MODULI SPACES OF TROPICAL CURVES ARE SIMPLY CONNECTED

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Abstract. We develop basic techniques for studying fundamental groups and singular homology of generalized Δ-complexes. As an application, we show that the moduli spaces of tropical curves Δ_g and $\Delta_{g,n}$ are simply connected, for $g \geq 1$. We also show that Δ_3 is homotopy equivalent to the 5-sphere, and that Δ_4 has 3-torsion in H_5.

1. Introduction

Moduli spaces of tropical curves appear naturally in algebraic geometry and topology, e.g., as the quotient of the simplicial completion of Outer Space by the action of outer automorphisms of free groups, as the quotient of the complex of curves by the action of the mapping class group, and as the dual complex of the boundary divisor in the Deligne-Mumford compactification of moduli spaces of stable curves. Their rational homology agrees with Kontsevich's graph homology, and with the top weight cohomology of moduli spaces of algebraic curves. Applications of these perspectives on the rational homology of moduli spaces of tropical curves include the proof that $\dim_{\mathbb{Q}} H^{4g-6}(M_g;\mathbb{Q})$ grows exponentially with g, and the verification of a formula conjectured by Zagier for the $\mathfrak{S}_n$-equivariant top weight Euler characteristic of $M_{g,n}$ [?, ?].

For nonnegative integers g and n, with $3g-3+n>0$, let $\Delta_{g,n}$ be the moduli space parametrizing stable tropical curves of genus g with n marked points and total volume 1. The homotopy type of $\Delta_{g,n}$ is an invariant of the algebraic moduli stack $\mathcal{M}_{g,n}$ [?]. The present paper initiates a study of its integral homology and homotopy groups. These are relatively simple when g is small. Indeed, $\Delta_{0,n}$ is homotopy equivalent to a wedge sum of spheres of dimension $n-4$, for $n \geq 4$. Moreover, $\Delta_{1,n}$ is contractible for $n = 1, 2$ and homotopy equivalent to a wedge sum of spheres of dimension $n-1$, for $n \geq 3$ [?], and the integral homology of $\Delta_{2,n}$ is torsion free [?]. We show that the complexity of the fundamental group does not increase with g.

Theorem 1.1. *The moduli space of tropical curves $\Delta_{g,n}$ is simply connected, for $g \geq 1$.*

We write Δ_g for $\Delta_{g,0}$. It is not difficult to see that Δ_2 is contractible and that Δ_3 has the rational homology of S^5. Here, we determine the homotopy type of Δ_3.

Theorem 1.2. *The moduli space Δ_3 is homotopy equivalent to S^5.*

By [?, Appendix A], Δ_4 has the rational homology of a point. However, we show that it is not contractible.

Theorem 1.3. *The reduced integral homology $\widetilde{H}_k(\Delta_4;\mathbb{Z})$ contains nontrivial 3-torsion for $k=5$ and nontrivial 2-torsion for $k=6,7$. It vanishes for $k \neq 5,6,7$.*

Theorem 1.3 gives the first example of torsion in the integral homology of moduli spaces of tropical curves. Also, while the rational homology of Δ_g is supported in the top $g-1$ degrees [**?**, Theorems 1.3 and 1.4], Theorem 1.3 shows that $H_*(\Delta_g;\mathbb{Z})$ can have torsion in lower degrees.

Our approach to these theorems starts from the construction of $\Delta_{g,n}$ as a *symmetric* Δ*-complex*, i.e., the geometric realization of a symmetric semi-simplicial set. Roughly speaking, just as a Δ-complex is obtained by gluing simplices along unions of faces, a symmetric Δ-complex is obtained by gluing quotients of simplices by subgroups of symmetric groups along unions of quotients of faces. See [**?**, Section 4] and Section 2 for a precise definition.

In Section 2 we examine the filtration of a symmetric Δ-complex by its p-skeleta, the unions of cells of dimension at most p. In Section 3 we show that the fundamental group is generated by loops in the 1-skeleton. However, for $k > 1$, we give examples showing that π_k (resp. H_k) is not generated by spheres (resp. cycles) in the k-skeleton. Moreover, even for $k = 1$, the relations among spheres (resp. cycles) in the k-skeleton are not necessarily generated by boundaries of balls (resp. chains) in the $(k+1)$-skeleton. In particular, the relations in π_1 among the classes of loops in the 1-skeleton are not always generated by boundaries of discs in the 2-skeleton.

