MOST BIG MAPPING CLASS GROUPS FAIL THE TITS ALTERNATIVE

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ABSTRACT. Let X be a surface, possibly with boundary. Suppose it has infinite genus or infinitely many punctures, or a closed subset which is a disk with a Cantor set removed from its interior. For example, X could be any surface of infinite type with only finitely many boundary components. We prove that the mapping class group of X does not satisfy the Tits Alternative. That is, $\mathrm{Map}(X)$ contains a finitely generated subgroup that is not virtually solvable and contains no nonabelian free group.

1. INTRODUCTION

Lanier and Loving gave examples of big mapping class groups that do not satisfy the Tits Alternative, and asked whether the same holds for every big mapping class group [9, Question 6]. We show that few if any big mapping class groups satisfy it:

Theorem 1.1. *Suppose X is a surface, possibly nonorientable and possibly with boundary. Also suppose that it satisfies one of the following:*

(i) *X has infinite genus;*
(ii) *X has infinitely many punctures;*
(iii) *X contains a closed subset homeomorphic to $D^2 - C$, where C is a Cantor set in the interior of the 2-disk.*

Then its mapping class group $\mathrm{Map}(X)$ does not satisfy the Tits Alternative. That is, $\mathrm{Map}(X)$ has a finitely generated subgroup $\tilde{G}$, which contains no nonabelian free group and no finite-index solvable subgroup.

Corollary 1.2. *Suppose X is a surface of infinite type, with only finitely many boundary components. Then $\mathrm{Map}(X)$ does not satisfy the Tits Alternative.*

For us, every surface X is connected and second countable, and may have boundary. When X is non-orientable, its *mapping class group*

$\mathrm{Map}(X)$ is defined as the group of self-homeomorphisms of X that fix the boundary pointwise, modulo isotopies that fix the boundary pointwise. When X is orientable we use the same definition except that the self-homeomorphisms are required to preserve orientation. A surface X has *finite type* or *infinite type* according to whether its fundamental group is finitely generated or not. A *small* resp. *big* mapping class group means the mapping class group of a finite-type resp. infinite-type surface. See [5] for general background on mapping class groups, and [1] for a survey of recent work on big mapping class groups. Although $\mathrm{Map}(X)$ is a topological group, its topology will play no role in this paper.

The *Tits Alternative* is a famous property of $\mathrm{GL}_n(k)$ over any field k, discovered by Tits [14]. Namely, every finitely generated subgroup is either "big" (contains a nonabelian free group) or "small" (contains a finite-index solvable subgroup). The point is that there is no "medium". One expresses this by saying that $\mathrm{GL}_n(k)$ satisfies the Tits Alternative. The property makes sense with any group in place of $\mathrm{GL}_n(k)$, and if it holds then we say that group satisfies the Tits Alternative.

Determining whether various groups satisfy the Tits Alternative is a major thread in geometric group theory. For example, it holds for the outer automorphism group of a finite-rank free group [3][4], and for many Artin groups [10]. It also holds for small mapping class groups, in fact Ivanov [8] and McCarthy [11] independently proved that these groups satisfy a "strong Tits alternative". This is defined the same way except with solvable replaced by abelian.

Lanier and Loving [9] showed that the mapping class group of an infinite-type open surface cannot have this stronger property. They also gave examples of big mapping class groups that do not satisfy even the classical Tits alternative. The first such example, not noted at the time, is probably due to Funar and Kapoudjian. They exhibited a big mapping class group that contains Thompson's group [6, Prop. 2.4][1, Prop. 5.30]. Further examples come from the work of Aougab, Patel and Vlamis [2], who constructed surfaces X such that every countable group embeds in $\mathrm{Map}(X)$.

The key idea in the proof of Theorem 1.1 is to construct $\tilde{G}$ together with a surjection $\tilde{G} \to G$, where G is Grigorchuck's group. We do this in such a way that the kernel is abelian, which allows us to transfer to $\tilde{G}$ the fact that G does not satisfy the Tits Alternative. We review the essential properties of G in Section 2. Section 3 gives our construction in case (iii) of the theorem, and Section 4 gives it for the other two

cases. Also see Section 4 for the definition of a puncture. Section 5 shows that these three cases are enough for Corollary 1.2, and makes a few remarks about the cases left open.

