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Research, Jan 2011–Dec 2015

26 January 2016

Kac–Moody theory

Like semisimple Lie algebras, Kac–Moody algebras have root systems, Weyl groups, and so on. The main difference is that Weyl groups can be infinite, for example Euclidean or hyperbolic reflection groups. The Euclidean case is well-understood. In the hyperbolic setting, I classified the 997 “interesting” rank 3 root systems [7]. Here “interesting” means that they have a chance to possess the automorphic properties of Borcherds’ monster and fake monster Lie algebras, as codified by Nikulin [19]. This work is related to my slightly earlier classification [2] of all integer quadratic forms of signature $(2, 1)$ whose isometry groups are generated by reflections, up to finite index.

I have also studied Kac–Moody groups over commutative rings. The foundational work is due to Tits [20], who generalized work of Chevalley and Kostant. He constructed these groups functorially in the ground ring, by means of a presentation that is huge, complicated and only implicitly-described. In fact I showed in [10] that practically speaking, it is impossible to even write down Tits’ presentation (for the E_{10} root system).

Happily, I was also able to simplify his presentation greatly, so that only the SL_2’s associated to the simple roots appear. One striking consequence is that the Steinberg group of any root system is the direct limit of the Steinberg groups associated to the pairs of simple roots. Another is that the Steinberg group and (often) the Kac-Moody group are finitely presented if the ground ring is finitely generated as a ring. (For brevity we have omitted some minor hypotheses that hold in most situations.)

For simply-laced hyperbolic root systems this was joint work with Lisa Carbone in [15]. My treatment of the wrinkles specific to affine systems will appear in [8]. The most general results are in [9]. All three papers rely on my refinement in [5] of Brink’s theorem [18] about centralizers of reflections in Coxeter groups.

Branched covers; Fake planes; Braid-like groups

◦ I showed that a certain condition on an incomplete metric space is strong enough to imply that its metric completion has negative (or nonpositive) curvature. For branched covers of Riemannian manifolds, one can deduce

such curvature bounds by checking an infinitesimal condition. This has applications to Artin groups, certain moduli spaces in algebraic geometry, and singularity theory. This work appeared in [4].

◦ Tathagata Basak and I investigated "braid-like groups": groups got by removing a hyperplane arrangement from (say) complex hyperbolic or Euclidean space, quotienting by a discrete group, and then taking the fundamental group. This construction generalizes a well-known description of the braid groups, and occurs naturally for many algebra-geometric moduli spaces. In [13] we gave a general method to find "geometric" generators for such groups. In an second paper, nearly complete, we use this to find such generators for the most complicated example known, coming from a complex reflection group acting on complex hyperbolic 13-space [14]. The particular interest in this example is its possible connection to the monster simple group, via an old conjecture of mine [1].

◦ Fumiharu Kato and I wrote two papers about 2-adic uniformization and fake projective planes. First we proved in [16] that exactly two discrete subgroups of $\mathrm{PGL}_3(\mathbb{Q}_2)$ have smallest possible covolume. Here $\mathbb{Q}_2$ is the field of 2-adic rational numbers. Mumford had used one of these groups in his famous construction of a fake projective plane. We used the other one to construct a different fake plane; this will appear in [17]. The advance here was technical: even though Mumford's construction requires a torsion-free subgroup, we made it work in the presence of torsion.

Brief mentions

◦ A simpler classification of Wolf's spherical space forms, and what appears to be their first listing without redundancy [11].

◦ The Conway–Sloane calculus for 2-adic quadratic forms is valid [12]. (Their calculus greatly streamlines the clumsy classifications discovered in the 1940's, but no one had published a proof.)

◦ In a construction of Higman (an amalgamation of three Baumslag-Solitar groups), many of the resulting groups collapse [6].

◦ In a Coxeter group W, the normalizer of a finite Coxeter subgroup W_0 falls into two pieces: a Coxeter group and the "nonreflection part" of the normalizer, which is the complicated part. I improved a bound of Borcherds on the cohomological dimension of the nonreflection part [3].

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