COMPLETIONS, BRANCHED COVERS, ARTIN GROUPS AND SINGULARITY THEORY

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Abstract. We study the curvature of metric spaces and branched covers of Riemannian manifolds, with applications in topology and algebraic geometry. Here curvature bounds are expressed in terms of the CAT(χ) inequality. We prove a general CAT(χ) extension theorem, giving sufficient conditions on and near the boundary of a locally CAT(χ) metric space for the completion to be CAT(χ). We use this to prove that a branched cover of a complete Riemannian manifold is locally CAT(χ) if and only if all tangent spaces are CAT(0) and the base has sectional curvature bounded above by χ. We also show that the branched cover is a geodesic space. Using our curvature bound and a local asphericity assumption we give a sufficient condition for the branched cover to be globally CAT(χ) and the complement of the branch locus to be contractible.

We conjecture that the universal branched cover of $\mathbb{C}^n$ over the mirrors of a finite Coxeter group is CAT(0). Conditionally on this conjecture, we use our machinery to prove the Arnol′d-Pham-Thom conjecture on $K(\pi, 1)$ spaces for Artin groups. Also conditionally, we prove the asphericity of moduli spaces of amply lattice-polarized K3 surfaces and of the discriminant complements of all the unimodal hypersurface singularities in Arnol′d's hierarchy.

1. Introduction

We are interested in when a branched cover M' of a complete Riemannian manifold M with sectional curvature $\leq \chi$ satisfies the same curvature bound. Our interest in this problem goes back to [3] and stems from its applications to the well-known $K(\pi, 1)$ problem for Artin groups, the topology of certain moduli spaces of K3 surfaces, and the topology of discriminant complements of singularities. A simple way to formulate the idea of a branched cover of a complete Riemannian manifold is to remove a subset Δ from M, leaving M_0, take a covering space

Date: August 6, 2011.

1991 *Mathematics Subject Classification.* Primary 51K10, Secondary 53C23, 57N65, 20F36, 14B07.

Partly supported by NSF grants DMS-024512 and DMS-0600112.

M_0' of M_0, and metrically complete it to get M'. So the study of this sort of branched covering is naturally phrased in terms of completions of metric spaces that satisfy some local curvature bounds.

A good setting for this situation uses the CAT(χ) inequality as the definition of an upper curvature bound. So we address a question more general than the original question about Riemannian manifolds. Namely, if X is a metric space, Δ a closed subset of it, and $X - \Delta$ is locally CAT(χ), under what conditions is X CAT(χ)? Here is our result when $\chi \leq 0$; the corresponding result for $\chi > 0$ is theorem 4.1.

Theorem 1.1 (=3.1; "CAT(χ) extension"). *Suppose $\chi \leq 0$, X is a complete geodesic space and Δ a nonempty closed convex subset. Assume also:* X Δ

(A) *every geodesic triangle with a vertex in Δ satisfies CAT(χ);*
(B) *local geodesics in the metric completion $\overline{T_cX}$ of the tangent space T_cX are unique, for every $c \in \Delta$; and*
(C) *there exists $\lambda > 0$ such that for all $x \in X - \Delta$, the closed ball with center x and radius $\lambda \cdot d(x, \Delta)$ is complete and CAT(χ).* λ

Then X is CAT(χ).

The hypotheses (A) and (B) are obviously necessary, but they are not sufficient. We think of (C) as a sort of "uniformly locally CAT(χ)" condition. To us this seems a simple and natural condition, particularly in light of our example 3.2.

One application of this theorem and its $\chi > 0$ analogue is to a simplicial complex X in which each simplex is given a metric of constant curvature χ. The key result here is that X is locally CAT(χ) just if every link in CAT(1). This was stated by Gromov [21, 4.2.A] and first proven in full generality by Bridson [7][8, II.5]. (Any of several minor regularity conditions is needed in order for X to be a geodesic space. Also, see [7, p. 377] for references to earlier work.) Using our results one can reprove Bridson's theorem by an induction on codimension (similar to but easier than our proof of theorem 5.1). The link condition is exactly the condition that the tangent spaces be CAT(0), and the inductive argument provides (C). So one can view theorem 1.1 as allowing the link condition to apply to metric spaces more general than metrized simplicial complexes.