We are grateful to Lanier and Loving for helpful correspondence and discussions, for many references, and for asking the question that inspired this work.

2. Grigorchuk's group

In this section we recall Grigorchuk's famous group and some of its many remarkable properties; see [7] for background. The properties we need are that it is a group of automorphisms of a rooted binary tree, and is finitely generated, infinite and torsion. Sadly, we do not use its most fascinating property, that it has intermediate growth.

We define T as the tree whose vertices are the finite binary words (sequences in $\{0, 1\}$). We indicate the empty word by $\emptyset$. The edges of T are the following: each word w is joined to $w0$ and $w1$, which are called the left and right *children* of w respectively. The nth *bit* of w means the nth term of the sequence w. This only makes sense when w has at least n terms.

Grigorchuk's group G is defined as the subgroup of $\mathrm{Aut}(T)$ generated by four specific transformations a, b, c, d. Each of these fixes the root $\emptyset$ of T. At other vertices the definitions are recursive, expressed in terms of an arbitrary word w (possibly empty):

$$a(0w) = 1w \qquad b(0w) = 0a(w) \qquad c(0w) = 0a(w) \qquad d(0w) = 0w$$
$$a(1w) = 0w \qquad b(1w) = 1c(w) \qquad c(1w) = 1d(w) \qquad d(1w) = 0b(w)$$

Flipping a bit in a word means changing that bit from 0 to 1 or vice-versa. So a acts by flipping the first bit of each nonempty word. We can also describe b, c, d in terms of bit-flipping. Namely, b acts on a word w by flipping the bit after the first 0 in w, just if the number of initial 1's in w is 0 or 1 mod 3. (If w has no 0's, or no bits after its first 0, then b fixes w.) For c we use the same definition but with "0 or 1" replaced by "0 or 2", and similarly for d using "1 or 2". An induction justifies these descriptions, pictured in Figure 1.

If $g \in \{a, b, c, d\}$ fixes a vertex v but exchanges the children of v, then we call v a *swap vertex* of g. One can also define the swap vertices as the boundary points of the fixed-point set in T. The only swap vertex of a is $\emptyset$. The swap vertices of b are the words $1 \cdots 10$, where the number of 1's is 0 or 1 mod 3. And similarly for c resp. d, with "0 or 1" replaced by "0 or 2" resp. "1 or 2".

Lemma 2.1. *G is not virtually solvable, and contains no nonabelian free group.*
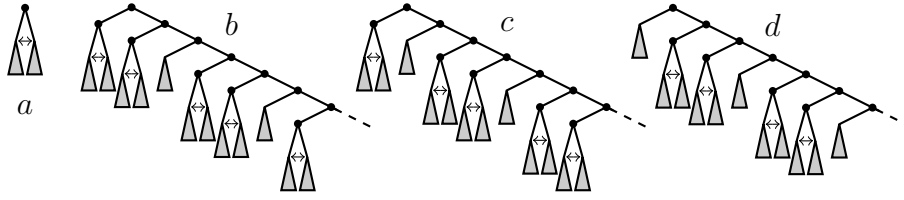
FIGURE 1. Generators for Grigorchuk's group G. Each 2-headed arrow represents the bodily exchange of the two indicated subtrees, drawn as shaded triangles. The other shaded triangles indicate pointwise-fixed subtrees.

Proof. The second claim follows from the fact that G is a torsion group. For the first, suppose G contains a solvable subgroup F of finite index. Having finite index in the finitely generated group G, F is also finitely generated. Every finitely generated solvable torsion group is finite. (The abelianization is finite, so the derived subgroup is finitely generated, hence finite by induction on solvable length.) So F is finite, which forces G to be finite, which it is not. $\square$

3. IF X CONTAINS $D^2 - (\text{CANTOR SET})$

In this section we suppose that X has a closed subset homeomorphic to $D^2 - C$, where D^2 is the 2-disk and C is a Cantor set in the interior of D^2. We will construct a subgroup $\tilde{G}$ of $\mathrm{Map}(X)$, supported on $D^2 - C$, which is not virtually solvable and contains no nonabelian free group. The idea is to interpret the generators of Grigorchuk's group G as automorphisms of X, and define $\tilde{G}$ as the subgroup of $\mathrm{Map}(X)$ generated by the mapping classes represented by these automorphisms.

