GLAUBERMAN'S ZJ THEOREM

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This is my attempt at a geodesic to Glauberman's ZJ-theorem, a technical tool in finite group theory. We only prove the basic version; see [1] for further exploitation of the underlying idea.

All groups considered are finite. For group elements w, x we write w^x for $x^{-1}wx$ and $[w, x]$ for $w^{-1}x^{-1}wx$. We nest commutator brackets to the left: $[w, x, y, z]$ means $[[[w, x], y], z]$. If some terms of such a commutator are groups rather than group elements, then we mean the group generated by all commutators of that form. For example, if X is a group then $[w, X, X]$ is generated by the $[w, x, x']$ with $x, x' \in X$.

Fix a prime p; the largest normal p-subgroup of a group G is written $O_p(G)$. And we write $\mathcal{A}(G)$ for the set of all abelian p-subgroups of G of largest possible order. The group they generate is called the Thompson subgroup $J(G)$ of G. Its center $Z(J(G))$ is often abbreviated to $ZJ(G)$.

Now suppose G acts on a p-group P. We define $O_p(G \curvearrowright P) \trianglelefteq G$ as the preimage of $O_p(G/C_G(P))$ under the natural map $G \to G/C_G(P)$. This notation is nonstandard but convenient; it can be pronounced "O_p of G's action on P". We say that $x \in G$ acts quadratically if $[P, x, x] = 1$. The action of G on P is called p-stable if every element of G that acts quadratically lies in $O_p(G \curvearrowright P)$. There is a simple "global" condition on G that guarantees this: that no subquotient is isomorphic to $\mathrm{SL}_2(\mathbb{F}_p)$. A proof can be extracted from that of [2, Lemma 6.3]. For example, this holds if p is odd and G has abelian or dihedral Sylow 2-subgroups.

Theorem 1 (ZJ-theorem)**.** *Suppose p is an odd prime and G is a group satisfying*

(a) $C_G(O_p(G)) \leq O_p(G)$ *and*

(b) *G acts p-stably on every normal p-subgroup of G.*

Then $ZJ(S)$ is characteristic in G, for any Sylow p-subgroup S of G.

Date: February 12, 2020.

2010 *Mathematics Subject Classification.* 20D25 (20D15, 20D20).

Supported by Simons Foundation Collaboration Grant 429818.

The original proof [2, Theorem A][3, Theorem 8.2.10] relies on Glauberman's replacement theorem [2, Theorem 4.1][3, Theorem 8.2.7]. We give simplified proofs of both theorems.

Lemma 2 (Properties of J)**.** *Suppose S is a p-group. Then*

(i) $ZJ(S) = \cap_{A\in\mathcal{A}(S)}A$.
(ii) *If $P \le S$ contains a member of $\mathcal{A}(S)$, then $ZJ(S) \le ZJ(P)$.*

Proof. (i) The inclusion $\cap_{A\in\mathcal{A}(S)}A \le ZJ(S)$ is obvious. For the reverse, suppose $x \in ZJ(S)$ and $A \in \mathcal{A}(S)$. Then x commutes with A, so $\langle A, x\rangle$ is abelian. The maximality of A among abelian groups forces $x \in A$.

(ii) The maximal order among abelian subgroups of P is the same as for S. So $\mathcal{A}(P) \subseteq \mathcal{A}(S)$. Then (i) gives

$$ZJ(S) = \cap_{A\in\mathcal{A}(S)}A \le \cap_{A\in\mathcal{A}(P)}A = ZJ(P). \qquad \square$$

The idea of using the following lemma to prove the Replacement Theorem is due to Kizmaz [4]. Our calculations are simpler than his.

Lemma 3. *Suppose p is odd and an abelian p-group A acts on a p-group B of class ≤ 2, centralizing $[B, B]$. Then $[b, A]$ is abelian, for any $b \in B$ satisfying $[b, A, A, A] = 1$.*

Proof. We must prove $[[b, x], [b, y]] = 1$ for all $x, y \in A$. We abbreviate nested commutators by using subscripts:

$$b_x = [b, x] \quad b_y = [b, y] \quad b_{xx} = [b, x, x] \quad b_{yy} = [b, y, y]$$
$$b_{xy} = [b, x, y] \quad b_{yx} = [b, y, x]$$

Consider the antisymmetric bilinear pairing $\lfloor\cdot,\cdot\rfloor : \bar{B} \times \bar{B} \to [B, B]$ defined by $\lfloor\bar{u}, \bar{v}\rfloor = [u, v]$, where $u, v \in B$ and bars indicate images in $\bar{B} = B/[B, B]$. We will use additive notation in $\bar{B}$ and $[B, B]$. We must prove $\lfloor\bar{b}_x, \bar{b}_y\rfloor = 0$. Note that $\bar{b}_{xy} = \bar{b}_{yx}$, because x and y commute and $\bar{B}$ is abelian.