It also promises to have applications to situations involving degenerate Riemannian metrics. For example, a celebrated result of Wolpert [42][43] is that Teichmüller space $\mathcal{T}$ is CAT(0) under the Weil-Petersen metric. Although this metric is Riemannian with nonpositive sectional curvature, it is not complete, so it was not clear that geodesics exist.

Wolpert proved that they do. Yamada [44] studied the metric near the boundary in order to show that component of each stratum of the boundary is convex. The geometry near the boundary is similar to our example 3.2, so it should be possible to develop these aspects of Teichmüller theory using our machinery.

Our main result on the curvature of branched covers of Riemannian manifolds is the expected one:

Theorem 1.2 (=5.1)**.** *Suppose M is a complete Riemannian manifold with sectional curvature $\leq \chi \in \mathbb{R}$ and $\Delta \subseteq M$ is locally the union of finitely many totally geodesic submanifolds of codimension 2. Suppose also that M_0' is a covering space of $M_0 := M - \Delta$ and M' is its metric completion. Then M' is locally CAT(χ) if and only if each of its tangent spaces is CAT(0).*

This reduces the question of local curvature bounds to an infinitesimal question. The point is that the tangent spaces to M' are branched covers of the tangent spaces to M (lemma 5.2), which are Euclidean. In some examples one can verify this tangent-space condition, but in general it seems to be quite hard. Our proof of the theorem is also surprisingly hard, requiring a delicate double induction. Charney and Davis [12, thm. 5.3] state a result similar to this one, but there is a gap in the proof; see the remark at the end of section 5.

Next we study the global geometry of M'. First we show that M' is a geodesic space (theorem 6.1). Then we prove the following theorem, which is our tool in applications. (We say a space is *aspherical* if its aspherical
homotopy groups $\pi_{n>1}$ are trivial.)

Theorem 1.3 (=6.3)**.** *Assume the hypotheses of theorem 1.2, with $\chi \leq 0$, M connected and M_0' the universal cover of M_0. If $T_xM - T_x\Delta$ is aspherical for all $x \in X$, and each tangent space to M' is CAT(0), then M' is CAT(χ)and M_0' is contractible.*

This theorem generalizes the main result of [3, lemma 3.3], which is the special case where Δ is locally modeled on the coordinate hyperplanes of $\mathbb{C}^n$.

We give three applications of this theorem, but unfortunately they are conditional on the following conjecture about finite Coxeter groups. Nevertheless, we view the unification of these problems and their reduction to this conjecture as progress.

Conjecture 1.4 (=7.1)**.** *Let W be a finite Coxeter group acting on $\mathbb{R}^n$ in the usual way, and let Δ be the union of the hyperplanes in $\mathbb{C}^n$ fixed by the reflections in W. Then the metric completion of the universal cover of $\mathbb{C}^n - \Delta$ is CAT(0).*

This is closely related to conjecture 3 of Charney and Davis [13], and in fact our approach to the Artin group $K(\pi, 1)$ problem is to show that our conjecture implies theirs. Combining our conjecture with Deligne's celebrated result [17] on the asphericity of mirror complements for finite Coxeter groups, we obtain:

Corollary 1.5 (=7.2)**.** *Assume conjecture 1.4. Also assume the situation of theorem 1.2, with $\chi \leq 0$, M connected and Δ locally modeled on the complexified mirror arrangements of finite Coxeter groups. Then the universal cover of $M - \Delta$ is contractible.*

We have already mentioned that the Arnol′d-Pham-Thom conjecture on $K(\pi, 1)$ spaces for Artin groups follows from conjecture 1.4:

Theorem 1.6 (=7.3)**.** *Assume conjecture 1.4. Let W be any Coxeter group, acting on its open Tits cone $C \subseteq \mathbb{R}^n$, and let M be its tangent bundle TC. Let Δ be the union of the tangent bundles to the mirrors of the reflections of W. Then $M - \Delta$ has contractible universal cover.*

In our other applications we will take M to be the Hermitian symmetric space $P\Omega$ associated to the orthogonal group $\mathrm{O}(2, n)$. The first application is to the moduli spaces of amply lattice-polarized K3 surfaces; here K is the "K3 lattice" $E_8^2 \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^3 \cong H^2(\text{any K3 surface}; \mathbb{Z})$.