We will work with the following description of $D^2 - C$ as an identification space, obtained by gluing pairs of pants together in the pattern of the tree T. Fix a pair of pants P, meaning a sphere minus the interiors of three pairwise disjoint closed disks. We suppose that the components of ∂P are called (in some order) the waist and the left and right cuffs of P. We fix a self-homeomorphism σ (for *s*wap) of P, which fixes the waist pointwise and exchanges the cuffs, such that σ^2 is the identity on each cuff. We choose a homeomorphism from the waist to S^1. We choose homeomorphisms from the cuffs to S^1 which are compatible with each other, in the sense that they are exchanged by σ.

We define W (for binary *W*ords) as the set of vertices of T, equipped with the discrete topology. We equip $P \times W$ with the product topology.

For each $w \in W$ we abbreviate $P \times \{w\}$ as P_w. We transfer our labeling (as waist or left/right cuff) of the components of ∂P to those of ∂P_w.

We define an equivalence relation $\sim$ on $P \times W$ by gluing the left resp. right cuff of each P_w to the waist of P_{w0} resp. P_{w1}. Formally, we declare that for each $w \in W$ we have $(l, w) \sim (l', w0)$ and $(r, w) \sim (r', w1)$, where l resp. r lies in the left resp. right cuff of P, and l' resp. r' is the corresponding point on the waist of P. (We fixed homeomorphisms from the waist and cuffs to S^1. Combining them gives homeomorphisms from the cuffs to the waist, which we take as the definitions of $l \mapsto l'$ and $r \mapsto r'$.) We write $X_\emptyset$ for $(P \times W)/\sim$, equipped with the quotient space topology. It is standard that $X_\emptyset \cong D^2 - C$, so we regard $X_\emptyset$ as a subsurface of X.

The curious notation $X_\emptyset$ is a special case of the following more general notation X_w, useful for referring to subsurfaces. If $w \in W$ then we define W_w as the set of binary words having w as a prefix. Equivalently, it consists of w and the vertices of T that lie below w. We define X_w as $(P \times W_w)/\sim$. If is easy to see that if $x \in W_w$ then $P_x \to P \times W_w \to X_w \to X_\emptyset$ is injective. So we will regard all P_x and X_w as subsets of $X_\emptyset \subseteq X$. In particular, X_w is a subsurface of X bounded by the waist of P_w.

Now suppose $g \in \{a, b, c, d\} \subseteq G$. We will define an automorphism $\hat{g}$ of X that is supported on $X_\emptyset$. We define it in terms of an automorphism of $P \times W$ that respects $\sim$. Namely, if $p \in P$ and $w \in W$, then

$$(3.1) \qquad \hat{g}(p, w) = \begin{cases} (\sigma(p), w) & \text{if } w \text{ is a swap vertex of } g \\ (p, g(w)) & \text{otherwise.} \end{cases}$$

After checking that this respects $\sim$, we may regard $\hat{g}$ as an automorphism of X.