Our strategy is to use the invariance of $\lfloor\cdot,\cdot\rfloor$ under the actions of x, y. Both x and y centralize $\bar{b}_{xx}$, $\bar{b}_{xy}$ and $\bar{b}_{yy}$. Conjugation by x sends $\bar{b}, \bar{b}_x, \bar{b}_y$ to $\bar{b} + \bar{b}_x$, $\bar{b}_x + \bar{b}_{xx}$ and $\bar{b}_y + \bar{b}_{yx}$ respectively, and similarly for y. Because x acts trivially on $[B, B]$ we have

$$\lfloor\bar{b}, \bar{b}_{xy}\rfloor = \lfloor\bar{b}^x, \bar{b}^x_{xy}\rfloor = \lfloor\bar{b} + \bar{b}_x, \bar{b}_{xy}\rfloor = \lfloor\bar{b}, \bar{b}_{xy}\rfloor + \lfloor\bar{b}_x, \bar{b}_{xy}\rfloor$$

Canceling proves $\lfloor\bar{b}_x, \bar{b}_{xy}\rfloor = 0$. The same invariance argument proves

$$\begin{aligned}\lfloor\bar{b}, \bar{b}_y\rfloor &= \lfloor\bar{b}^x, \bar{b}^x_y\rfloor = \lfloor\bar{b} + \bar{b}_x, \bar{b}_y + \bar{b}_{yx}\rfloor \\ &= \lfloor\bar{b}, \bar{b}_y\rfloor + \lfloor\bar{b}, \bar{b}_{yx}\rfloor + \lfloor\bar{b}_x, \bar{b}_y\rfloor + \lfloor\bar{b}_x, \bar{b}_{yx}\rfloor\end{aligned}$$

We already saw that the last term on the right vanishes. The first term cancels with the left side, leaving $\lfloor\bar{b}, \bar{b}_{yx}\rfloor + \lfloor\bar{b}_x, \bar{b}_y\rfloor = 0$. Symmetry in

x and y gives $\lfloor \bar{b}, \bar{b}_{xy} \rfloor + \lfloor \bar{b}_y, \bar{b}_x \rfloor = 0$. Because $\bar{b}_{xy} = \bar{b}_{yx}$ this shows $\lfloor \bar{b}_x, \bar{b}_y \rfloor = \lfloor \bar{b}_y, \bar{b}_x \rfloor$. The antisymmetry of $\lfloor \cdot, \cdot \rfloor$ gives $2\lfloor \bar{b}_x, \bar{b}_y \rfloor = 0$. This implies $\lfloor \bar{b}_x, \bar{b}_y \rfloor = 0$ because p is odd. □

Theorem 4 (Glauberman Replacement)**.** *Suppose p is odd, S is a p-group, $B \trianglelefteq S$ has class ≤ 2, and $[B, B] \leq ZJ(S)$. Then either every member of $\mathcal{A}(S)$ contains B, or B normalizes one that does not.*

Proof. It suffices to prove: if there exists $A \in \mathcal{A}(S)$ that does not contain B, and for which $N_B(A) < B$, then there exists $A^* \in \mathcal{A}(S)$, also not containing B, with $N_B(A^*)$ strictly larger than $N_B(A)$.

Set $N := AN_B(A)$. This is not all of AB, since B does not normalize A. So there exists $b \in AB$ such that $\langle N, b \rangle$ contains N of index p. Because N already surjects to AB/B, b may be chosen to lie in B. By construction we have $N \trianglelefteq \langle N, b \rangle$. From $[B, B] \leq ZJ(S) \leq A$ we see $A \cap B \trianglelefteq B$, hence $A \cap B \leq A \cap A^b$.

From $A \trianglelefteq N \trianglelefteq \langle N, b \rangle$ we get $A^b \trianglelefteq N$. So $H := AA^b \trianglelefteq N$. Furthermore, A and A^b normalize each other, so $[H, H] \leq (A \cap A^b) \leq Z(H)$. In particular, H has class ≤ 2. The identity $(aa'^{-1})(a')^b = a[a', b]$, for any $a, a' \in A$, shows that H is also equal to $A[A, b]$.