In Section 4 we describe the spectral sequence associated to the filtration by p-skeleta. Taken with rational coefficients, this spectral sequence recovers the rational cellular homology theory developed in [**?**]. With integer coefficients, it provides new information on torsion in homology of symmetric Δ-complexes. We refine this approach by considering an analogous spectral sequence relative to a subcomplex, which is essential for the proof of Theorem 1.3. The idea of computing homology of moduli spaces of tropical curves relative to large contractible subcomplexes is not new; contractibility of $\Delta_{g,n}^{br}$, the closure of the locus of tropical curves with bridges, is essential to the main results in [**?**]. Here we use an even larger contractible subcomplex, given by Theorem 6.1.

Our applications involve moduli spaces of tropical curves, which are symmetric Δ-complexes. Nevertheless, the techniques we develop apply to more general filtered spaces, in which a p-skeleton is obtained from a $(p-1)$-skeleton as the mapping cone for a continuous map from a finite disjoint union of quotients of $(p-1)$-spheres by finite subgroups of the orthogonal group. We call these *generalized CW-complexes*, and Theorems 3.1, 4.2, and 4.3 are proved for this more general class of spaces. Generalized CW-complexes that are not symmetric Δ-complexes appear naturally in algebraic geometry as dual complexes of boundary divisors in non-simplicial toroidal compactifications, such as the Voronoi compactifications of moduli spaces of abelian varieties.

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2. Structure of symmetric Δ-complexes

Let I be the category with objects $[p] = \{0, \ldots, p\}$, for $p \geq 0$, with morphisms given by injective maps. Recall that a symmetric Δ-complex is a functor $X\colon I^{\mathrm{op}} \to \mathsf{Sets}$. Such a functor is determined by a set $X_p = X([p])$ for each $p \geq 0$, actions of the symmetric group $\mathfrak{S}_{p+1}$ on X_p for all p, and face maps $d_i\colon X_p \to X_{p-1}$ for $p \geq 1$, obtained by applying the functor X to the unique order-preserving injective map $[p-1] \to [p]$ whose image does not contain i. The face maps satisfy the usual simplicial identities as well as a compatibility with the symmetric group action.

An injection $\theta\colon [p] \to [q]$ determines an inclusion of standard simplices $\theta_*\colon \Delta^p \to \Delta^q$, whose image is the p-face with vertices corresponding to the image of θ. The *geometric realization* of a symmetric Δ-complex X is

$$|X| = \Big(\coprod_{p=0}^{\infty} X_p \times \Delta^p\Big)/\sim, \tag{2.0.1}$$

where $\sim$ is the equivalence relation generated by $(x, \theta_* a) \sim (\theta^* x, a)$. Each $x \in X_p$ determines a map of topological spaces $x\colon \Delta^p \to |X|$, which factors through the quotient of Δ^p by the stabilizer $H_x < \mathfrak{S}_{p+1}$, and X may be recovered from the topological space $|X|$ together with this set of maps from simplices.

Let $\underline{X}_p \subset X_p$ be a subset consisting of one representative of each $\mathfrak{S}_{p+1}$-orbit. Then $|X|$ is partitioned into *cells*

$$|X| = \coprod_p \coprod_{x \in \underline{X}_p} (\Delta^p)^\circ / H_x,$$

each isomorphic to the quotient of an open simplex by a linear finite group action. Note that the closure of each p-cell meets only finitely many cells, each of dimension less than p. The properties of this stratification are hence closely analogous to the properties of a CW-complex, except that the cells are quotients of open balls by finite groups, rather than ordinary open balls. We capture this analogy with the following definition.

Definition 2.1. *A* generalized CW-complex[1] *is a Hausdorff topological space X together with a partition into locally closed* cells*, such that, for each cell $C \subset X$, there is a continuous map from the quotient of the closed unit ball in some $\mathbb{R}^p$ by a finite subgroup of the orthogonal group such that*

(1) The quotient of the open unit ball maps homeomorphically onto C, and
(2) The image meets only finitely many cells, each of dimension less than p.

We require furthermore that a subset of X is closed if and only if its intersection with closure of each cell is closed.

We say that a generalized CW-complex is *finite* if it has only finitely many cells. All of the applications we consider involve only finite generalized CW-complexes.

