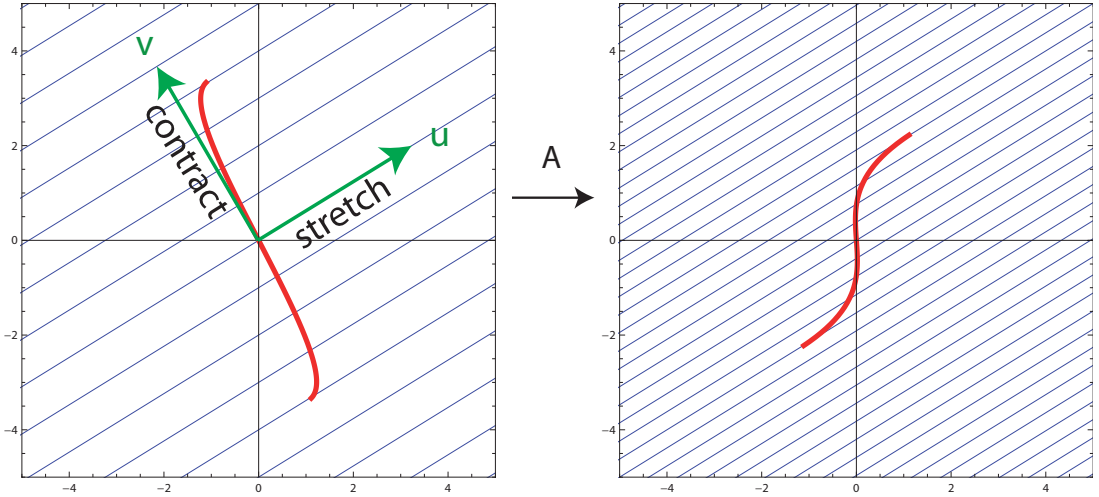
- Consider $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) : \mathbb{R}^2 \to \mathbb{R}^2$.
- Descends to a homeomorphism $\varphi : S \to S$ where $S = (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$ is the once-punctured torus.
- A has eigenvectors u and v with eigenvalues $\lambda > 1$ and $\frac{1}{\lambda} < 1$.
- Lines parallel to u **foliate** (partition) $\mathbb{R}^2 - \mathbb{Z}^2$. Projecting these lines onto S gives a foliation $\mathcal{F}^u$ of S into **leaves** (curves).
- Measure μ on $\mathcal{F}$ assigns a positive real number to an arc transverse to $\mathcal{F}$ on S.



- Consider $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) : \mathbb{R}^2 \to \mathbb{R}^2$.
- Descends to a homeomorphism $\varphi : S \to S$ where $S = (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$ is the once-punctured torus.
- A has eigenvectors u and v with eigenvalues $\lambda > 1$ and $\frac{1}{\lambda} < 1$.
- Lines parallel to u **foliate** (partition) $\mathbb{R}^2 - \mathbb{Z}^2$. Projecting these lines onto S gives a foliation $\mathcal{F}^u$ of S into **leaves** (curves).
- Measure μ on $\mathcal{F}$ assigns a positive real number to an arc transverse to $\mathcal{F}$ on S.
- $\varphi(\mathcal{F}^u) = \mathcal{F}^u$ and $\varphi(\mu_u) = \lambda\mu_u$.



- Consider $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) : \mathbb{R}^2 \to \mathbb{R}^2$.
- Descends to a homeomorphism $\varphi : S \to S$ where $S = (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$ is the once-punctured torus.
- A has eigenvectors u and v with eigenvalues $\lambda > 1$ and $\frac{1}{\lambda} < 1$.
- Lines parallel to u **foliate** (partition) $\mathbb{R}^2 - \mathbb{Z}^2$. Projecting these lines onto S gives a foliation $\mathcal{F}^u$ of S into **leaves** (curves).
- Measure μ on $\mathcal{F}$ assigns a positive real number to an arc transverse to $\mathcal{F}$ on S.
- $\varphi(\mathcal{F}^u) = \mathcal{F}^u$ and $\varphi(\mu_u) = \lambda\mu_u$.
- Starting with v instead of u gives another measured foliation $(\mathcal{F}^s, \mu_s)$.



- A **singular foliation** $\mathcal{F}$ of a surface S is a foliation of S away from a finite set P of *singular points*. Points of P are locally modelled on a k-prong with $k \geq 3$. Punctures of S are locally modelled on a k-prong with $k \geq 1$.

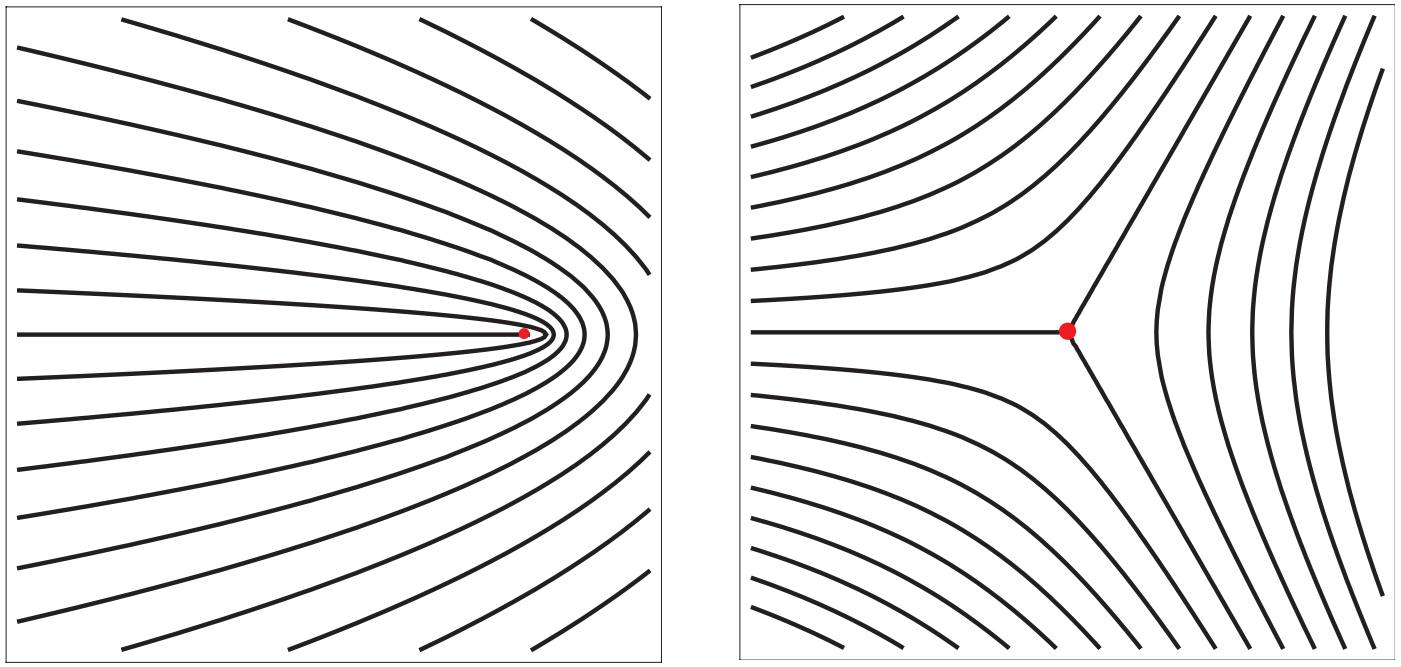


Figure : k-prong for $k = 1$ and $k = 3$, singular point (or puncture) shown in red.

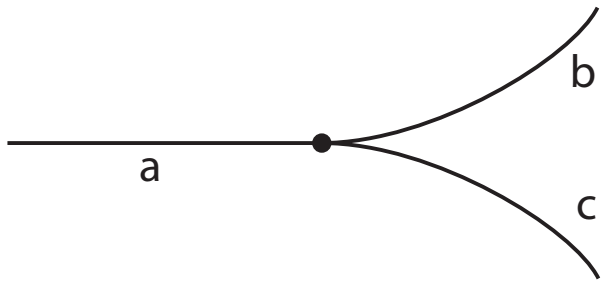
A homeomorphism $\varphi : S \to S$ (with $(S) < 0$), is **pseudo-Anosov** if there exist measured foliations $(\mathcal{F}^u, \mu_u)$, $(\mathcal{F}^s, \mu_s)$ of S with transverse leaves and a common set of singular points with $\varphi(\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \lambda\mu_u)$ and $\varphi(\mathcal{F}^s, \mu_s) = (\mathcal{F}^s, \frac{1}{\lambda}\mu_s)$, where $\lambda > 1$ is called the **dilatation** of φ.

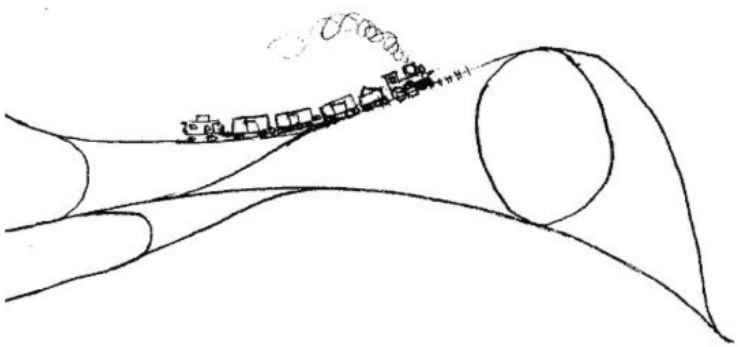
- $(\mathcal{F}^u, \mu_u)$ is called the **unstable** foliation of φ and $(\mathcal{F}^s, \mu_s)$ is called the **stable** foliation.
- A finite segment of a leaf of $\mathcal{F}^u$ can be thought of as stretched by a factor of λ.

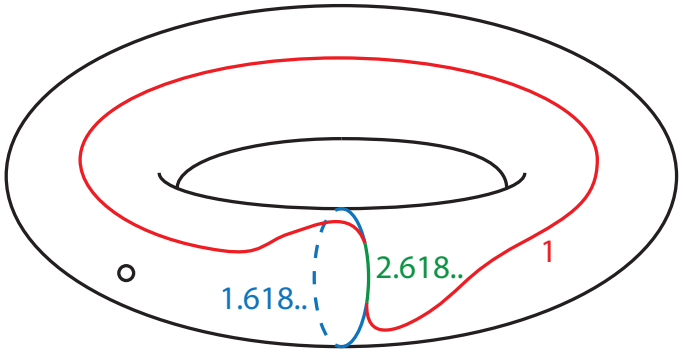
Train tracks

- A (measured) **train track** on a surface S is a finite 1-complex $\tau \subset S$ with C^1 embedded edges and a positive real number assigned to each edge such that each vertex is modelled on a *switch* (see Figure).

a, b, c > 0
Switch condition:
a = b + c



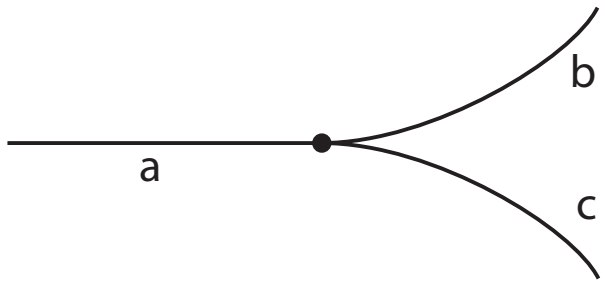




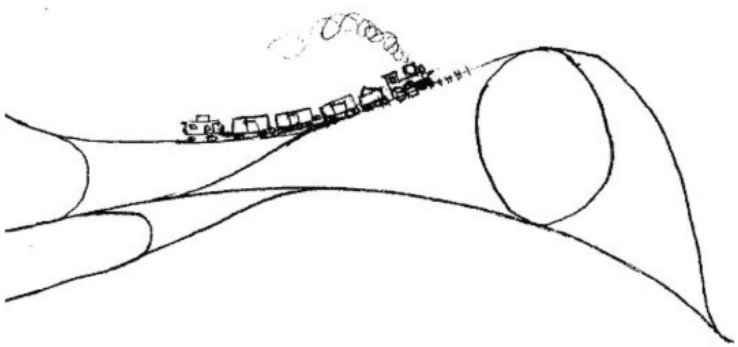
Train tracks

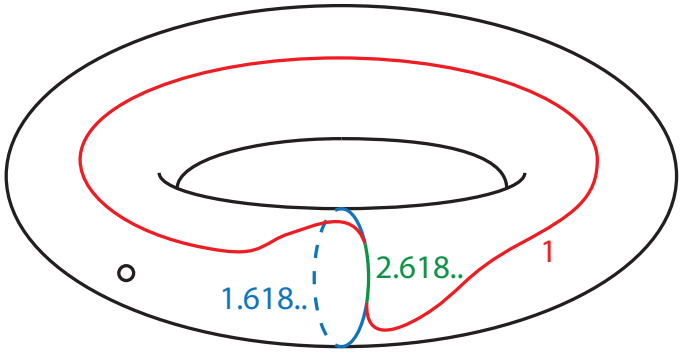
- A (measured) **train track** on a surface S is a finite 1-complex $\tau \subset S$ with C^1 embedded edges and a positive real number assigned to each edge such that each vertex is modelled on a *switch* (see Figure).

a, b, c > 0
Switch condition:
a = b + c



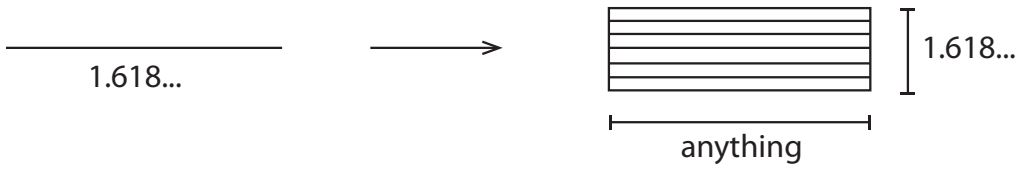
- We require that each component of $S \backslash \tau$ is a disk with at least 3 cusps, or a punctured disk with at least 1 cusp.