Theorem 1.7 (=7.4)**.** *Assume conjecture 1.4. Suppose M is an integer quadratic form of signature $(1, t)$ with a fixed embedding in K. Then the moduli space of amply M-polarized K3 surfaces $(X, j : M \to \operatorname{Pic} X)$, for which the composition $M \to \operatorname{Pic} X \to H^2(X)$ is isomorphic to $M \to K$, has contractible orbifold universal cover.*

For orientation we remark that a K3 surface amply polarized by the 1-dimensional lattice $\langle 4 \rangle$ is the same thing as a smooth quartic surface in $\mathbb{C}P^3$. The global Torelli theorem for lattice-polarized K3s says that the spaces of amply lattice-polarized K3s are exactly the sort of space to which our techniques apply. A similar situation arises in Bridgeland's study of stability conditions on K3 surfaces [9]; see the remark after theorem 7.4.

Our final application is to the discriminant complements of singularities. The discriminant complement of a singularity is essentially the space of all deformations of the singularity that are deformed enough for the singularity to become smooth. (See section 8 for precise definitions.) The nature of the discriminant complement has been central to singularity theory since Brieskorn's famous paper [10] on the discriminants of the simple (A_n, D_n and E_n) singularities. In Arnol′d's hierarchy of hypersurface singularities, the singularities one step more complicated than the simple ones are the "unimodal" singularities. There are three kinds: simply-elliptic, cusp and exceptional.

Theorem 1.8 (=8.1+8.3)**.** *Assume conjecture 1.4. Then the discriminant complement of any unimodal hypersurface singularity is aspherical.*

The corresponding theorem for simple singularities is due to Deligne [17] and we use his result in our proof. We also rely on a great deal of work by Looijenga: his work provides descriptions of the discriminant complements to which our techniques can be adapted. One interesting connection to the Arnol′d-Pham-Thom conjecture is that the Tits cone of the $Y_{p,q,r}$ Coxeter group plays a central role in the treatment of cusp singularities.

Our methods apply to many singularities other than the unimodal ones, but we have confined discussion of these to a few remarks in section 8. An interesting twist that comes up in the case of triangle singularities is that the relevant hyperplane arrangements in $P\Omega$ need not be locally modeled on those of finite Coxeter groups. However, it seems likely that our methods still apply. See the remark after lemma 8.2.

The paper is organized as follows. Section 2 gives background on CAT(χ) geometry. Sections 3 and 4 give the CAT(χ) extension theorem in the $\chi \leq 0$ and $\chi > 0$ cases respectively. The proof of the $\chi \leq 0$ case is quite complicated. It could be simplified if one is willing to make more assumptions, such as extendibility of geodesics, or that distinct geodesics have distinct directions in the tangent space. The $\chi > 0$ case follows quite easily from the $\chi \leq 0$ case.

In sections 5–6 we treat the local and global properties of branched covers of Riemannian manifolds. In section 7 we discuss our conjecture about finite Coxeter groups and treat the Arnol′d-Pham-Thom conjecture and the asphericity of moduli spaces of amply lattice-polarized K3s. Section 8, on singularity theory, is the longest and has a different flavor from the earlier sections. This is because of the complexity of the discriminants. Even Looijenga's elegant descriptions of them require some work before we can apply our machinery. We have also given more detail than strictly necessary in hope of inspiring those who work on Artin groups to look also at the fundamental groups of these discriminant complements. These groups are like Artin groups, but different and perhaps better. See [39] for presentations in the simply elliptic and cusp cases.