Definition 2.2. *The p-skeleton of a generalized CW-complex, denoted $X^{(p)} \subset X$ is the union of its cells of dimension at most p.*

[1]The terminology mirrors the *generalized cone complexes* in [?], which are built out of quotients of rational polyhedral cones by finite groups. See Example 2.4.

Suppose X has p-cells $\{C_i : i \in I\}$. Let $G_i \leq O(p)$ be a subgroup of the orthogonal group for which there is a continuous map $B^p/G_i \to X$ taking $(B^p)^\circ/G_i$ homeomorphically onto C_i. Note that B^p/G_i is cone shaped around the image of the origin, and that $X^{(p)}$ is naturally identified with the mapping cone of the attaching map

$$\coprod_{i \in I} \partial B^p/G_i \to X^{(p-1)}.$$

The total space X is the direct limit of its skeleta $X^{(p)}$.

Example 2.3. Let X be a symmetric Δ-complex. Then $|X|$, together with its partition into cells, is a generalized CW-complex.

Example 2.4. Let Δ be a generalized cone complex, as defined in [**?**], obtained as the colimit of a diagram of rational polyhedral cones with face maps. Then the geometric realization $|\Delta|$ is partitioned into cells by the images of the relative interiors of the cones in the diagram. The link of the vertex, with its induced partition into cells, is a generalized CW-complex.

Remark 2.5. Example 2.4 shows that generalized CW-complexes appear naturally in algebraic geometry, as dual complexes of boundary divisors in toroidal embeddings. In cases where the toroidal embedding is non-simplicial, as is the case in such well-studied examples as the perfect cone and second Voronoi toroidal compactifications of moduli spaces of abelian varieties, the resulting generalized CW-complex is not a symmetric Δ-complex. While the applications to moduli spaces of tropical curves stated in the introduction require only the special case of symmetric Δ-complexes, we expect that generalized CW-complexes will be useful for future applications, e.g., for studying the top weight cohomology of the moduli space of abelian varieties. Since the proofs of our basic technical results (Theorems 3.1, 4.2, and 4.3) work equally well for generalized CW-complexes, we present them in this greater level of generality.

3. The fundamental group is generated by the 1-skeleton

The key technical step in the proof of Theorem 1.1 is the following cellular approximation theorem in dimension 1, showing that the fundamental group of $\Delta_{g,n}$ is generated by loops in its 1-skeleton.

Theorem 3.1. *Let X be a finite generalized CW-complex with $x \in |X^{(1)}|$. Then the natural map $\pi_1(|X^{(1)}|, x) \to \pi_1(|X|, x)$ is surjective.*

Proof. It suffices to show that, if X is obtained from a subcomplex X' by attaching a single cell of dimension $n \geq 2$, then the induced map $\pi_1(|X'|, x) \to \pi_1(|X|, x)$ is surjective. We may assume X and X' are connected. In this case, $|X|$ is the identification space obtained from $|X'|$ and B^n/G by identifying the points of S^{n-1}/G with their images in $|X'|$ under the attaching map, where $G \leq O(n)$ is finite. We write 0 for the origin in B^n. We express $|X|$ as the union of the open sets $|X| - \{0\}$ and $(B^n - S^{n-1})/G$. Their intersection is homeomorphic to $\mathbb{R} \times (S^{n-1}/G)$ and in particular is connected. By van Kampen's theorem, $\pi_1(|X|)$ is generated by the images under inclusion of $\pi_1(|X| - \{0\})$ and $\pi_1((B^n - S^{n-1})/G)$.

The first of these is the same as the image under inclusion of $\pi_1(|X'|)$. This is because the radial deformation retraction of $B^n - \{0\}$ onto S^{n-1} is G-equivariant and therefore induces a deformation retraction from $(B^n - \{0\})/G$ into S^{n-1}/G. Since $\pi_1((B^n - S^{n-1})/G)$ is trivial, $\pi_1(|X|)$ is generated by the image under the inclusion of $\pi_1(|X'|)$, as required. □

It follows from Theorem 3.1 that H_1 is also generated by loops in the 1-skeleton of a finite-dimensional generalized CW-complex. This surjectivity on H_1 and π_1 looks like a weak result, but it is the only general result of this nature that one can expect. Suppose $k > 0$. We will exhibit a symmetric Δ-complex whose dimension n is larger than $k+1$, such that the inclusion of $|X^{(n-1)}|$ into $|X|$ induces non-injective maps on H_k and π_k and non-surjective maps on H_{k+1} and π_{k+1}. We begin with generalized CW-complex examples because they make the essentials more visible. See Remark 6.2 for an example in nature of a symmetric Δ-complex X, such that kth homology and homotopy groups of the k-skeleton does not surject to that of X.