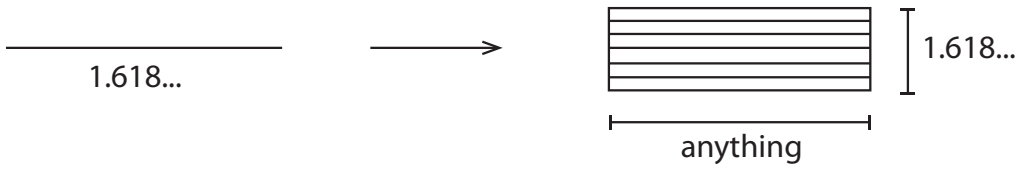
Train track to foliation

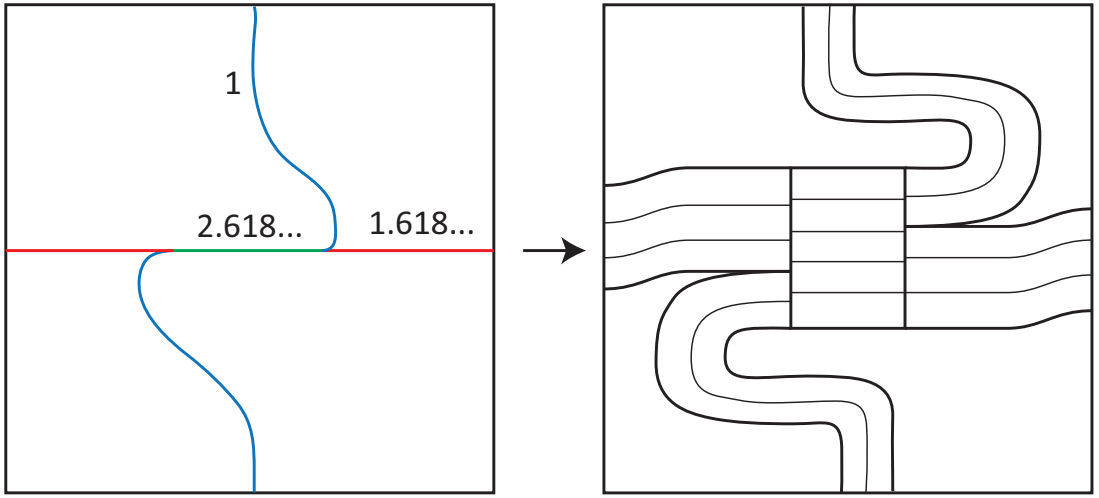
- Step 1: Replace each edge of τ by a foliated Euclidean rectangle.



Train track to foliation

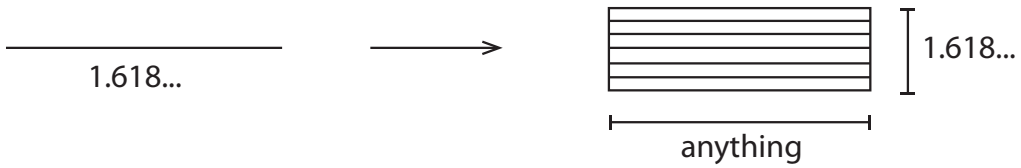
- Step 1: Replace each edge of τ by a foliated Euclidean rectangle.



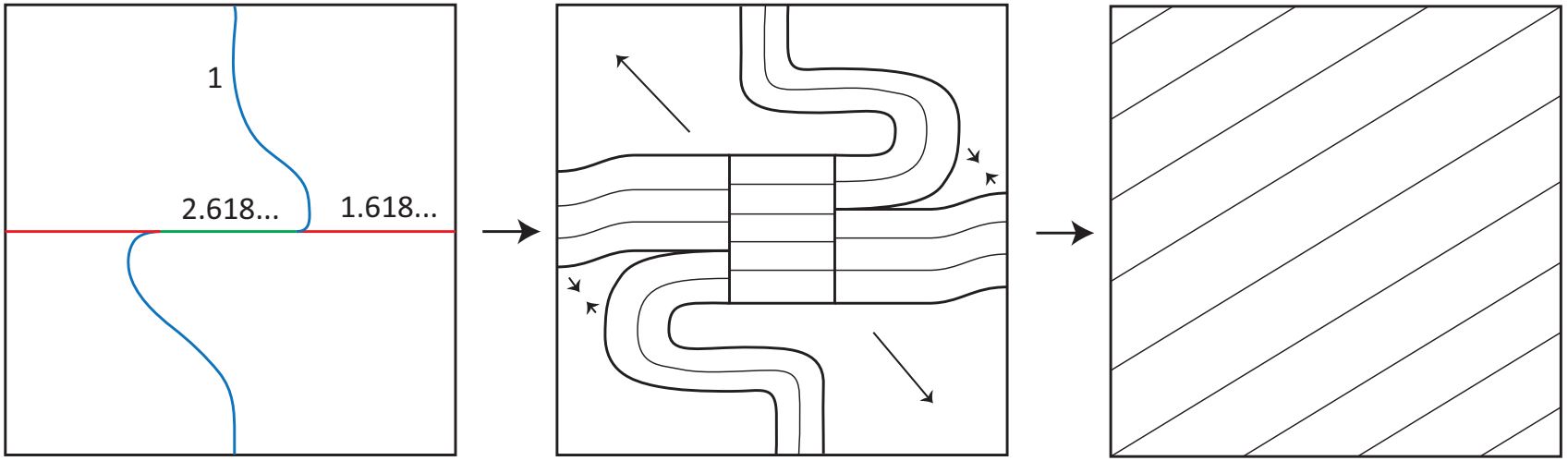


Train track to foliation

- Step 1: Replace each edge of τ by a foliated Euclidean rectangle.

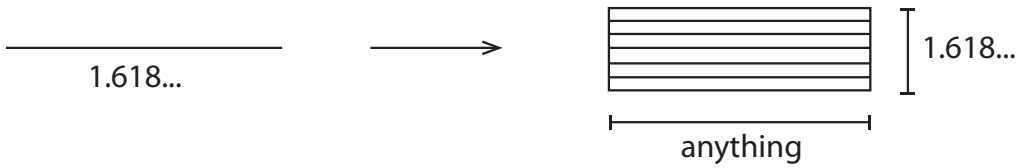


- Step 2: Perform identifications of the boundary of foliated regions to get a foliation $(\mathcal{F}, \mu)$.

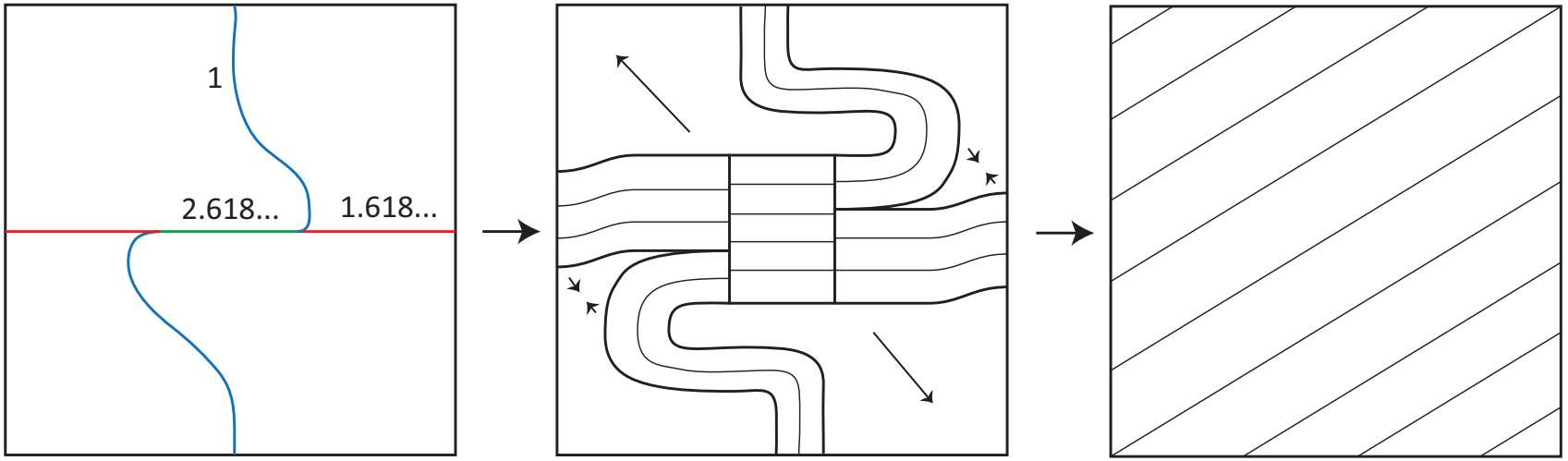


Train track to foliation

- Step 1: Replace each edge of τ by a foliated Euclidean rectangle.

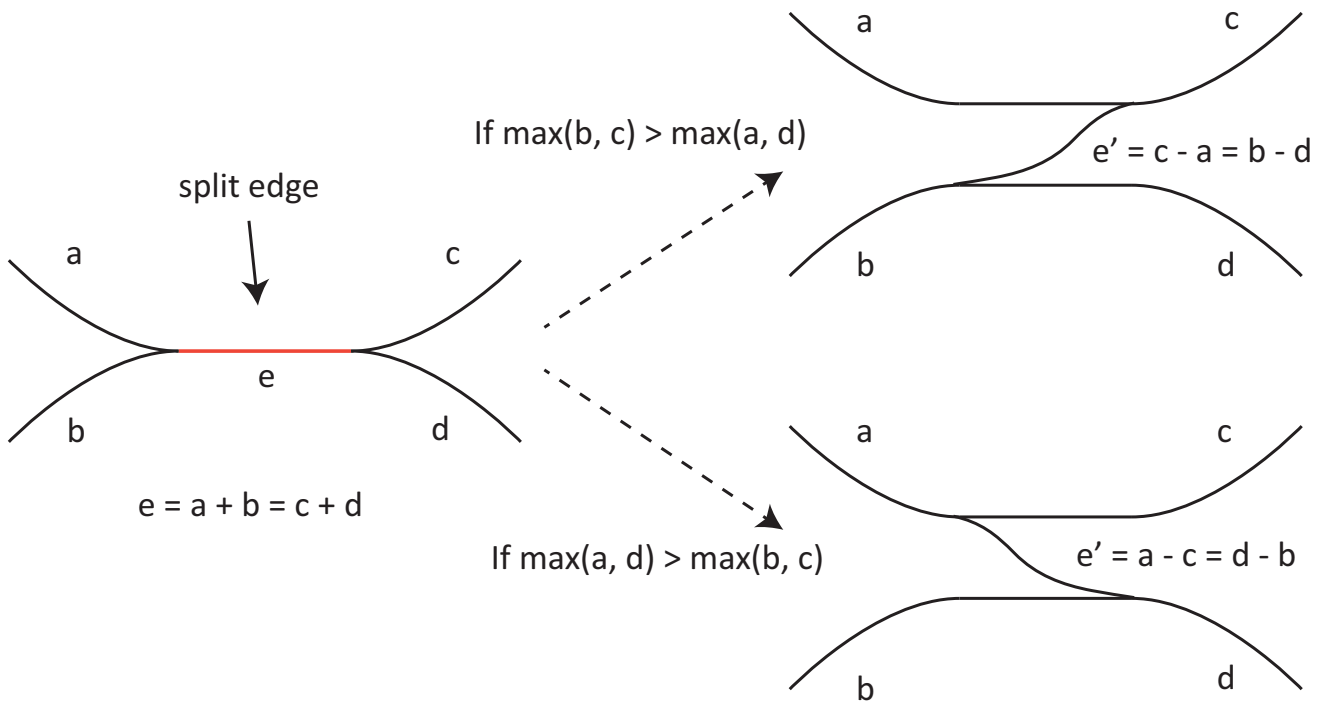


- Step 2: Perform identifications of the boundary of foliated regions to get a foliation $(\mathcal{F}, \mu)$.
- We say that τ **carries** the foliation $(\mathcal{F}, \mu)$.