A visual version of this construction, entirely optional, appears in Figure 2, which shows X_w for w a swap vertex of g. The same construction should be carried out simultaneously for every swap vertex. The figure shows four frames of an isotopy of X_w inside $\mathbb{R}^3$, whose time 1 map we take as the definition of $\hat{g}$. Although X_w is the union of infinitely many pairs of pants, we have only drawn enough of them to illustrate the idea. You should imagine attaching another 8 pairs of pants on the left, then attaching 16 more pairs, then another 32, and so on, limiting to the Cantor set which is the end space of X_w. The top right pair of pants is P_w, and we imagine it is made of rubber, deforming as needed to accomodate the motion of the surfaces X_{w0} and X_{w1} attached to its cuffs, which are made of steel. The isotopy moves X_{w0} and X_{w1} rigidly (ie, by translations) until they trade places. Strictly
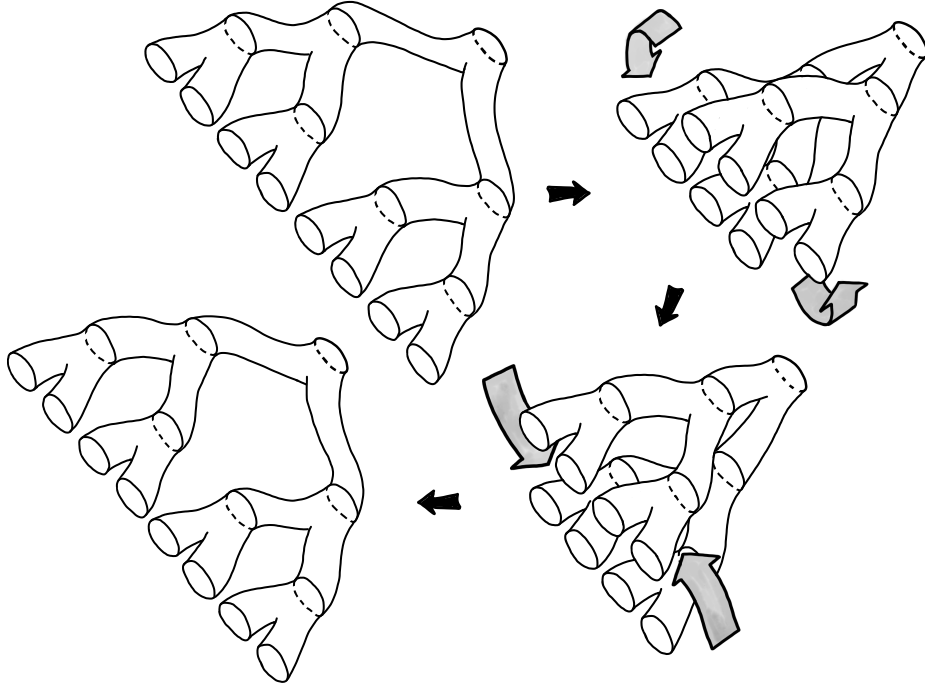
FIGURE 2. Our construction of $\hat{g}$. Here g is one of Grig-
orchuk's generators a, b, c, d, and the surface shown rep-
resents $X_w \subseteq X$, where $w \in W$ is a swap vertex of g. We
define $\hat{g}$ (on X_w) as the end result of the pictured isotopy
in $\mathbb{R}^3$. See the text for details.

speaking, X_{w0} and X_{w1} should appear identical, so that the time 1 map
of the isotopy is a self-map of the pictured surface. But we have drawn
them slightly differently, so the reader can follow their motions more
easily. It is easy to see that the restriction of $\hat{g}^2$ to P_w is (isotopic to)
the product of the left-handed Dehn twist around the waist and the
right-handed Dehn twists around the cuffs.

Now we return to the formal development. We define $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in$
$\mathrm{Map}(X)$ as the mapping classes of $\hat{a}, \hat{b}, \hat{c}, \hat{d}$, and

$$\hat{G} = \langle \hat{a}, \hat{b}, \hat{c}, \hat{d} \rangle \subseteq \mathrm{Homeo}(X) \qquad \tilde{G} = \langle \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \rangle \subseteq \mathrm{Map}(X)$$

By construction, $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ permute the $P_w \subseteq X$ in the same way that
$a, b, c, d \in G$ permute the vertices w of T. It follows that $\hat{a}, \hat{b}, \hat{c}, \hat{d}$
permute the ends of $X_\emptyset$ the same way that a, b, c, d permute the ends
of T. The same holds with $\tilde{a}, \ldots, \tilde{d}$ in place of $\hat{a}, \ldots, \hat{d}$, because the
action of $\mathrm{Homeo}(X)$ (or $\mathrm{Homeo}^+(X)$ if X is orientable) on the ends
of X factors through $\mathrm{Map}(X)$. Also, $\tilde{G}$ fixes every end of X not coming

from $X_\emptyset$. We have shown that the image of $\tilde{G}$, under the action of $\mathrm{Map}(X)$ on the ends of X, is isomorphic to G.

Lemma 3.1. *The surjection $\tilde{G} \to G$ has abelian kernel.*

Proof. Let U be the union of a family of mutually disjoint annular neighborhoods of the waists of the P_w, with w varying over W. We claim that every $\alpha \in \hat{G}$, that acts trivially on the end space of X, is supported on U (up to isotopy). Given this, it follows that the kernel of $\tilde{G} \to G$ lies in the image of $\mathrm{Map}(U) \to \mathrm{Map}(X)$. Then the lemma follows from the fact that $\mathrm{Map}(U)$ is a direct product of copies of $\mathbb{Z}$.