I would like to thank the Japan Society for the Promotion of Science, the Clay Mathematics Institute and Kyoto University for their support and hospitality during this work. I am very grateful to E. Looijenga for helping me understand some of the singularity theory and to K. Wirthmüller for unbinding and scanning his dissertation in order to send it to me.

2. Background and Conventions

For experts we summarize our nonstandard conventions as follows. (1) Geodesics need only be parameterized proportionally to arclength, not necessarily by arclength. (2) We use the "strong" definition of the CAT(χ) inequality, and allow triangles with local geodesics as edges. (3) Lemma 2.2 slightly refines standard results on deformations of geodesics. (4) Our formulation of the tangent space is equivalent to other treatments but the phrasing may be different.

Let (X, d) be a metric space. $B_R(x)$ denotes the open R-ball about x. When we speak of points being *within* some $\varepsilon > 0$, we mean that their distance is less than ε.

(X, d)
$B_R(x)$
within

If γ is a continuous function from an interval $I = [a, b]$ to X, then we call γ a *path* from its *initial endpoint* $\gamma(a)$ to its *final endpoint* $\gamma(b)$. Its *length* $\ell(\gamma)$ is

$$\ell(\gamma) := \sup \sum_{i=1}^{n} d\big(\gamma(t_{i-1}), \gamma(t_i)\big) \in [0, \infty]$$

path
initial endpoint
final endpoint
length
$\ell(\gamma)$

where the supremum is over all finite sequences $a = t_0 \leq \cdots \leq t_n = b$. We say γ has *speed* $\leq \sigma$ if the restriction of γ to any subinterval has length at most σ times the length of the subinterval. X is called a *length space* and d a *path metric* if the distance between any two points is the infimum of the lengths of the paths joining them.

speed
length space
path metric

We call γ a *geodesic* of speed $\sigma \geq 0$ if $d\big(\gamma(s), \gamma(t)\big) = \sigma \cdot d(s, t)$ for all $s, t \in I$. We call γ a *local geodesic* of speed σ if this holds locally on I. We call γ a geodesic if it is a geodesic of some speed, and similarly for local geodesics. X is a *geodesic space* if any two of its points can be joined by a geodesic. Often it is convenient to be sloppy and forget the parameterization, identifying a geodesic or even a local geodesic with its image. When there is no ambiguity about which local geodesic is intended, we will often indicate it by specifying its endpoints, for example $\overline{xy}$. A subset $C \subseteq X$ is called *convex* if any two of its points

geodesic
local geodesic
geodesic space
$\overline{xy}$
convex

may be joined by a geodesic in X and every geodesic joining them lies in C.

A *triangle* T in X with vertices x, y, z means a choice of local geo--desics $\overline{xy}$, $\overline{yz}$, $\overline{zx}$ joining them in pairs. We call these the *edges* of T, and call T a *geodesic triangle* if they are geodesics. Most references discuss only geodesic triangles, but at times we will be trying to show that a given local geodesic is actually a geodesic, and the more general formulation will be useful. When there is no ambiguity about which edges are intended, we will sometimes specify T by naming its vertices.

triangle
edge
geodesic triangle

Now suppose $\chi \leq 0$ and let X_χ be the complete connected simply-connected surface of constant curvature χ—the Euclidean or hyperbolic plane if $\chi = 0$ or -1. A *comparison triangle* T' for T means a geodesic triangle in X_χ whose edges $\overline{x'y'}$, $\overline{y'z'}$ and $\overline{z'x'}$ have the same lengths as those of T. A comparison triangle exists if and only if the lengths of T's edges satisfy the triangle inequality, and in this case T' is unique up to isometry of X_χ. In particular, every geodesic triangle has a comparison triangle.