Using bars for images in $H/(A \cap A^b)$, obviously we have $\bar{A} \cdot \overline{[A, b]} = \bar{H}$. On the other hand, $[A, b]$ lies in $H \cap B$, and $\overline{H \cap B}$ meets $\bar{A}$ trivially because $A \cap B \leq A \cap A^b$. That is: $\overline{[A, b]}$ meets every coset of $\bar{A}$ in $\bar{H}$, yet lies in $\overline{H \cap B}$, which contains at most one point of each coset. It follows that $\overline{[A, b]}$ and $\overline{H \cap B}$ coincide and form a complement to $\bar{A}$. Therefore

$$A^* := (A \cap A^b)[A, b] = (A \cap A^b)(H \cap B)$$

is a complement to A in H, modulo $A \cap A^b$. Since A^b is another such complement, we have $A^*/(A \cap A^b) \cong A^b/(A \cap A^b)$ and therefore $|A^*| = |A|$.

To prove $A^* \in \mathcal{A}(S)$ it remains to prove A^* abelian. Because $A \cap A^b$ is central in H, it is enough to prove $[A, b]$ abelian. This follows from lemma 3, which applies because $[B, B] \leq A$ and

$$[\, b, A, A, A] \leq [H, A, A] \leq [Z(H), A] = 1.$$

The following observations finish the proof. First, A^* lies in N, hence does not contain B. Second, N normalizes A^* because it normalizes A, A^b, H and B. Third, b normalizes A^* because $[A \cap A^b, b] \leq [A, b]$ and $[[A, b], b] \leq [B, B] \leq A \cap A^b$. So $N_B(A) < N_B(A^*)$. □

Proof of the ZJ theorem. The Frattini argument shows that $\mathrm{Aut}(G)$ is generated by inner automorphisms and automorphisms that preserve S.

The latter obviously preserve $ZJ(S)$, so it suffices to prove $ZJ(S) \trianglelefteq G$. Using induction starting with $1 \trianglelefteq G$, it is enough to prove:

$$\text{if } \exists\, W \trianglelefteq G \text{ with } W < ZJ(S), \text{ then } \exists\, B \trianglelefteq G \text{ with } W < B \leq ZJ(S).$$

Given such W, we define X as the subgroup of $ZJ(S)$ corresponding to the fixed-point subgroup of S in $ZJ(S)/W$. So X is normal in S, and is strictly larger than W because $ZJ(S)$ is. To complete the proof we will show that $B := \langle X^G \rangle \trianglelefteq G$ lies in $ZJ(S)$. Supposing it does not, we will derive a contradiction.

Step 1: $B \leq O_p(G)$. Being a subgroup of $ZJ(S)$, X is abelian. Together with $X \trianglelefteq S$ this gives $[O_p(G), X, X] \leq [X, X] = 1$. Because G acts p-stably on $O_p(G)$, X lies in $O_p(G \curvearrowright O_p(G))$. This equals $O_p(G)$ because $C_G(O_p(G))$ is a p-group by hypothesis. Since X lies in $O_p(G)$, its G-conjugates do too.

Step 2: $[B, B] \leq W \leq Z(B)$. By the definition of X, S acts trivially on X/W. In particular $O_p(G)$ does. Conjugation shows that $O_p(G)$ acts trivially (mod W) on every G-conjugate of X, hence trivially on B/W. That is, $[B, O_p(G)] \leq W$. And $W \leq Z(B)$ because B is generated by abelian groups that contain W.

Step 3: Set $H = O_p(G \curvearrowright B)$ and $P = H \cap S$. Then $ZJ(S) \leq ZJ(P)$. The previous step showed that B has class ≤ 2 and that $[B, B] \leq ZJ(S)$. So we may apply the replacement theorem. Since we are supposing $B \not\leq ZJ(S)$, its second alternative must hold: B normalizes but does not lie in some $A \in \mathcal{A}(S)$. Then $[B, A, A] \leq [A, A] = 1$, and the p-stability of G's action on B implies $A \leq H$. Together with $A \leq S$ this gives $A \leq P$, ie $A \in \mathcal{A}(P)$. Then lemma 2(ii) shows $ZJ(S) \leq ZJ(P)$.

Step 4: $B \leq ZJ(P)$. The Frattini argument and the definition of H show

$$G = HN_G(P) = C_G(B)PN_G(P) \leq C_G(X)N_G(P).$$

So the G-conjugates of X are the same as the $N_G(P)$-conjugates. It therefore suffices to show that every $N_G(P)$-conjugate of X lies in $ZJ(P)$. This is immediate from $X \leq ZJ(S) \leq ZJ(P) \trianglelefteq N_G(P)$.

Contradiction: By $B \leq ZJ(P)$, every member of $\mathcal{A}(P)$ contains B. But in step 3 we found one which does not. □

Our method of "growing" the normal subgroup from W to B derives from Stellmacher's construction [5][6] of a group similar to ZJ.

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