Example 3.2 (Attaching a generalized n-cell can kill π_1 and H_k for odd $k < n-1$)**.** Suppose $n > 2$ and let $G \leq O(n)$ be generated by the negation map, and consider B^n/G. The CW complex structure on $|X'| = S^{n-1}/G \cong \mathbb{R}P^{n-1}$ is not important. We regard B^n/G as a single generalized n-cell, attached to $|X'|$ in the obvious way. The result is contractible. Therefore, attaching this generalized n-cell kills not just $H_{n-1}(\mathbb{R}P^{n-1};\mathbb{Z})$ but also $\pi_1(\mathbb{R}P^{n-1}) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_i(\mathbb{R}P^{n-1};\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ where $i < n-1$ is positive and odd. So attaching a generalized n-cell can kill π_1 and $H_{\text{odd}<n-1}$.

Example 3.3 (Attaching a generalized n-cell can kill π_k and H_k for $2 \leq k < n-1$)**.** As a variation on the previous example, take $2 \leq k \leq n-2$ and let $g \in O(n)$ act on $\mathbb{R}^n$ by fixing pointwise a $(k-1)$-dimensional subspace and acting by negation on its orthogonal complement. Writing $G \cong \mathbb{Z}/2\mathbb{Z}$ for the group generated by g, again B^n/G is contractible. But this time $|X'| = S^{n-1}/G$ is the join of $\mathbb{R}P^{n-k}$ and S^{k-2}. Taking the join of a space with S^{k-2} is the same as (unreducedly) suspending it $k-1$ times. So the reduced homology of $|X'|$ is obtained by shifting the reduced homology of $\mathbb{R}P^{n-k}$ up $k-1$ degrees. In particular, the first nonzero homology group of $|X'|$ in positive degree is $H_k(|X'|) \cong H_1(\mathbb{R}P^{n-k}) \cong \mathbb{Z}/2\mathbb{Z}$. Since we chose $k \geq 2$, we always suspend at least once, so $|X'|$ is simply connected. Because of this, the Hurewicz theorem shows that the first nontrivial homotopy group of $|X'|$ is $\pi_k(|X'|) \cong H_k(|X'|) \cong \mathbb{Z}/2\mathbb{Z}$. So the maps on H_k and π_k induced by $|X'| \to |X|$ are not injective, even though X is obtained from X' by attaching a single generalized n-cell.

The next few examples are quotients of a standard simplex. Let $G \leq \mathfrak{S}_{n+1}$. Then the quotient $X = \Delta^n/G$ naturally inherits the structure of a symmetric Δ-complex, with X_q being the set of G-orbits of q-faces of Δ^n. Then $|X^{(n-1)}|$ is the quotient of the boundary of the simplex by G. The geometric realization $|X|$ is contractible because it is the cone on $|X^{(n-1)}|$.

Example 3.4 (Example 3.3 for symmetric Δ-complexes)**.** The phenomena of Example 3.3 also occur for symmetric Δ-complexes. Suppose $k \geq 3$ is given, choose ℓ satisfying $3 \leq \ell \leq k$, and set $n = k + \ell - 1$. Consider the subgroup G of $O(n)$ generated by the involution g that fixes all but the first 2ℓ vertices of Δ^n, which it permutes by $(01)(23)\cdots(2\ell{-}2\ 2\ell{-}1)$.

This makes sense since $2\ell \leq n+1$. We make the affine span of $\Delta^n \subseteq \mathbb{R}^{n+1}$ into a vector space V by taking its barycenter as the origin. It is easy to see that g acts on V by negating an ℓ-dimensional subspace and pointwise fixing its orthogonal complement, which has dimension $k-1$. We can identify the unit sphere in V with the boundary of Δ^n by radial projection. This extends to a G-equivariant identification of Δ^n with the unit ball in V. Therefore every feature of the previous example carries over to the symmetric Δ-complex Δ^n/G. In particular, the maps on π_k and H_k induced by $|(\Delta^n/G)^{(n-1)}| \to |\Delta^n/G|$ are not injective. We needed the restriction $\ell \geq 3$ to get $n > k+1$, and $\ell \leq k$ so that the permutation of the vertices of Δ^n makes sense. Hence these examples only occur for $k \geq 3$. For $k = 1, 2$ we refer to the following examples.