Now we prove the claim. Because α permutes the P_w, and acts trivially on the ends of $X_\emptyset$, it sends each P_w to itself. Considering their intersections shows that α preserves the waist of every P_w. After an isotopy supported in $X_\emptyset$ and preserving every P_w, we may suppose α fixes each waist pointwise. The mapping class group of a pair of pants is generated by the Dehn twists around its boundary components. Applying this separately to each P_w shows that α is isotopic to an automorphism of X that is supported in U. $\square$

Lemma 3.2. *Suppose $f : \tilde{G} \to G$ is a surjection of groups with solvable kernel. Then*
   (i) *$\tilde{G}$ is virtually solvable if and only if G is.*
   (ii) *$\tilde{G}$ contains a nonabelian free group if and only if G does.*

Proof. (i) Obvious. (ii) First suppose G contains a subgroup freely generated by two elements x, y, and choose any lifts $\tilde{x}, \tilde{y}$ of them in $\tilde{G}$. These lifts satisfy no nontrivial relations, because if they did then x, y would also. So $\tilde{x}, \tilde{y}$ generate a nonabelian free subgroup of $\tilde{G}$. Now suppose $\tilde{G}$ contains a nonabelian free group $\tilde{F}$. Being nonabelian and free, $\tilde{F}$ lacks normal solvable subgroups, so it meets $\mathrm{Ker}(f)$ trivially. Therefore $f(\tilde{F}) \cong \tilde{F}$ is a nonabelian free subgroup of G. $\square$

Lemma 3.3. *With X as above, $\mathrm{Map}(X)$ does not satisfy the Tits alternative.*

Proof. It is enough to show that $\tilde{G} \subseteq \mathrm{Map}(X)$ is not virtually solvable and contains no nonabelian free group. By Lemmas 3.1 and 3.2 it is enough to prove this with G in place of $\tilde{G}$. We did this in Lemma 2.1. $\square$

4. If X has infinitely many handles or punctures

In this section we consider a surface X with infinite genus, or infinitely many punctures. Again we will build a subgroup $\tilde{G}$ of $\mathrm{Map}(X)$
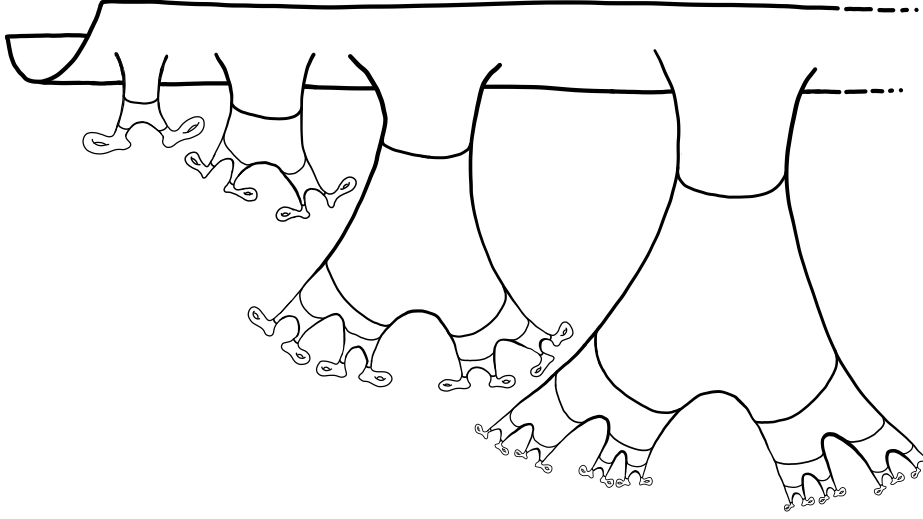
FIGURE 3. The subsurfaces $X_1, X_2, X_3, X_4, \ldots$ of X.

that is not virtually solvable and contains no nonabelian free group. The method is a variation on the previous section. We will focus on the infinite-genus case, and then modify the argument for the case of infinitely many punctures.