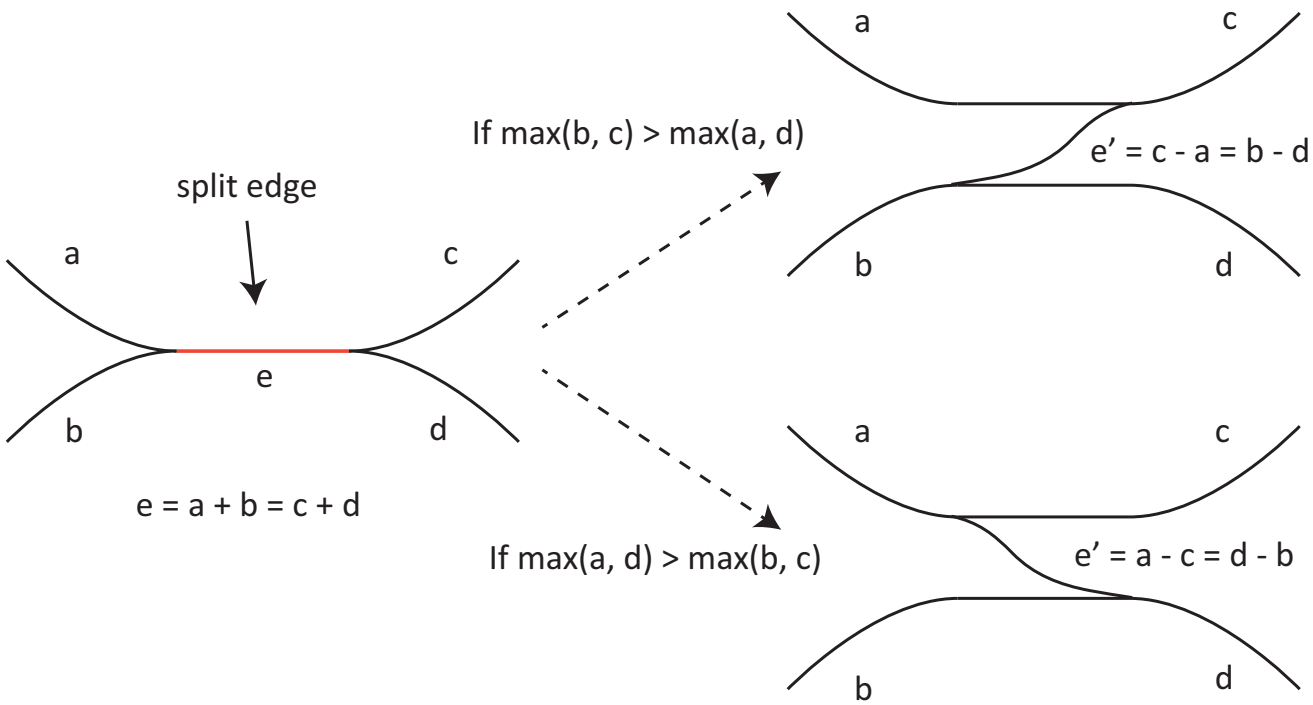
Splitting of train tracks

- A **split** of a train track τ on a surface S is a move which produces a train track τ' by 'splitting' a *large* edge of τ in one of two ways.



Splitting of train tracks

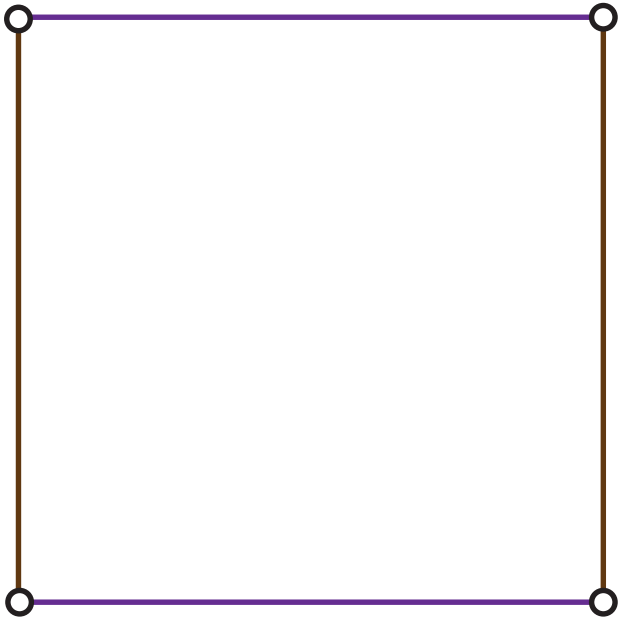
- A **split** of a train track τ on a surface S is a move which produces a train track τ' by 'splitting' a *large* edge of τ in one of two ways.
- A **maximal split** is a move which splits every edge of maximal measure (weight) simultaneously. We write $\tau \rightharpoonup \tau'$ to denote a maximal split.



Dual triangulation to a train track

- Let τ be a train track on a surface S.
- Assume that every component of $S \backslash \tau$ is homeomorphic to a *punctured* disk. Then there is an ideal triangulation of S dual to τ.

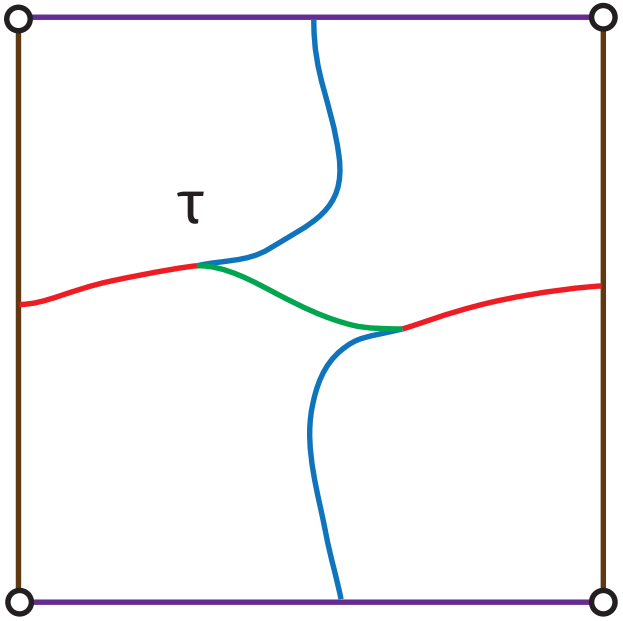
S = punctured torus



Dual triangulation to a train track

- Let τ be a train track on a surface S.
- Assume that every component of $S \backslash \tau$ is homeomorphic to a *punctured* disk. Then there is an ideal triangulation of S dual to τ.

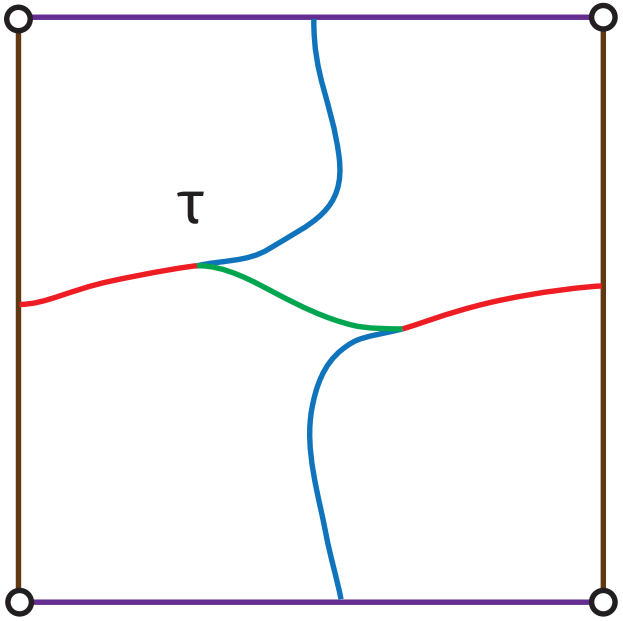
S = punctured torus



Dual triangulation to a train track

- Let τ be a train track on a surface S.
- Assume that every component of $S \backslash \tau$ is homeomorphic to a *punctured* disk. Then there is an ideal triangulation of S dual to τ.

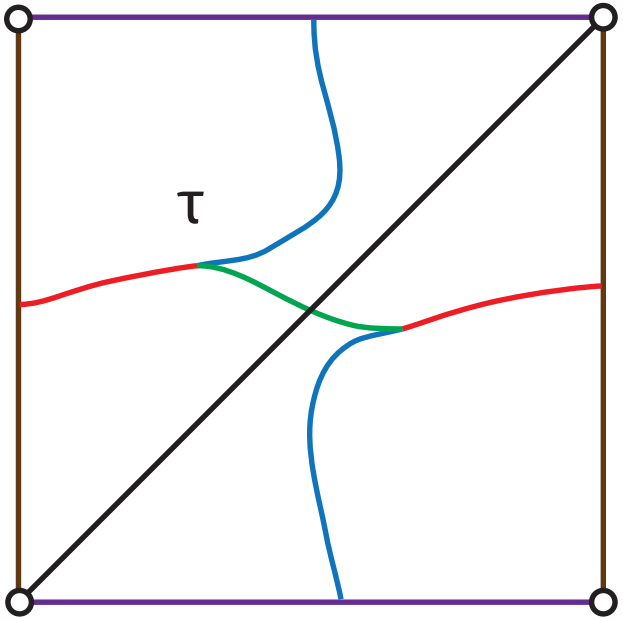
S = punctured torus



Dual triangulation to a train track

- Let τ be a train track on a surface S.
- Assume that every component of $S \backslash \tau$ is homeomorphic to a *punctured* disk. Then there is an ideal triangulation of S dual to τ.

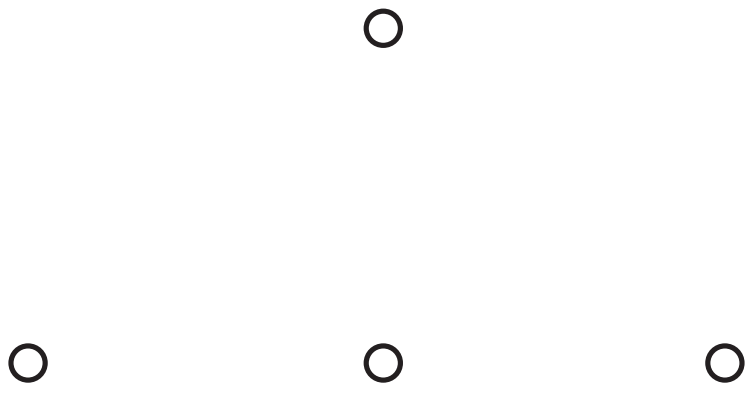
S = punctured torus



Dual triangulation to a train track

- Let τ be a train track on a surface S.
- Assume that every component of $S \backslash \tau$ is homeomorphic to a *punctured* disk. Then there is an ideal triangulation of S dual to τ.

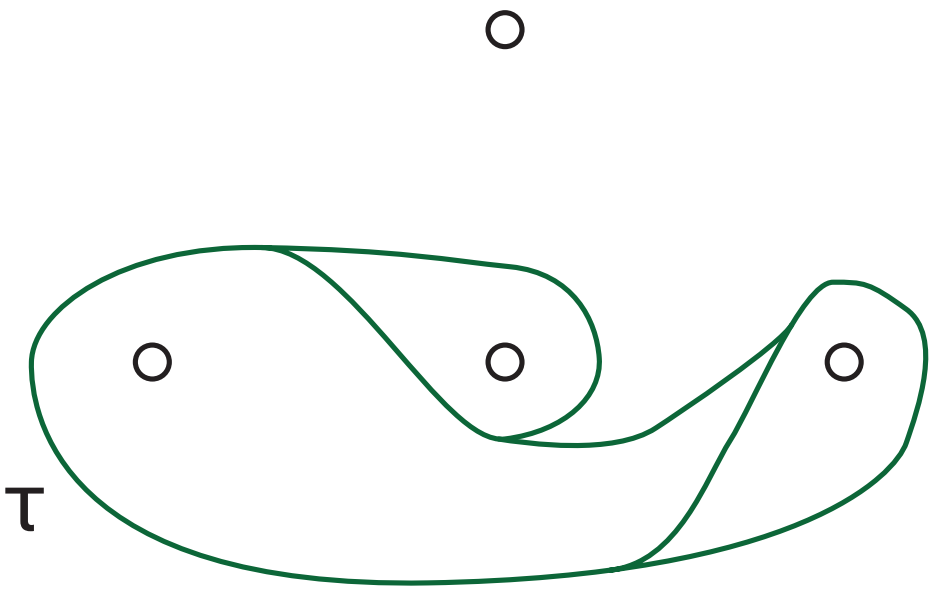
$$S = (\mathbb{R}^2 \cup \infty) \setminus \{4 \text{ points}\}$$



Dual triangulation to a train track

- Let τ be a train track on a surface S.
- Assume that every component of $S \backslash \tau$ is homeomorphic to a *punctured* disk. Then there is an ideal triangulation of S dual to τ.