Example 3.5 (Attaching a symmetric 3-simplex can kill π_1 and H_1)**.** We construct a 3-dimensional contractible symmetric Δ-complex whose 2-skeleton is not simply connected. Take $G \cong \mathbb{Z}/4\mathbb{Z}$ to be generated by an element of $O(3)$ that permutes the vertices of Δ^3 cyclically. The quotient of the boundary can be worked out by taking one facet as a fundamental domain for G and working out the edge pairings. It turns out to be $\mathbb{R}P^2$. Taking $X = \Delta^3/G$, it follows that $|X|$ is simply connected, even though its 2-skeleton is not: $\pi_1(|X^{(2)}|) \cong \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$.

Example 3.6 (Attaching a symmetric 4-simplex can kill π_2 and H_2)**.** We construct a 4-dimensional contractible symmetric Δ-complex whose 3-skeleton is simply connected and has nontrivial H_2 (hence π_2). Take $X = \Delta^4/G$ where $G \cong \mathbb{Z}/4\mathbb{Z}$ fixes one vertex and permutes the others cyclically. This example contains the previous one; the quotient of the boundary of Δ^4 is the unreduced suspension of $\mathbb{R}P^2$. The suspension points are the fixed points of G, namely the fixed vertex and the barycenter of the opposite facet.

Example 3.7 (Non-surjectivity of π_k and H_k)**.** Take X as in any of the previous examples, and define Z as the generalized CW complex or symmetric Δ-complex obtained by identifying the two copies of X along their $(n-1)$-skeleta. So $Z^{(n-1)} = X'$ in the notation of the previous examples. Let $k > 1$ be the smallest degree for which $H_k(|X'|) \neq 0$. Since Z is the (unreduced) suspension of X', we have $H_{k+1}(|Z|) \neq 0$. The Hurewicz theorem shows that $\pi_{k+1}(|Z|)$ is also nonzero. On the other hand, the natural maps $H_{k+1}(|X'|) \to H_{k+1}(|Z|)$ and $\pi_{k+1}(|X'|) \to \pi_{k+1}(|Z|)$ are the zero maps because $|X'| \to |Z|$ factors through the contractible space $|X|$. So the induced maps on H_{k+1} and π_{k+1} are not surjective.

4. Computing homology using the filtration by skeleta

Let A be an abelian group. Any finite filtration of a topological space Y by subspaces

$$\emptyset = Y^{-1} \subset Y^0 \subset Y^1 \subset \cdots \subset Y^n = Y$$

induces a filtration on the singular chain complex with coefficients in A,

$$0 \subset C(Y^0; A) \subset C(Y^1; A) \subset \cdots \subset C(Y^n; A) = C(Y; A).$$

This filtration on $C(Y)$ gives rise to a spectral sequence with

$$E_0^{p,q} = C_{p+q}(Y^p, Y^{p-1}; A) = C_{p+q}(Y^p; A)/C_{p+q}(Y^{p-1}; A),$$

and

$$E_1^{p,q} = H_{p+q}(Y^p, Y^{p-1}; A),$$

that converges to

$$E_\infty^{p,q} = \frac{\operatorname{im}(H_{p+q}(Y^p; A) \to H_{p+q}(Y; A))}{\operatorname{im}(H_{p+q}(Y^{p-1}; A) \to H_{p+q}(Y; A))}.$$

Remark 4.1. The higher differentials in this spectral sequence may be understood as follows. An element of $E^1_{p,q}$ is represented by a $(p+q)$-chain σ in Y^p whose boundary $\partial\sigma$ is a $(p+q-1)$-cycle in Y^{p-1}. If $\partial\sigma$ is contained in Y^{p-r} but not in Y^{p-r-1}, then $[\sigma]$ survives to E_r, and $d_r \colon E_r^{p,q} \to E_r^{p-r,q+r-1}$ maps $[\sigma]$ to $[\partial\sigma]$ in the surviving subquotient of $H_{p+q-1}(Y^{p-r}, Y^{p-r-1}; A)$.

This spectral sequence will be our main tool for understanding the homology of generalized CW-complexes. When applied to the filtration by skeleta, it gives the following.