Suppose X is a surface with infinite genus. Take X_1 to be a compact subsurface with genus 2 and one boundary component. Then take X_2 to be a compact subsurface of $X - X_1$ with genus 4 and one boundary component. Then take X_3 to be a compact subsurface of $X - (X_1 \cup X_2)$ with genus 8 and one boundary component. Continuing in this fashion, we obtain an infinite sequence X_n of mutually disjoint compact subsurfaces of X, where X_n has genus 2^n and one boundary component. As suggested by Figure 3, we express each X_n as the union of $2^n - 1$ pairs of pants, arranged like a binary tree, and 2^n handles.

To express this precisely, recall W, P and P_w (with $w \in W$) from Section 3. For each integer n we write $W_n \subseteq W$ for the set of binary words of length n, $W_{<n}$ for those of length $< n$, and $W_{\leq n}$ for those of length $\leq n$.

We also fix a compact surface S (for Shoe) with genus 1 and one boundary component. We will call ∂S the rim of S; a shoemaker would say collar, but that already has a meaning in topology. We fix a homeomorphism from the rim to the circle S^1.

For each $n \geq 1$, we identify X_n with the quotient of $(P \times W_{<n}) \cup (S \times W_n)$ by the following equivalence relation $\sim$, similar to the one in Section 3. We abbreviate $S \times \{w\}$ as S_w, for each $w \in W_n$. We refer to the S_w as the *shoes*. First, for each $w \in W_{<n-1}$, we glue the cuffs of P_w

to the waists of P_{w0} and P_{w1}. Formally, we declare that $(l, w) \sim (l', w0)$ and $(r, w) \sim (r', w1)$, where l resp. r lies in the left resp. right cuff of P, and l' resp. r' is the corresponding point on the waist of P. Second, for each $w \in W_{n-1}$ we glue the cuffs of P_w to the rims of S_{w0} and S_{w1}. Formally, we declare that $(l, w) \sim (l', w0)$ and $(r, w) \sim (r', w1)$, where l, r are as before, but now l', r' are the corresponding points on the rim of S.

For each $n \geq 1$, and each $w \in W_{<n}$, we have introduced a subsurface P_w of X_n. Since all the X_n lie in X, we have introduced infinitely many subsurfaces of X, all called P_w. To distinguish them when necessary, we write $P_{n,w}$ for the one that lies in X_n. The same issue does not arise for the shoes S_w: if $w \in W$ has length n, then only X_n contains S_w.

For $g \in \{a, b, c, d\} \subseteq G$, we now define an automorphism $\hat{g}$ of X. It is the identity outside the X_n, and for each fixed n we define $\hat{g}$ on X_n as follows. The idea is to act on X_n "in the same way that g acts on the top $n + 1$ levels of the tree T". Formally, we define $\hat{g}$ as an automorphism of $(P \times W_{<n}) \cup (S \times W_n)$ that respects $\sim$. For each $w \in W_{<n}$ we define $\hat{g}$ on P_w by (3.1). And for $w \in W_n$ we define $\hat{g}$ on S_w by $\hat{g}(p, w) = (p, g(w))$. After checking that this respects $\sim$, we may regard $\hat{g}$ as defined on X_n. This completes the construction of $\hat{g} \in \mathrm{Homeo}(X)$. If X is orientable then clearly $\hat{g}$ lies in $\mathrm{Homeo}^+(X)$.

We define $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathrm{Map}(X)$ as the mapping classes of $\hat{a}, \hat{b}, \hat{c}, \hat{d}$, and

$$\hat{G} = \langle \hat{a}, \hat{b}, \hat{c}, \hat{d} \rangle \subseteq \mathrm{Homeo}(X) \qquad \tilde{G} = \langle \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \rangle \subseteq \mathrm{Map}(X)$$

Lemma 4.1. *The image of $\tilde{G}$ in $\mathrm{Aut}(H_1(X))$ is isomorphic to G.*

Proof. Because $\hat{G}$ fixes $X - \bigcup_n X_n$ pointwise, it acts trivially on the homology classes supported there. So it is enough to work out the action of $\tilde{G}$ on

$$H_1\left(\bigcup X_n\right) = \bigoplus_{n \geq 1} H_1(X_n) = \bigoplus_{n \geq 1} \bigoplus_{w \in W_n} H_1(S_w) = \bigoplus_{w \in W - \{\emptyset\}} H_1(S_w)$$