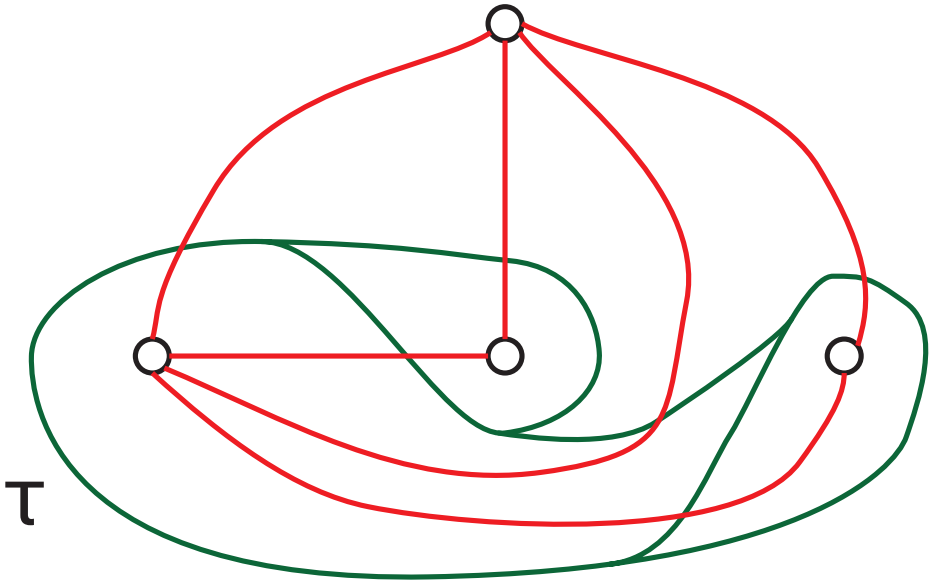
$$S = (\mathbb{R}^2 \cup \infty) \setminus \{4 \text{ points}\}$$



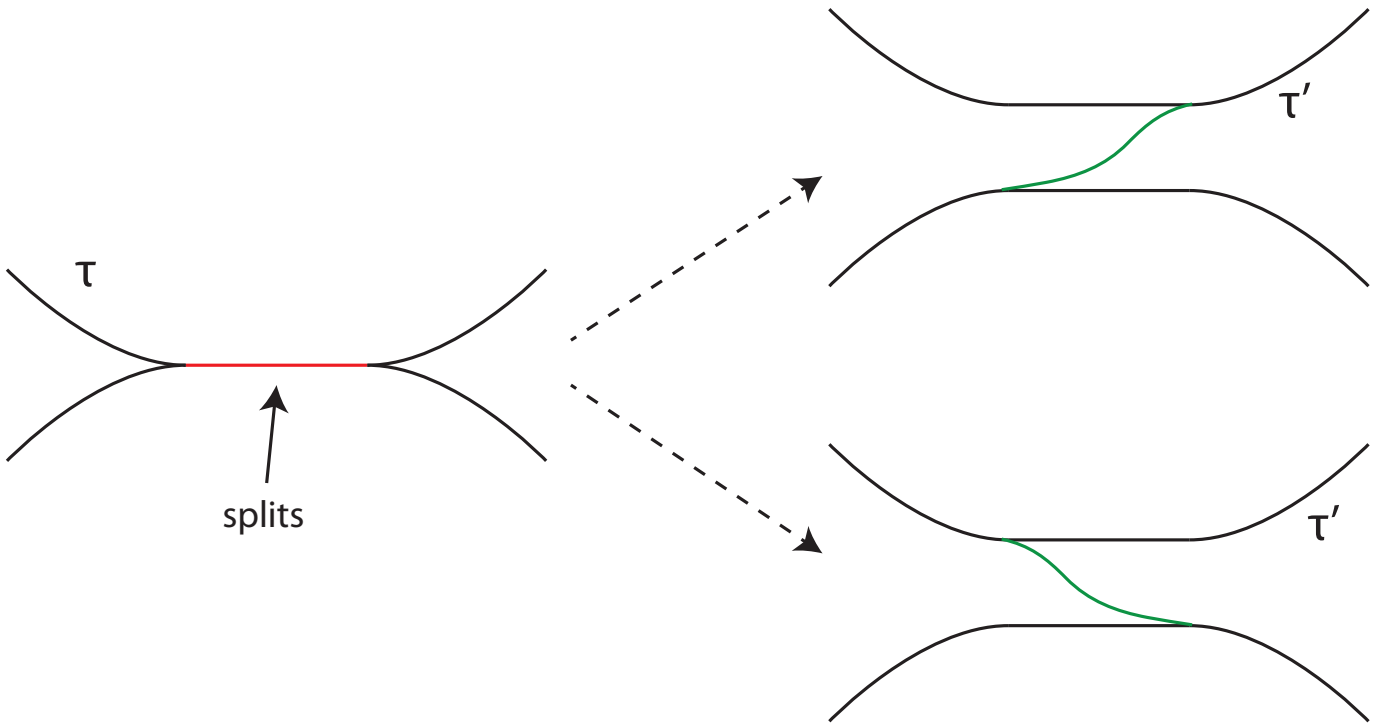
Dual triangulation to a train track

- Let τ be a train track on a surface S.
- Assume that every component of $S \backslash \tau$ is homeomorphic to a *punctured* disk. Then there is an ideal triangulation of S dual to τ.

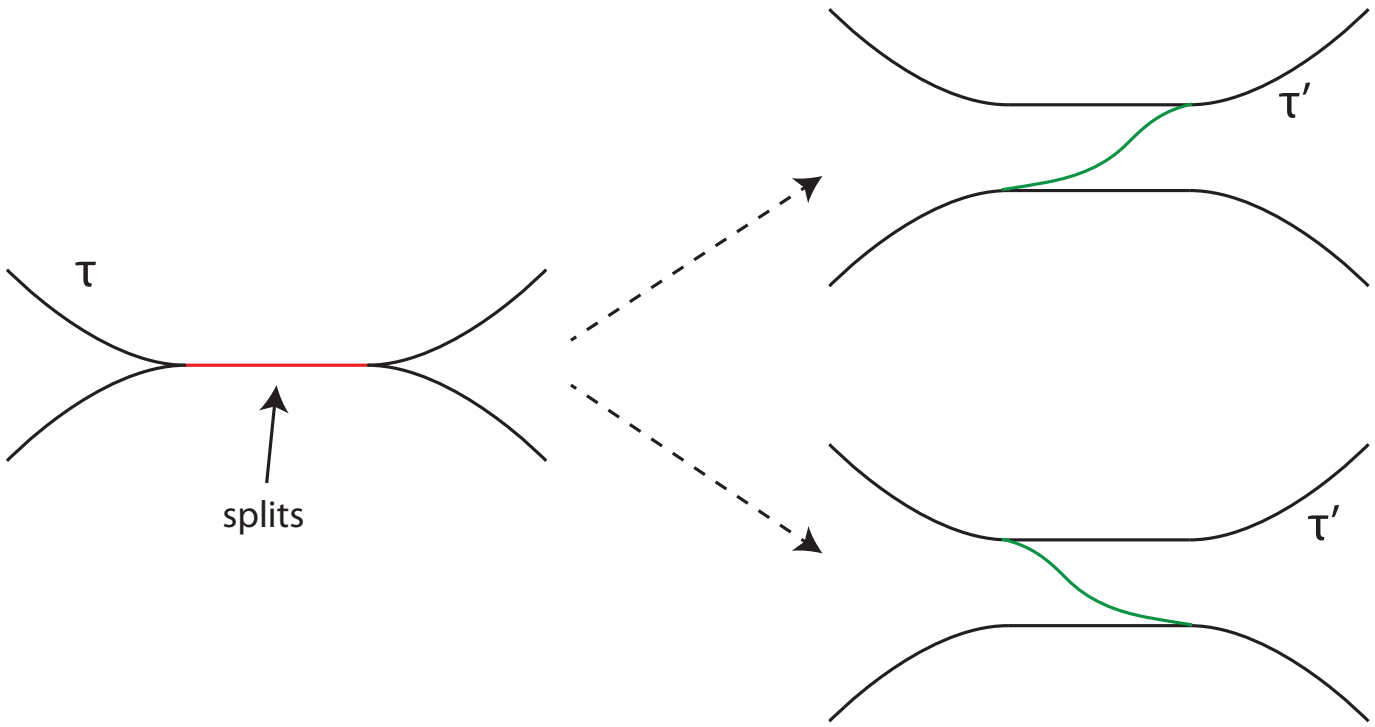
$S = (R^2 \cup \infty) \setminus \{4 \text{ points}\}$



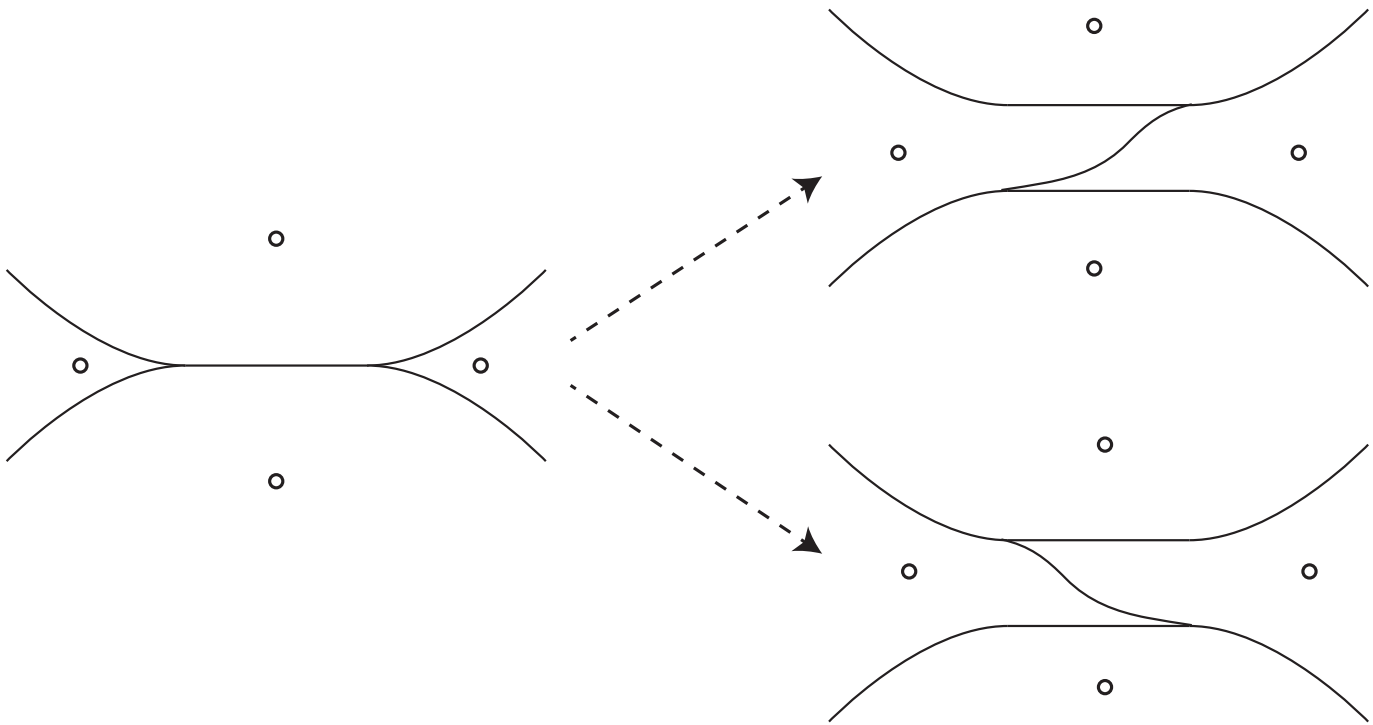
- What happens to the dual ideal triangulation when a train track splits?
- A diagonal exchange occurs!