Theorem 4.2. *Let X be a finite-dimensional generalized CW-complex whose p-cells are $\{C_i \cong (B^p)^\circ/G_i : i \in I^p\}$, for some finite subgroups $G_i \subset O(p)$. Then there is a spectral sequence with*

$$E_1^{0,q} = H_q(|X^{(0)}|; A), \quad \text{and} \quad E_1^{p,q} = \bigoplus_{i \in I^p} \widetilde{H}_{p+q-1}(S^{p-1}/G_i; A),$$

for $p \geq 1$, that converges to

$$E_\infty^{p,q} = \frac{\operatorname{im}(H_{p+q}(|X^{(p)}|; A) \to H_{p+q}(|X|; A))}{\operatorname{im}(H_{p+q}(|X^{(p-1)}|; A) \to H_{p+q}(|X|; A))}.$$

Proof. Consider the spectral sequence associated to the filtration of X by its skeleta $X^{(p)}$, which has $E_1^{p,q} = H_{p+q}(|X^{(p)}|, |X^{(p-1)}|; A)$. For $p > 0$, each connected component of $|X^{(p)}|/|X^{(p-1)|}$ is homeomorphic to the wedge sum of unreduced suspensions $S(S^{(p-1)}/G_i)$ associated to p-cells $C_i = B^p/G_i$ in that component, and the theorem follows. □

If we take $A = \mathbb{Q}$, then $E^1_{p,q}$ vanishes for $q \neq 0$, so we get a single chain complex, the $q = 0$ row of E^1, that computes the rational homology of $|X|$. When X is a symmetric Δ-complex, this is precisely the rational cellular chain complex presented in [**?**].

For our applications to $\Delta_{g,n}$, which typically has many cells and contains a large contractible subcomplex, we use the following variant of Theorem 4.2.

Theorem 4.3. *Let X be a finite-dimensional generalized CW-complex, let $Z \subset X$ be a subcomplex, and let $\{C_i \cong (B^p)^\circ/G_i : i \in I^p\}$, be the cells of X that are not in Z. Then there is a spectral sequence with*

$$E_1^{0,q} = H_q(|Z| \cup X^{(0)}; A), \quad \text{and} \quad E_1^{p,q} = \bigoplus_{i \in I^p} \widetilde{H}_{p+q-1}(S^{p-1}/G_i; A),$$

for $p \geq 1$, that converges to

$$E_\infty^{p,q} = \frac{\operatorname{im}(H_{p+q}(|X^{(p)}|; A) \to H_{p+q}(|X|; A))}{\operatorname{im}(H_{p+q}(|X^{(p-1)}|; A) \to H_{p+q}(|X|; A))}.$$

Proof. The argument is identical to the proof of Theorem 4.2, using the filtration of X by the subcomplexes $W^{(p)} = Z \cup X^{(p)}$. □

5. Quotients with reflections

As explained above, a generalized CW-complex is filtered by its p-skeleta, and the associated graded of the singular homology is the abutment of a spectral sequence whose E_1-page is a direct sum of homology groups of quotients of spheres by finite subgroups of orthogonal groups. The topology of such quotients of spheres is understood in only a few special cases. Swartz considered the case where the subgroup of the orthogonal group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^k$ [?], and Lange classified the cases where the quotient is a PL-sphere [?] or a topological sphere [?]. See also [?] for applications of Lange's results to quotients of S^{n-1} by subgroups of the permutation group $\mathfrak{S}_{n+1}$.

Proposition 5.1. *Let $H < O(n)$ be a finite subgroup that contains a reflection. Then S^{n-1}/H is contractible.*

Proof. Let $K < H$ be the subgroup generated by reflections. Each reflection in K fixes a hyperplane pointwise, and we let C be a chamber, i.e., the closure of a component of the complement of the union of these hyperplanes. Every point in $\mathbb{R}^n$ is K-equivalent to exactly one point in C [?, V.3.3], and hence $C \to \mathbb{R}^n/K$ is a homeomorphism. Furthermore, since K is normal in H, we have homeomorphisms

$$S^{n-1}/H \cong (S^{n-1}/K)/(H/K) \cong (S^{n-1} \cap C)/H_C,$$

where $H_C < H$ is the stabilizer of C. We now show that $(S^{n-1} \cap C)/H_C$ is contractible.