By construction, $\hat{G}$ permutes the shoes, hence the summands on the right. In particular, an element of $\hat{G}$ that preserves a summand $H_1(S_w)$ must carry the corresponding shoe S_w to itself. Furthermore, the definition of the generators of $\hat{G}$ on the shoes implies: an element of $\hat{G}$ that preserves some S_w must act trivially on it. Therefore: any element of $\hat{G}$, that preserves every summand $H_1(S_w)$, must act trivially on every S_w, hence trivially on $H_1(X)$.

Another way to say this is that the image of $\hat{G}$ in $\mathrm{Aut}(H_1(X))$ is the same as the image of $\hat{G}$ in the group of permutations of the shoes. By construction, $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ permute the shoes in the same way that $a, b, c, d \in$

G permute the words $w \in W - \{\emptyset\}$. So the image of $\hat{G}$ in $\mathrm{Aut}(H_1(X))$ is a copy of G. The action on $H_1(X)$ factors through $\mathrm{Map}(X)$, so the same is true with $\tilde{G}$ in place of $\hat{G}$. □

Lemma 4.2. *The surjection $\tilde{G} \to G$ has abelian kernel.*

Proof. We will write f for the surjection $\hat{G} \to G$. We must show that the image of $\mathrm{Ker}(f)$ in $\tilde{G}$ is abelian. So suppose $\alpha \in \mathrm{Ker}(f)$. The previous proof shows that α acts trivially on every shoe S_w. Together with the fact that α permutes the $P_{n,w}$, this shows that α preserves every $P_{n,w}$.

From here one can follow the proof of Lemma 3.1. We take U to be the union of disjoint annular neighborhoods of the waists of the $P_{n,w}$ (with $n \geq 1$ and $w \in W_{<n}$) and the rims of the S_w (with $w \in W - \{\emptyset\}$). Arguing as for Lemma 3.1 shows that α is isotopic to a homeomorphism supported on U. Because $\mathrm{Map}(U)$ is abelian, it follows that $\mathrm{Ker}(f)$ has abelian image in $\tilde{G}$. □

Lemma 4.3. *The mapping class group of an infinite-genus surface does not satisfy the Tits Alternative.*

Proof. Mimic the proof of Lemma 3.3, using Lemma 4.2 in place of Lemma 3.1. □

To deal with punctures in place of handles, we first define punctures. Let E be the end space of X; we equip $X \cup E$ with its standard topology. For us, A *puncture* of X means an end e for which there exists an embedding of the 2-disk into $X \cup E$, that sends the origin to e and sends no other point into E.

Lemma 4.4. *If X has infinitely many punctures, then its mapping class group does not satisfy the Tits Alternative.*

Proof sketch. Choose an infinite sequence of distinct ends e_n, and disks in $X \cup E$ centered at them, as above. By shrinking the nth disk, we may suppose it is disjoint from its predecessors. So any two of the disks are disjoint. With this preparation, it is easy to construct a sequence of disjoint closed sets $X_1, X_2, \ldots$ in X, such that X_n is a disk with 2^n punctures. We decompose X_n as in the infinite-genus case, except that we use once-punctured disks in place of the shoes. The surface automorphisms $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ and the group $\hat{G}$ they generate are defined as before. One replaces Lemma 4.1 by an analysis of how $\hat{G}$ permutes the ends of X. This is simpler than Lemma 4.1 and similar to what we did in Section 3. The only difference is that the ends of $\bigcup_n X_n$ are indexed by the vertices of T, whereas there the ends of $X_\emptyset$ were indexed by the

ends of T. The rest of the proof is the same as in the infinite-genus case. □

5. The main results

Theorem 1.1 is the union of Lemmas 3.3, 4.3 and 4.4. For Corollary 1.2, suppose X has infinite type but only finitely many boundary components. The theorem below shows that X has one of the features (i)–(iii) in Theorem 1.1. Quoting that theorem shows that $\mathrm{Map}(X)$ does not satisfy the Tits Alternative. Lanier and Loving used the same trichotomy to construct the surfaces mentioned in the introduction, whose mapping class groups fail to satisfy the *strong* Tits Alternative.