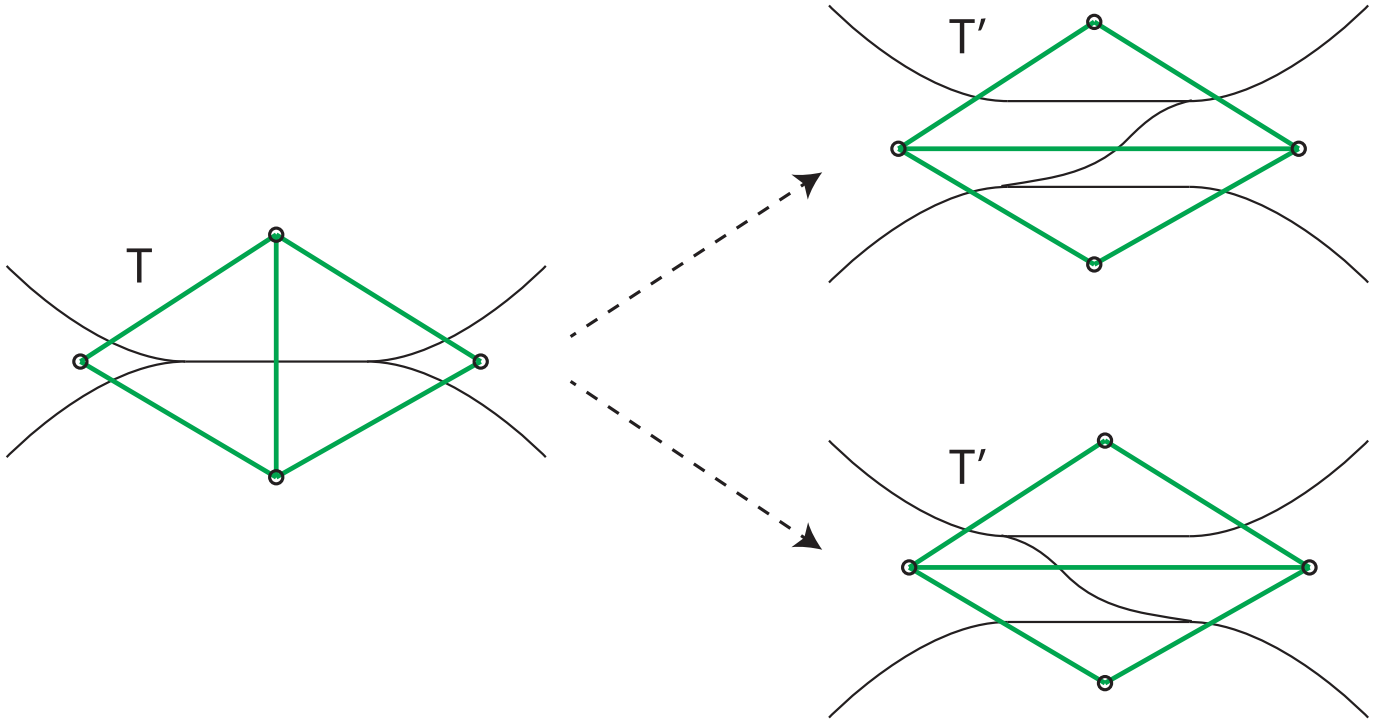
- What happens to the dual ideal triangulation when a train track splits?
- A diagonal exchange occurs!



- What happens to the dual ideal triangulation when a train track splits?
- A diagonal exchange occurs!



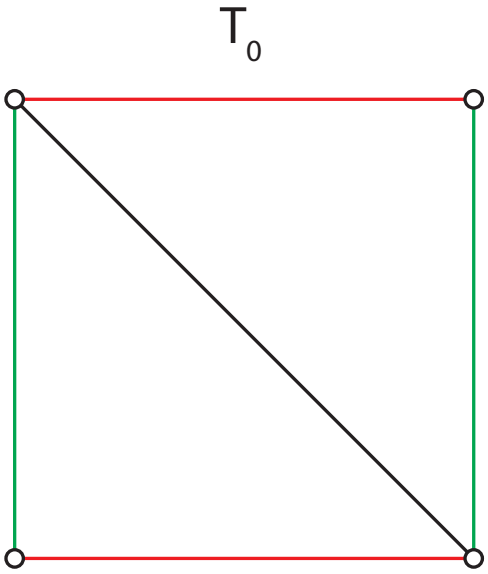
- What happens to the dual ideal triangulation when a train track splits?
- A diagonal exchange occurs!



Triangulating a mapping torus

A **layered** triangulation of a mapping torus M_φ, where $\varphi : S \to S$ is a homeomorphism, is one obtained as follows:

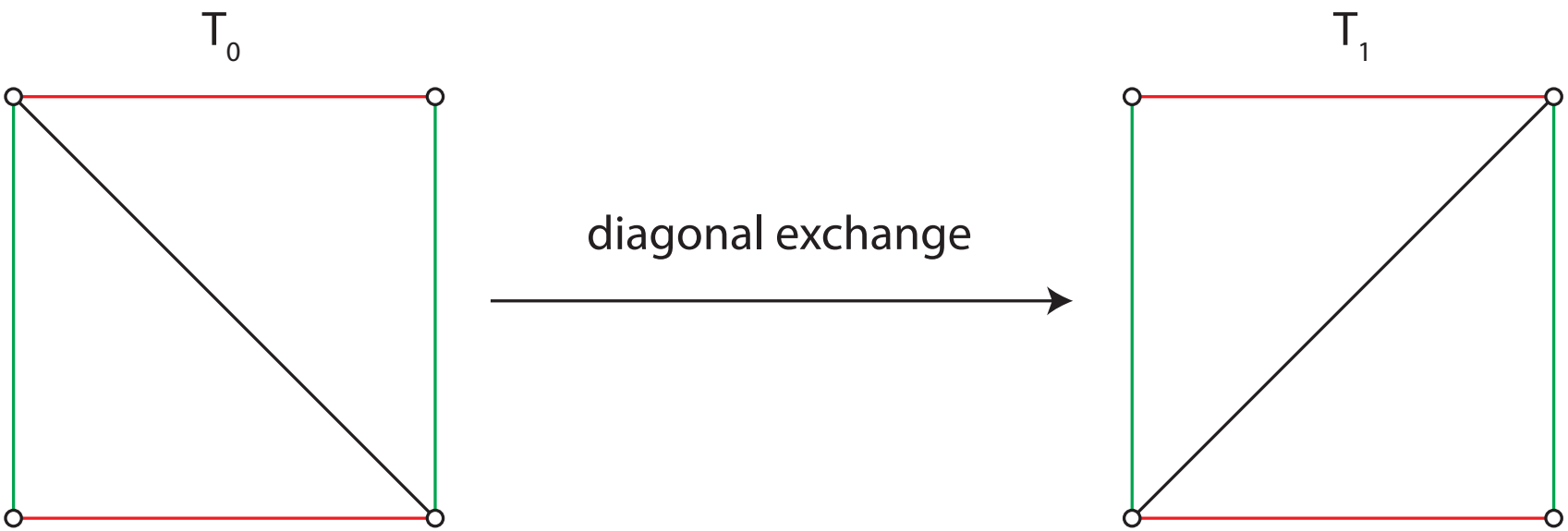
1. Start with an ideal triangulation T_0 of the surface S.



Triangulating a mapping torus

A **layered** triangulation of a mapping torus M_φ, where $\varphi : S \to S$ is a homeomorphism, is one obtained as follows:

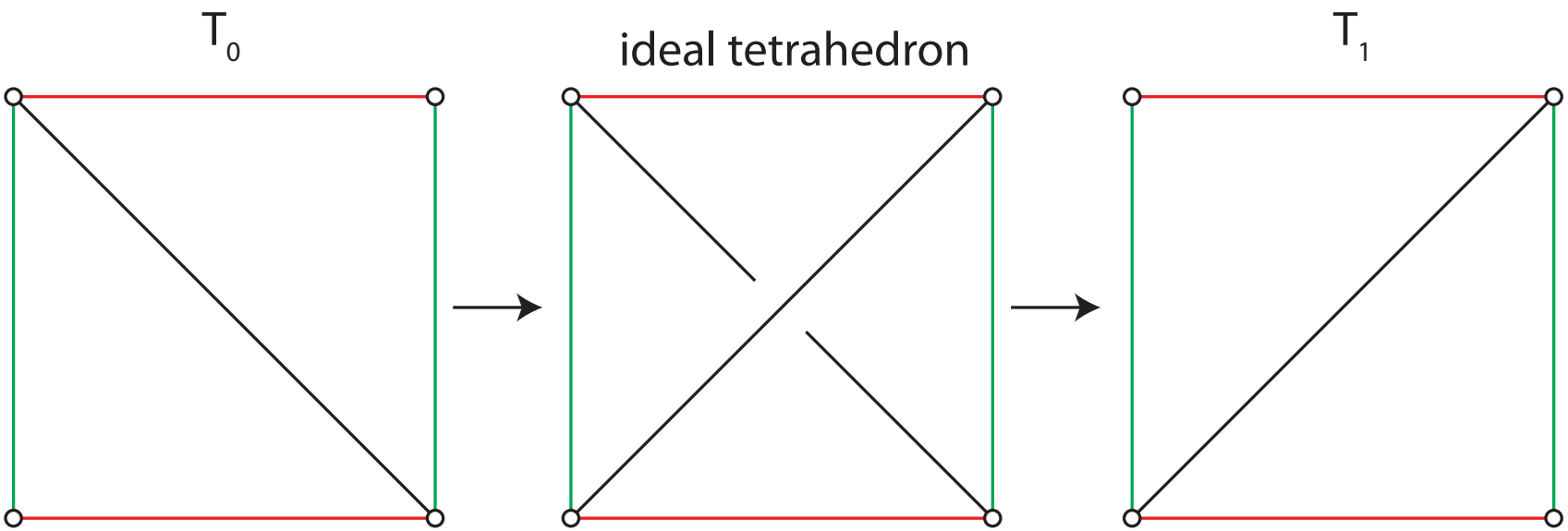
1. Start with an ideal triangulation T_0 of the surface S.
2. Pick an edge and replace it by the 'other' diagonal edge to get a triangulation T_1 of S.



Triangulating a mapping torus

A **layered** triangulation of a mapping torus M_φ, where $\varphi : S \to S$ is a homeomorphism, is one obtained as follows:

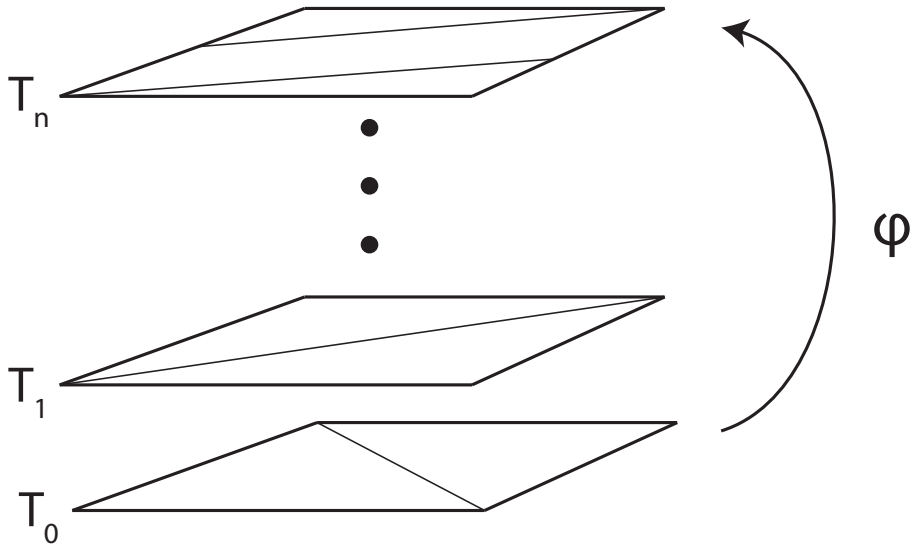
1. Start with an ideal triangulation T_0 of the surface S.
2. Pick an edge and replace it by the 'other' diagonal edge to get a triangulation T_1 of S.
3. Insert a tetrahedron which interpolates between T_0 and T_1.



Triangulating a mapping torus

A **layered** triangulation of a mapping torus M_φ, where $\varphi : S \to S$ is a homeomorphism, is one obtained as follows:

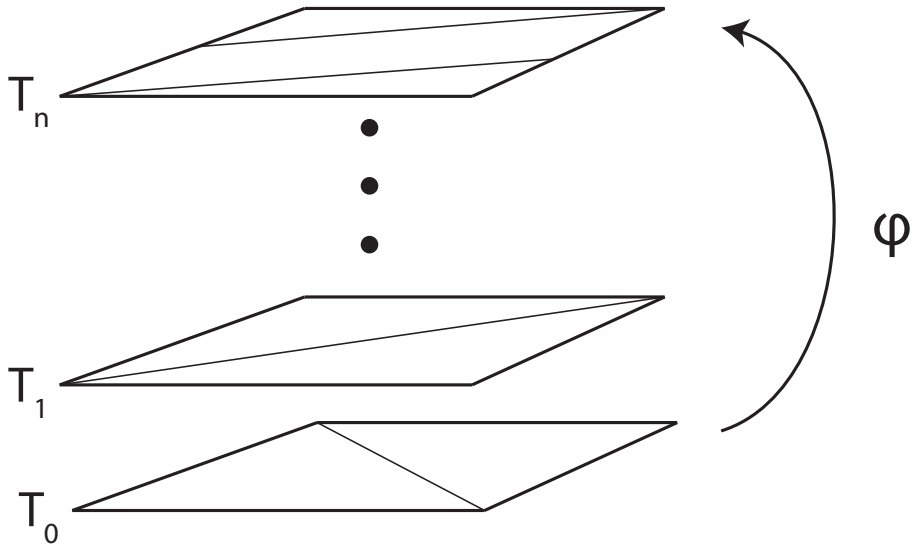
1. Start with an ideal triangulation T_0 of the surface S.
2. Pick an edge and replace it by the 'other' diagonal edge to get a triangulation T_1 of S.
3. Insert a tetrahedron which interpolates between T_0 and T_1.
4. Repeat this to get a sequence of triangulations $T_0, T_1, \ldots, T_n$ such that $\varphi(T_0) = T_n$.