There is a point v in the interior of C that is invariant under H_C; one can choose any point in the interior of C and take the sum of the points in its H_C-orbit. Then, composing straight line flow toward v with radial projection to the unit sphere gives a deformation retraction of $S^{n-1} \cap C$ to the point $v/|v|$. This deformation retraction is H_C-equivariant, and hence induces a deformation retraction from $(S^{n-1} \cap C)/H_C$ to a point. □

6. Applications to moduli spaces of tropical curves

One strategy for studying the topology of $\Delta_{g,n}$ is to identify large contractible subcomplexes. The largest contractible subcomplex identified in [?, Theorem 1.1] is the locus $\Delta_{g,n}^{br}$ of tropical curves with bridges, cut vertices, loop edges, repeated markings, or vertices of positive weight; it is the closure of the locus of tropical curves with bridges. We now use Proposition 5.1 to produce an even larger contractible subcomplex.

Theorem 6.1. *The subcomplex $\Delta_{g,n}^{bm} \subset \Delta_{g,n}$ parametrizing tropical curves with bridges, cut vertices, loops, repeated markings, vertices of positive weight, or multiple edges is contractible, for $g \geq 1$.*

The superscript *bm* reflects the fact that this subcomplex is the closure of the locus of tropical curves with *bridges* or *multiple* edges.

Proof. First, note that contracting an edge in a graph with a multiple edge produces a graph with either a loop edge or a multiple edge. Hence $\Delta_{g,n}^{bm}$ is a subcomplex. It is the closure of the locus of tropical curves with bridges or multiple edges.

Note that $\Delta_{g,n}^{bm}$ is obtained from the subcomplex $\Delta_{g,n}^{br}$ as an iterated mapping cone, for a finite sequence of continuous maps from quotients of spheres $S^{p-1}/\operatorname{Aut}(G)$, where G

is a graph with multiple edges. We claim that each of these quotients is contractible. Interchanging a pair of edges between the same endpoints acts by a reflection, and hence the quotient $S^{p-1}/\operatorname{Aut}(G)$ is contractible, by Proposition 5.1.

Observe that if Y is contractible and $f\colon Y \to Z$ is continuous, then the inclusion of Z into the mapping cone of f is a homotopy equivalence. Applying this observation to the iterated mapping cone construction discussed above, we see that the inclusion of $\Delta_{g,n}^{br}$ in $\Delta_{g,n}^{bm}$ is a homotopy equivalence. The subcomplex $\Delta_{g,n}^{br}$ is contractible [**?**, Theorem 1.1], and the theorem follows. □

We now show that $\Delta_{g,n}$ is simply connected, for $g \geq 1$.

Proof of Theorem 1.1. First, we claim that the 1-skeleton of $\Delta_{g,n}$ is contained in $\Delta_{g,n}^{bm}$. To see this, note that $\Delta_{g,n}^{(1)}$ parametrizes stable tropical curves with 1 or 2 edges. Either one of the edges is a bridge, all of the edges are loops, or there are two edges that together form a loop. In particular, the underlying graph has either a loop, bridge, or multiple edges. This proves the claim. Next, recall that the fundamental group is generated by loops in $\Delta_{g,n}^{(1)}$, by Theorem 3.1. If $g \geq 1$, then all such loops can be contracted in $\Delta_{g,n}^{bm}$, by Theorem 6.1, and hence $\Delta_{g,n}$ is simply connected. □

Next, we show that Δ_3 is homotopy equivalent to S^5.

Proof of Theorem 1.2. First, note that Δ_3 is homotopy equivalent to Δ_3/Δ_3^{bm}, by Theorem 6.1. Enumerating stable graphs of genus 3 shows that the only one without bridges, cut vertices, loops, multiple edges, or vertices of positive weight is the complete graph K_4. The cell of Δ_3 corresponding to K_4 is the quotient of the open 5-simplex, whose vertices correspond to the 2-element subsets of a 4-element set, by the permutation group $\mathfrak{S}_4$.