Theorem 5.1. *Suppose X is an infinite-type surface with only finitely many boundary components. Then X has infinite genus, or infinitely many punctures, or contains a closed subset homeomorphic to a disk with a Cantor set removed from its interior.*

We call a surface X *planar* if every circle embedded in $X - \partial X$ is separating.

Lemma 5.2. *Suppose X is a planar surface and E its space of ends. Suppose e is an isolated point of E and lies outside the closure of ∂X (in $X \cup E$). Then e is a puncture.*

Proof. Straightforward. □

Proof of Theorem 5.1. The arguments are like those of Richards [13] in his proof of Kerékjártó's theorem classifying noncompact surfaces. We would like to quote his results, but they do not allow boundary. A generalization due to Prishlyak and Mischenko [12] allows boundary, but I could not follow it.

If X is nonorientable, then we can find an embedded Möbius band M_1. If $X - M_1$ is nonorientable, then it contains an embedded Möbius band M_2. Repeating this process with $X - (M_1 \cup M_2)$, and so on, we find a sequence of mutually disjoint embedded Möbius bands. Suppose first that this construction does not terminate, so the sequence is infinite. The connected sum of three crosscaps is the same as the connected sum of a torus and a single crosscap, so X has infinite genus.

This leaves the case that the sequence terminates. Then X is the connected sum of an orientable surface X_1 and finitely many crosscaps. If the genus of X_1 is infinite then we are done, so suppose it is finite. Gathering the handles and crosscaps together, we see that X is the connected sum of a planar surface X_2 and a closed surface. Because X has infinite type, X_2 does too.

We write E for the end space of X, which is also the end space of X_2. It is infinite, or else X_2 would have finite type. Its intersection with $\overline{\partial X}$ (the closure taken in $X \cup E$) is finite, because X has finitely many boundary components, each with at most two ends. So $E - \overline{\partial X}$ is infinite. The finiteness of $E \cap \overline{\partial X}$ also shows that a point of $E - \overline{\partial X}$ is isolated in $E - \overline{\partial X}$ if and only if it is isolated in E. So we may speak unambiguously of isolated ends.

If there are infinitely many isolated points of $E - \overline{\partial X}$, then we are done because Lemma 5.2 shows that each is a puncture. So suppose otherwise. Because E is infinite, $E - \overline{\partial X}$ has a non-isolated point e. Choose an embedded circle $C \subseteq X_2$ that separates e from ∂X_2 and the isolated ends. Of the two components of $X_2 - C$, one has e among its ends. Write D for the union of that component and C, and write E_0 for the end space of D. Because E_0 has no isolated points, it is perfect and therefore a Cantor set. Because X_2 is planar, a standard subdivision-into-pairs-of-pants argument shows that D is a disk with a Cantor set removed from its interior. $\qquad\square$

If X has infinitely many boundary components, then our constructions may fail. The reason is that the homeomorphisms used to define $\mathrm{Map}(X)$ must fix ∂X pointwise, so that they cannot permute the components of ∂X. But even if one allowed permutations in some way, some surfaces would still escape our methods:

Example 5.3. Let $D_{n \in \mathbb{Z}}$ be the closed disk in $\mathbb{R}^2$ with radius $1/3$ and center $(n, 0)$. Let C_n be a Cantor set in ∂D_n. Let X be the plane, minus the union of the C_n and the interiors of the D_n. Then X has countably many boundary components $B_{k \geq 1}$, each a copy of $\mathbb{R}$. From X remove k points from each B_k, and write Y for what remains. Then every self-homeomorphism of Y sends every component of ∂Y to itself. (The key step is that every self-homeomorphism of Y sends every B_k to itself. To see this, say that two components of ∂Y *abut* if some end of Y lies in both their closures. Under the equivalence relation this generates, two components of ∂Y are equivalent if and only if they lie in the same B_k. Every self-homeomorphism preserves the unique equivalence class of each size $k > 1$, hence every B_k.) It seems likely that the central quotient of $\mathrm{Map}(Y)$ is the end-preserving subgroup of $\mathrm{Map}(Y - \partial Y)$, ie a version of the pure braid group on infinitely many strands.

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