Triangulating a mapping torus

A **layered** triangulation of a mapping torus M_φ, where $\varphi : S \to S$ is a homeomorphism, is one obtained as follows:

1. Start with an ideal triangulation T_0 of the surface S.
2. Pick an edge and replace it by the 'other' diagonal edge to get a triangulation T_1 of S.
3. Insert a tetrahedron which interpolates between T_0 and T_1.
4. Repeat this to get a sequence of triangulations $T_0, T_1, \ldots, T_n$ such that $\varphi(T_0) = T_n$.
5. Glue T_0 to T_n via φ.



Periodic splitting sequence

Theorem (Agol)

Let τ be a train track carrying the unstable foliation $\mathcal{F}^u$ of a pseudo-Anosov homeomorphism $\varphi : S \to S$. There exist $n, m > 0$ such that

$$\tau \rightharpoonup \tau_1 \rightharpoonup \tau_2 \rightharpoonup \cdots \rightharpoonup \tau_n \rightharpoonup \cdots \rightharpoonup \tau_{n+m},$$

such that $\varphi(\frac{1}{\lambda}\tau_n) = \tau_{n+m}$, where $\lambda > 1$ is the dilatation of φ. We call $\tau_n \rightharpoonup \cdots \rightharpoonup \tau_{n+m} = \varphi(\frac{1}{\lambda}\tau_n)$ a **periodic splitting sequence**.

Periodic splitting sequence

Theorem (Agol)

Let τ be a train track carrying the unstable foliation $\mathcal{F}^u$ of a pseudo-Anosov homeomorphism $\varphi : S \to S$. There exist $n, m > 0$ such that

$$\tau \rightharpoonup \tau_1 \rightharpoonup \tau_2 \rightharpoonup \cdots \rightharpoonup \tau_n \rightharpoonup \cdots \rightharpoonup \tau_{n+m},$$

such that $\varphi(\frac{1}{\lambda}\tau_n) = \tau_{n+m}$, where $\lambda > 1$ is the dilatation of φ. We call $\tau_n \rightharpoonup \cdots \rightharpoonup \tau_{n+m} = \varphi(\frac{1}{\lambda}\tau_n)$ a **periodic splitting sequence**.

Idea of proof: Train tracks which carry the same foliation have a common maximal split. We have that $\varphi(\frac{1}{\lambda}\tau)$ carries $\varphi(\mathcal{F}^u, \frac{1}{\lambda}\mu_u) = (\mathcal{F}^u, \mu_u)$. So τ and $\varphi(\frac{1}{\lambda}\tau)$ have a common maximal split, namely $\varphi(\frac{1}{\lambda}\tau_n) = \tau_{n+m}$ for some $n, m > 0$.

- Assume that the unstable foliation $\mathcal{F}^u$ of $\varphi : S \to S$ has no singular points. We construct a layered triangulation of M_φ.

- Assume that the unstable foliation $\mathcal{F}^u$ of $\varphi : S \to S$ has no singular points. We construct a layered triangulation of M_φ.
- **Note:** if $\mathcal{F}^u$ has singular points on S, we can remove those singular points to get a surface S°. Then φ restricts to $\varphi^\circ : S^\circ \to S^\circ$, and we can triangulate M_{φ° instead.

- Assume that the unstable foliation $\mathcal{F}^u$ of $\varphi : S \to S$ has no singular points. We construct a layered triangulation of M_φ.
- **Note:** if $\mathcal{F}^u$ has singular points on S, we can remove those singular points to get a surface S°. Then φ restricts to $\varphi^\circ : S^\circ \to S^\circ$, and we can triangulate M_{φ° instead.
- Then to each train track τ_i in the periodic splitting sequence there is a dual ideal triangulation T_i of the surface.

$$\tau_n \rightharpoonup \tau_{n+1} \rightharpoonup \cdots \rightharpoonup \tau_{n+m} = \varphi(\frac{1}{\lambda}\tau_n)$$

$$T_n \to T_{n+1} \to \cdots \to T_{n+m} = \varphi(T_n).$$

- Assume that the unstable foliation $\mathcal{F}^u$ of $\varphi : S \to S$ has no singular points. We construct a layered triangulation of M_φ.
- **Note:** if $\mathcal{F}^u$ has singular points on S, we can remove those singular points to get a surface S°. Then φ restricts to $\varphi^\circ : S^\circ \to S^\circ$, and we can triangulate M_{φ° instead.
- Then to each train track τ_i in the periodic splitting sequence there is a dual ideal triangulation T_i of the surface.

$$\tau_n \rightharpoonup \tau_{n+1} \rightharpoonup \cdots \rightharpoonup \tau_{n+m} = \varphi(\frac{1}{\lambda}\tau_n)$$

$$T_n \to T_{n+1} \to \cdots \to T_{n+m} = \varphi(T_n).$$

- Each maximal split $\tau_i \rightharpoonup \tau_{i+1}$ corresponds to a sequence of diagonal exchanges, for which we attach tetrahedra interpolating between T_i and T_{i+1}.

- Assume that the unstable foliation $\mathcal{F}^u$ of $\varphi : S \to S$ has no singular points. We construct a layered triangulation of M_φ.
- **Note:** if $\mathcal{F}^u$ has singular points on S, we can remove those singular points to get a surface S°. Then φ restricts to $\varphi^\circ : S^\circ \to S^\circ$, and we can triangulate M_{φ° instead.
- Then to each train track τ_i in the periodic splitting sequence there is a dual ideal triangulation T_i of the surface.

$$\tau_n \rightharpoonup \tau_{n+1} \rightharpoonup \cdots \rightharpoonup \tau_{n+m} = \varphi(\frac{1}{\lambda}\tau_n)$$

$$T_n \to T_{n+1} \to \cdots \to T_{n+m} = \varphi(T_n).$$

- Each maximal split $\tau_i \rightharpoonup \tau_{i+1}$ corresponds to a sequence of diagonal exchanges, for which we attach tetrahedra interpolating between T_i and T_{i+1}.
- Since $\tau_{n+m} = \varphi(\frac{1}{\lambda}\tau_n)$ we have $T_{n+m} = \varphi(T_n)$. So we can glue T_n to T_{n+m}.

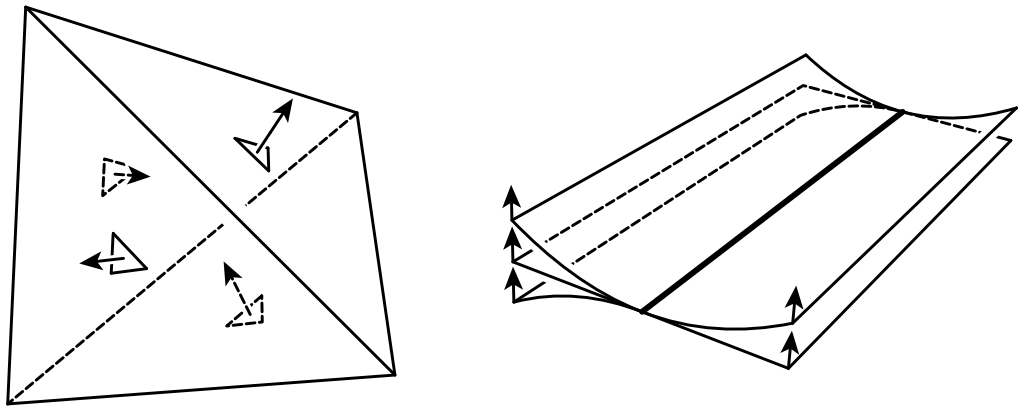
- Assume that the unstable foliation $\mathcal{F}^u$ of $\varphi : S \to S$ has no singular points. We construct a layered triangulation of M_φ.
- **Note:** if $\mathcal{F}^u$ has singular points on S, we can remove those singular points to get a surface S°. Then φ restricts to $\varphi^\circ : S^\circ \to S^\circ$, and we can triangulate M_{φ° instead.
- Then to each train track τ_i in the periodic splitting sequence there is a dual ideal triangulation T_i of the surface.

$$\tau_n \rightharpoonup \tau_{n+1} \rightharpoonup \cdots \rightharpoonup \tau_{n+m} = \varphi(\frac{1}{\lambda}\tau_n)$$

$$T_n \to T_{n+1} \to \cdots \to T_{n+m} = \varphi(T_n).$$

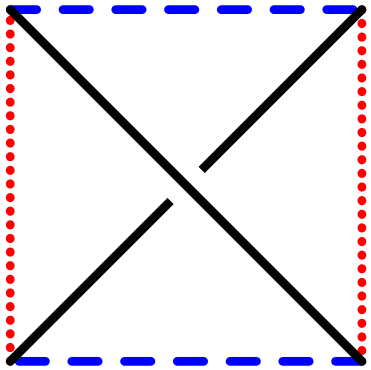
- Each maximal split $\tau_i \rightharpoonup \tau_{i+1}$ corresponds to a sequence of diagonal exchanges, for which we attach tetrahedra interpolating between T_i and T_{i+1}.
- Since $\tau_{n+m} = \varphi(\frac{1}{\lambda}\tau_n)$ we have $T_{n+m} = \varphi(T_n)$. So we can glue T_n to T_{n+m}.
- We have constructed a layered triangulation of M_φ.

A **taut tetrahedron** is an ideal tetrahedron with a coorientation assigned to each face, such that precisely two faces are cooriented into the tetrahedron, and precisely two are cooriented outwards. Each edge is assigned an angle of either π if the coorientations on the adjacent faces agree, or 0 if they disagree.



A **taut triangulation** of M is an ideal triangulation of M with a coorientation assigned to each ideal triangle so that each tetrahedron is taut and the sum of the angles around each edge is 2π.

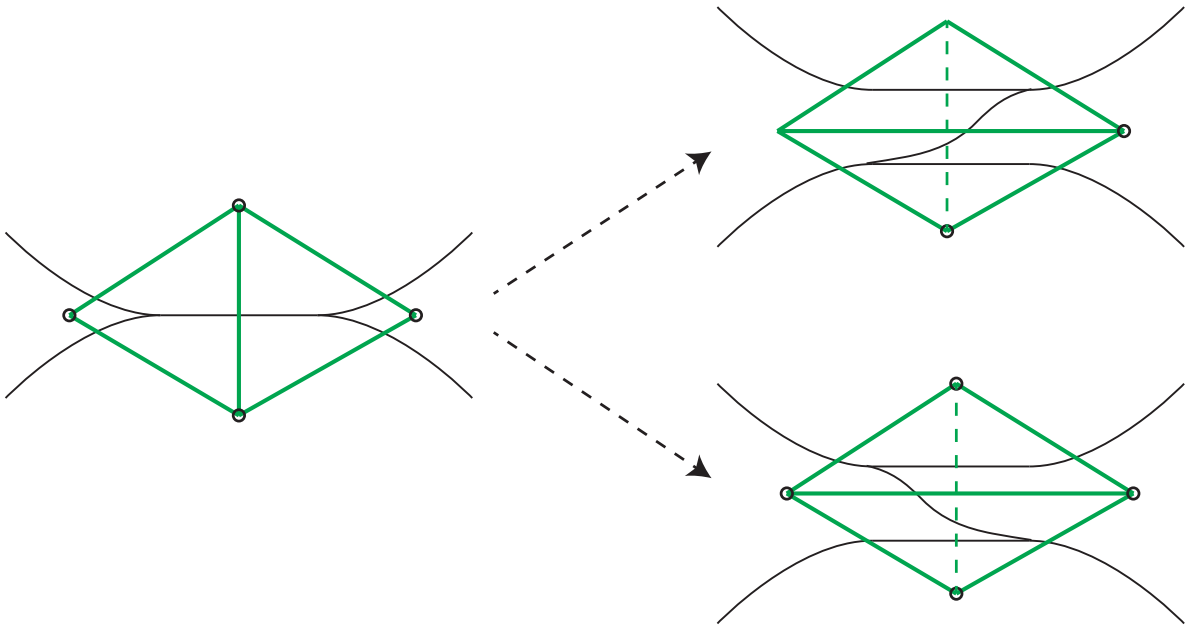
We colour the zero angle edges blue and red as shown. Then the red edges are called **right veering** and the blue edges are called **left veering**.