It follows Δ_3/Δ_3^{bm} is homeomorphic to the unreduced suspension $S(S^4/\mathfrak{S}_4)$. Each transposition in $\mathfrak{S}_4$ acts by a double transposition on the vertices of the 5-simplex. This is a rotation in the sense of [**?**], meaning that its fixed-point set has codimension 2. Any quotient of S^{n-1} by a finite rotation group is PL-homeomorphic to S^{n-1} [**?**]. In particular, $S^4/\mathfrak{S}_4$ is homeomorphic to S^4, and hence Δ_3 is homotopy equivalent to $S(S^4) \cong S^5$, as required. □

We conclude by showing that Δ_4 has nontrivial 3-torsion in H_5 and nontrivial 2-torsion in H_6 and H_7. The proof uses Theorems 4.3 and 6.1, together with explicit computations of the integral homology of certain quotients of spheres S^{p-1} by subgroups of the permutation group $\mathfrak{S}_{p+1}$. These computations were carried out with computer assistance. We considered the Δ-complex structure on $S^{p-1}/\mathfrak{S}_{p+1}$ induced by barycentric subdivision on the boundary of Δ^p, and used a python script to generate matrices for the simplicial chain complex. We then used Magma [**?**] to compute the homology. The code may be found at the following link.

`https://github.com/dcorey2814/homologyQuotientSpheres.git`

Proof of Theorem 1.3. We compute the E_1-page of the spectral sequence given by Theorem 4.3, using the contractible subcomplex $Z = \Delta_4^{bm}$.

By enumerating the stable graphs of genus 4, we see that there are precisely 3 cells in Δ_4 that are not contained in Δ_4^{bm}. These are the edge graph G of a square pyramid, the edge graph G' of a triangular prism, and the complete bipartite graph $K_{3,3}$. For each graph, we computed the reduced homology of the corresponding sphere quotient. The nonzero reduced homology groups are as follows.

$$\widetilde{H}_k(S^6/\operatorname{Aut}(G);\mathbb{Z}) = \begin{cases} \mathbb{Z}/4\mathbb{Z}, & \text{for } k=4, \\ \mathbb{Z}/2\mathbb{Z}, & \text{for } k=5; \end{cases}$$

$$\widetilde{H}_k(S^7/\operatorname{Aut}(G');\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{for } k=5, \\ \mathbb{Z}/2\mathbb{Z}, & \text{for } k=6; \end{cases}$$

$$\widetilde{H}_k(S^7/\operatorname{Aut}(K_{3,3});\mathbb{Z}) = \begin{cases} \mathbb{Z}/3\mathbb{Z}, & \text{for } k=4, \\ \mathbb{Z}/4\mathbb{Z}, & \text{for } k=5, \\ \mathbb{Z}/2\mathbb{Z}, & \text{for } k=6. \end{cases}$$

Consider the spectral sequence given by Theorem 4.3 for $X = \Delta_4$ and $Z = \Delta_4^{bm}$. The nonzero terms on the E_1-page, aside from $E_1^{0,0} = \mathbb{Z}$, are:

$$\begin{aligned} E_1^{7,-2} &= \mathbb{Z}/4\mathbb{Z}, \\ E_1^{7,-1} &= \mathbb{Z}/2\mathbb{Z}, \end{aligned}$$

and

$$\begin{aligned} E_1^{8,-3} &= \mathbb{Z}/3\mathbb{Z}, \\ E_1^{8,-2} &= \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \\ E_1^{8,-1} &= \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

There are no differentials between nonzero terms on E_r, for $r > 1$, so the sequence degenerates at E_2. The only differentials between nonzero terms on E_1 are $E_1^{8,-2} \to E_1^{7,-2}$ and $E_1^{8,-1} \to E_1^{7,-1}$. Each source is larger than its target, so we conclude that both $E_\infty^{8,-2}$ and $E_\infty^{8,-1}$ contain nontrivial 2-torsion. It follows that $H_6(\Delta_4;\mathbb{Z})$ and $H_7(\Delta_4;\mathbb{Z})$ contain nontrivial 2-torsion. Similarly, we see that $E_\infty^{8,-3} = \mathbb{Z}/3\mathbb{Z}$ and conclude that $H_5(\Delta_4;\mathbb{Z})$ contains nontrivial 3-torsion. □

Remark 6.2. In the proof of Theorem 1.3, we have seen that $\Delta_4^{(6)}$ is contained in a contractible subcomplex Δ_4^{bm}. However, Δ_4 has nontrivial H_5 and H_6. We conclude that $H_5(\Delta_4;\mathbb{Z})$ and $H_6(\Delta_4;\mathbb{Z})$ are not generated by cycles in the 5-skeleton and 6-skeleton, respectively. By the Hurewicz Theorem, we see also that $\pi_5(\Delta_4;\mathbb{Z})$ is not generated by maps of spheres into the 5-skeleton.

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