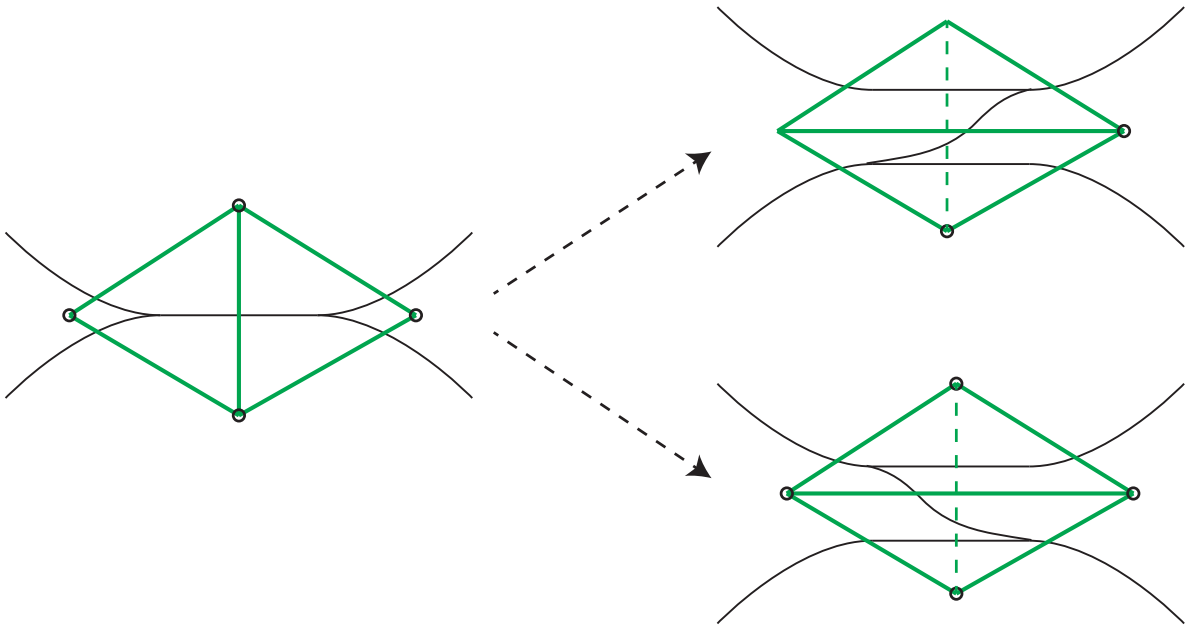
As viewed from a red edge the triangles at the red edge move to the right going from bottom to top. As viewed from a blue edge the triangles at the edge move to the left going from bottom to top.

A **veering triangulation** of M is a taut triangulation $\mathcal{T}$ with an assignment of red or blue to the edges of $\mathcal{T}$ so that the zero angles of each tetrahedron are coloured as above.

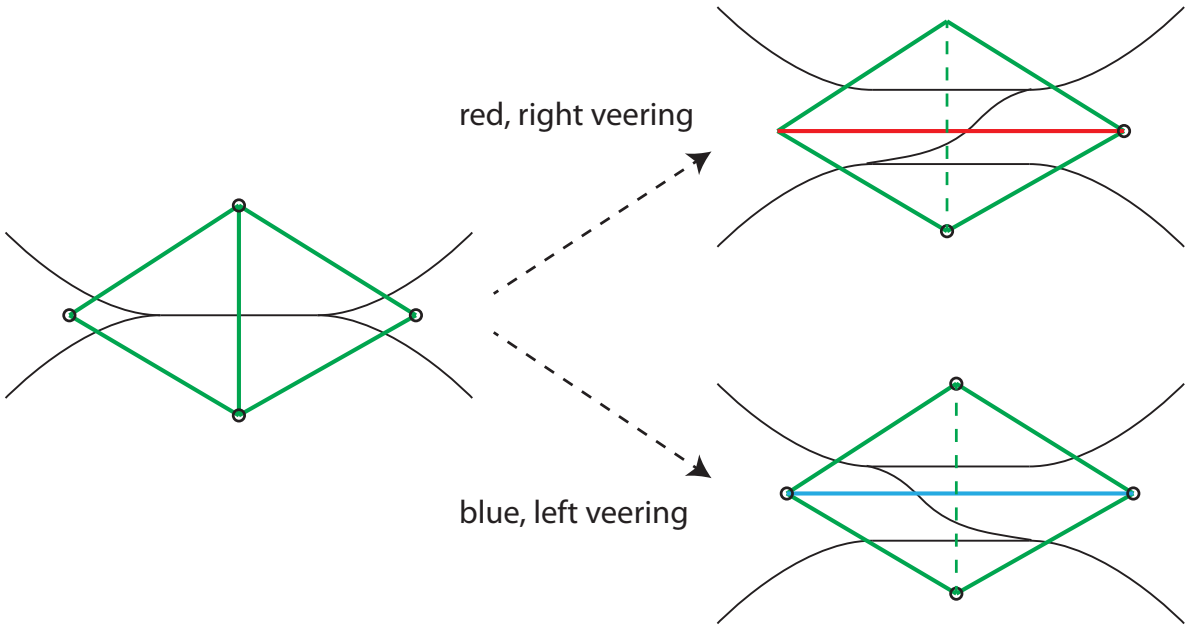
- The fact that this construction produces veering triangulations follows from the combinatorics of train tracks.



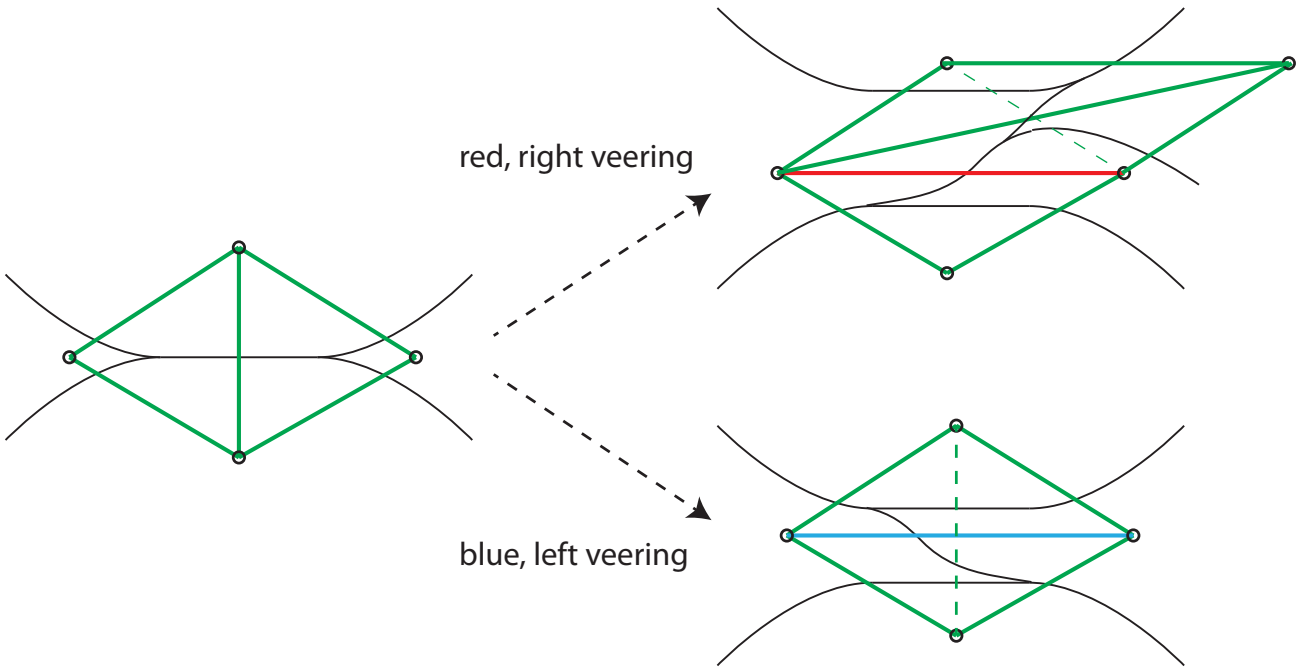
- The fact that this construction produces veering triangulations follows from the combinatorics of train tracks.
- Each tetrahedron is naturally taut.



- The fact that this construction produces veering triangulations follows from the combinatorics of train tracks.
- Each tetrahedron is naturally taut.
- When an edge is born do we colour it red or blue?



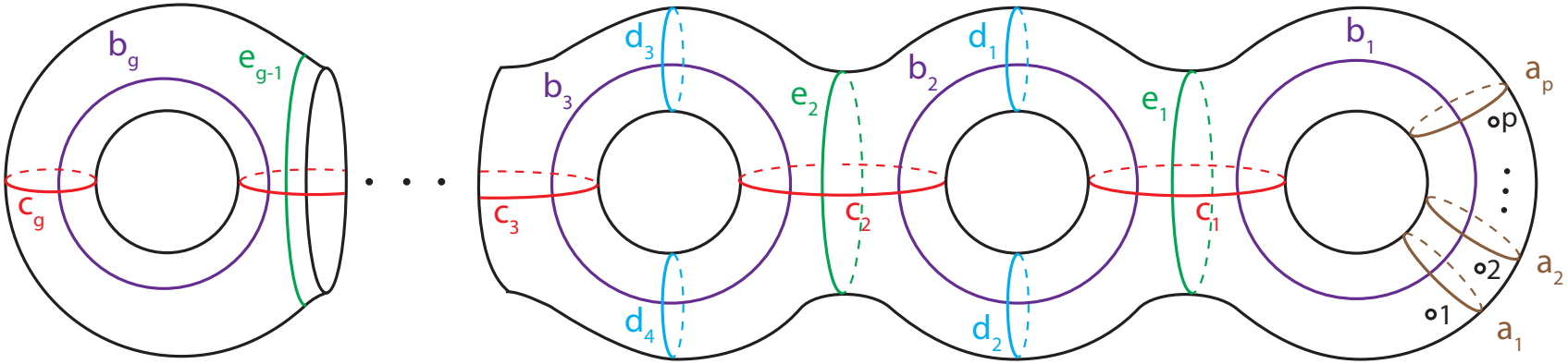
- The fact that this construction produces veering triangulations follows from the combinatorics of train tracks.
- Each tetrahedron is naturally taut.
- When an edge is born do we colour it red or blue?



- We wrote a computer program called *Veering* which automates this construction of veering triangulations.

- We wrote a computer program called *Veering* which automates this construction of veering triangulations.
- Relies on the program *Trains* by Toby Hall, an implementation of the Bestvina-Handel algorithm for constructing a train track carrying the unstable foliation of a pseudo-Anosov homeomorphism.

- We wrote a computer program called *Veering* which automates this construction of veering triangulations.
- Relies on the program *Trains* by Toby Hall, an implementation of the Bestvina-Handel algorithm for constructing a train track carrying the unstable foliation of a pseudo-Anosov homeomorphism.
- Homeomorphisms are specified as a composition of Dehn twists in the curves shown below, and permutations of the punctures.



- The shape of an oriented *hyperbolic* ideal tetrahedron in $\mathbb{H}^3$ is determined by a complex number $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$.
- Given an oriented ideal triangulation of a hyperbolic 3-manifold M, we can realise each tetrahedron T_i as a hyperbolic tetrahedron determined by z_i with $\mathrm{Im}(z) > 0$.
- The hyperbolic tetrahedra glue together coherently to give the complete hyperbolic structure on M if and only if they satisfy a system of equations in the z_i's called **Thurston's gluing and completeness equations**.
- If there exists such a solution, the ideal triangulation is called **geometric**.

Q: Are veering triangulations always geometric?

A *necessary* (but insufficient!) condition for an (oriented) ideal triangulation to be geometric is that it admits a **strict angle structure**, that is, we can find positive volume ideal hyperbolic tetrahedron shapes on the tetrahedra so that the angle around each edge is 2π.

Hodgson, Rubinstein, Segerman and Tillmann introduced a larger class of 'veering triangulations' and showed:

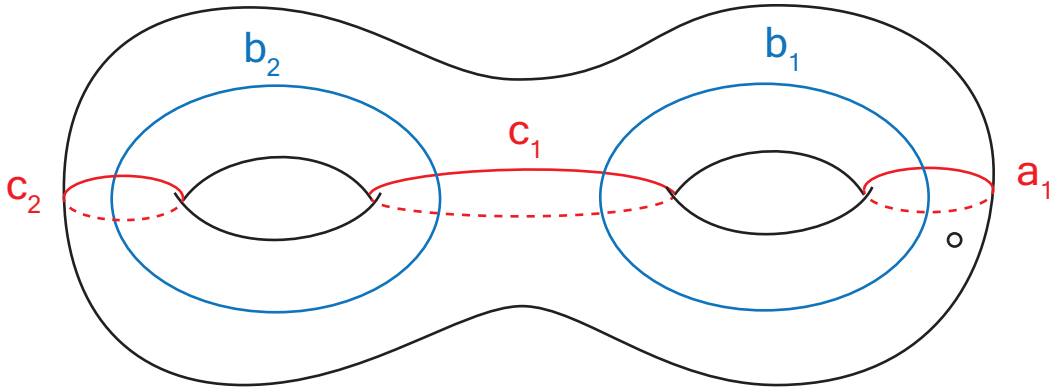
Theorem (HRST)

Veering triangulations admit strict angle structures.

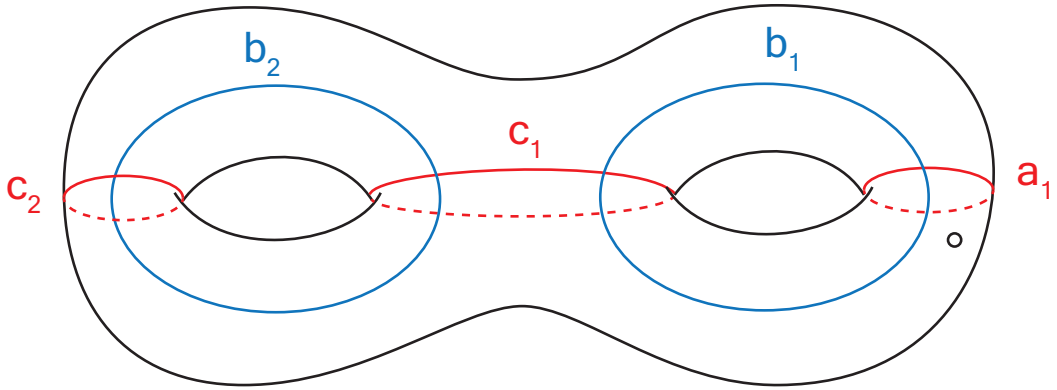
(Gueritaud and Futer have given another proof of this.)

- We found examples of veering triangulations which are **not** geometric.

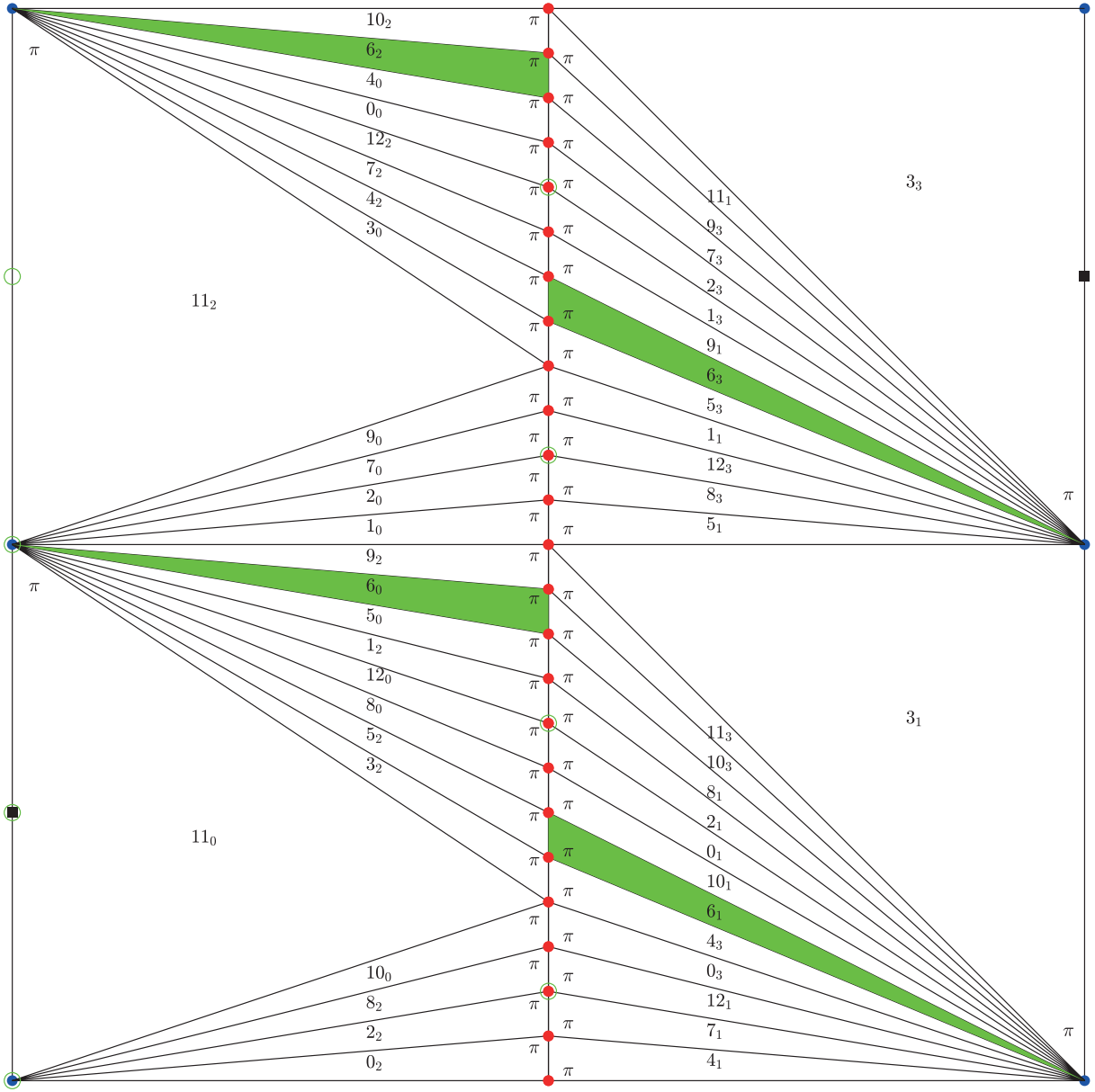
- We found examples of veering triangulations which are **not** geometric.
- The smallest example we've found is a 13 tetrahedron triangulation of the bundle M_φ, where φ is the composition $T_{c_1} \circ T_{b_2} \circ T_{a_1} \circ T_{a_1} \circ T_{a_1} \circ T_{b_1} \circ T_{a_1}$ of Dehn twists of the once punctured genus 2 surface (T_γ is a Dehn twist in γ.)



- We found examples of veering triangulations which are **not** geometric.
- The smallest example we've found is a 13 tetrahedron triangulation of the bundle M_φ, where φ is the composition $T_{c_1} \circ T_{b_2} \circ T_{a_1} \circ T_{a_1} \circ T_{a_1} \circ T_{b_1} \circ T_{a_1}$ of Dehn twists of the once punctured genus 2 surface (T_γ is a Dehn twist in γ.)
- M_φ is the manifold s479 in the SnapPea census. The dilatation of φ is approx 2.89005.., and M_φ has hyperbolic volume 4.85117...

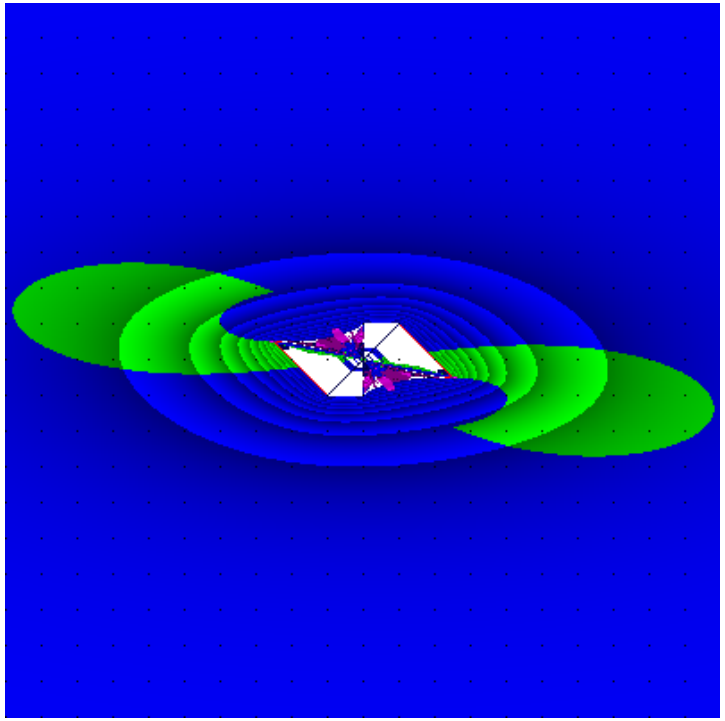


Here is the the triangulation of the cusp (the shaded tetrahedron is negatively oriented for the complete hyperbolic structure.)



- Although there are non-geometric veering triangulations, we might still hope for weaker conclusions, e.g.
  Question: Given a veering triangulation, is there a (possibly incomplete) hyperbolic structure obtained by gluing together positive volume ideal tetrahedra? (This would give a point in Thurston's hyperbolic Dehn surgery space.)

- Although there are non-geometric veering triangulations, we might still hope for weaker conclusions, e.g.
  Question: Given a veering triangulation, is there a (possibly incomplete) hyperbolic structure obtained by gluing together positive volume ideal tetrahedra? (This would give a point in Thurston's hyperbolic Dehn surgery space.)
- **Example:** Here is the Dehn surgery space for the above example, with green indicating all tetrahedra positively oriented, blue some negatively oriented solutions (as found by SnapPy.)



- We constructed all veering triangulations (up to conjugation and inversion) of a once-punctured genus 2 surface, given by a product of at most 7 Dehn twists in the curves shown previously.

- We constructed all veering triangulations (up to conjugation and inversion) of a once-punctured genus 2 surface, given by a product of at most 7 Dehn twists in the curves shown previously.
- Out of about 600 triangulations produced, 48 are numerically reported non-geometric by the computer program *SnapPy*.

- We constructed all veering triangulations (up to conjugation and inversion) of a once-punctured genus 2 surface, given by a product of at most 7 Dehn twists in the curves shown previously.
- Out of about 600 triangulations produced, 48 are numerically reported non-geometric by the computer program *SnapPy*.
- In the 13 tetrahedron example we verified rigorously that it is non-geometric, expect all 48 to be non-geometric.

- We constructed all veering triangulations (up to conjugation and inversion) of a once-punctured genus 2 surface, given by a product of at most 7 Dehn twists in the curves shown previously.
- Out of about 600 triangulations produced, 48 are numerically reported non-geometric by the computer program *SnapPy*.
- In the 13 tetrahedron example we verified rigorously that it is non-geometric, expect all 48 to be non-geometric.

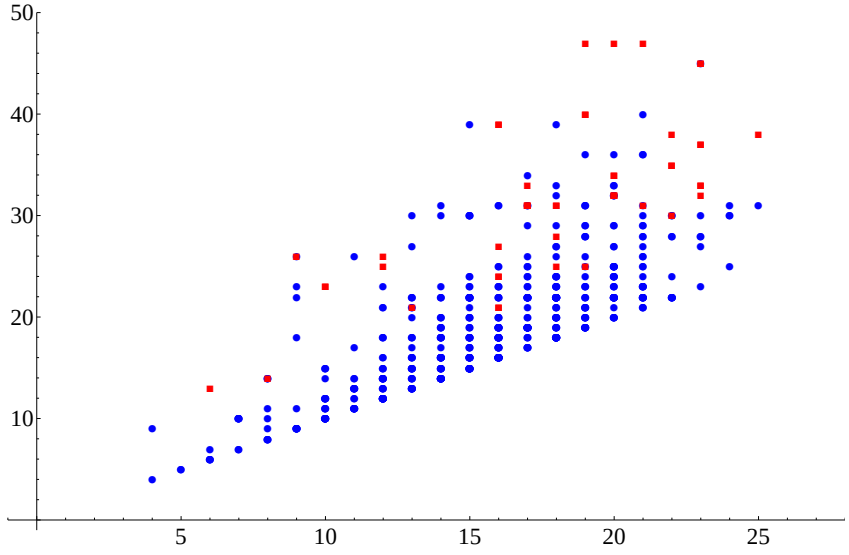


Figure : Vertical axis: no. of tetrahedra in the veering triangulation. Horizontal axis: no. of tetrahedra after the veering triangulation is simplified by SnapPy.

Conjugacy in mapping class group

- Let φ and ψ be pseudo-Anosov homeomorphisms of a surface S.

Conjugacy in mapping class group

- Let φ and ψ be pseudo-Anosov homeomorphisms of a surface S.
- Denote by $\mathrm{MCG}(S)$ the mapping class group of S.

Conjugacy in mapping class group

- Let φ and ψ be pseudo-Anosov homeomorphisms of a surface S.
- Denote by $\mathrm{MCG}(S)$ the mapping class group of S.

Question: are φ and ψ conjugate in $\mathrm{MCG}(S)$, that is, does there exist $h \in \mathrm{MCG}(S)$ such that

$$\varphi = h \circ \psi \circ h^{-1}?$$

Conjugacy in mapping class group

- Let φ and ψ be pseudo-Anosov homeomorphisms of a surface S.
- Denote by $\mathrm{MCG}(S)$ the mapping class group of S.

Question: are φ and ψ conjugate in $\mathrm{MCG}(S)$, that is, does there exist $h \in \mathrm{MCG}(S)$ such that

$$\varphi = h \circ \psi \circ h^{-1}?$$

- φ and ψ are conjugate if and only if they have 'combinatorially isomorphic' periodic splitting sequences.

Conjugacy in mapping class group

- Let φ and ψ be pseudo-Anosov homeomorphisms of a surface S.
- Denote by $\mathrm{MCG}(S)$ the mapping class group of S.

Question: are φ and ψ conjugate in $\mathrm{MCG}(S)$, that is, does there exist $h \in \mathrm{MCG}(S)$ such that

$$\varphi = h \circ \psi \circ h^{-1}?$$

- φ and ψ are conjugate if and only if they have 'combinatorially isomorphic' periodic splitting sequences.
- *Veering* implements this algorithmically.

Thanks!

Veering webpage: `http://www.ms.unimelb.edu.au/~veering/`