χ
X_χ
comparison triangle

Suppose that T has a comparison triangle. Then to each point of an edge of T', there is a corresponding point on the corresponding edge of T. (This correspondence is usually formulated in the other direction; we do it this way since the edges of T may cross themselves.) We say that T satisfies the *CAT(χ) inequality* if for any two edges of T' and points v', w' on them, the inequality

CAT(χ) inequality

$$d_X(v, w) \leq d_{X_\chi}(v', w') \tag{2.1}$$

holds, where v and w are the corresponding points of the corresponding edges of T. We say X is a *CAT(χ) space* if it is geodesic and every geodesic triangle satisfies this inequality. We say X is *locally CAT(χ)*, or has *curvature* $\leq \chi$, if each point has a neighborhood which is CAT(χ).

CAT(χ) space
locally CAT(χ)
curvature $\leq \chi$

This definition is sometimes called the "strong" form of the CAT(χ) inequality, because some treatments restrict one of v, w to be a vertex of the triangle. It is well-known that all triangles satisfy the strong form if and only if all triangles satisfy the weak one [19, ch. 3]. We prefer the strong form because it admits the Alexandrov subdivision lemma.

The Cartan-Hadamard theorem for CAT(χ) spaces asserts that if X is complete, simply connected and locally CAT(χ) then it is CAT(χ), hence contractible [8, chap. II.4]. This is the main reason we care about CAT(χ) spaces.

It would be too much to ask for all triangles to behave like geodesic triangles. But Alexandrov's subdivision lemma still holds, with the same proof. See [8, lemma I.2.16].

Lemma 2.1 (Alexandrov Subdivision). *Let T be a triangle in a metric space, with vertices x, y and z. Suppose w is a point of $\overline{yz}$, that $\overline{xw}$ is a local geodesic, and that both triangles xwy and xwz have comparison triangles and satisfy CAT(χ). Then T also has a comparison triangle and satisfies CAT(χ).* □

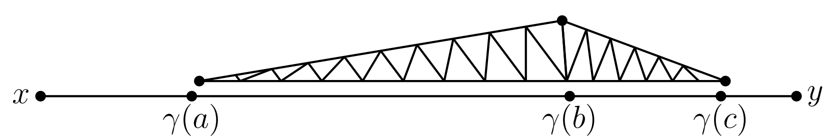
The following is a slight extension of standard results about deforming local geodesics.

Lemma 2.2. *Let X be a metric space, $x, y \in X$, and $\gamma : [0,1] \to X$ a local geodesic from x to y. Suppose $R > 0$ is such that the closed R-ball around each point of γ is complete and CAT(χ). Then*

(a) *for all $x_0 \in B_{R/2}(x)$ and $y_0 \in B_{R/2}(y)$, there exists a unique local geodesic γ_0 from x_0 to y_0 that is uniformly within $R/2$ of γ;*
(b) *γ_0 minimizes length among all paths from x_0 to y_0 that are uniformly within $R/2$ of γ;*
(c) *if $x_1 \in B_{R/2}(x)$ and $y_1 \in B_{R/2}(y)$, then $t \mapsto d(\gamma_0(t), \gamma_1(t))$ is convex. In particular, $d(\gamma_0(t), \gamma_1(t)) \le \max\{d(x_0, x_1), d(y_0, y_1)\}$ for all t.*

Also, suppose $0 \le a \le b \le c \le 1$ and that T is a triangle whose edges are uniformly within $R/2$ of the restrictions of γ to $[a, b]$, $[b, c]$ and $[a, c]$. Then T admits a comparison triangle and satisfies CAT(χ).

Proof sketch. The final claim is a consequence of repeated use of Alexandrov's lemma. One should think of T as a "long thin triangle" and subdivide it as suggested by the following figure:



The hard part of the lemma is the existence of γ_0, which is proven in [8, lemma II.4.3] and in [1, thm. 2]. Specifically, if $x_0 \in B_{R/2}(x)$ and $y_0 \in B_{R/2}(y)$ then there is a unique local geodesic $\gamma_0 : [0,1] \to X$ from x_0 to y_0 for which the function $t \mapsto d(\gamma(t), \gamma_0(t))$ is convex. This implies existence in (a).

Now suppose β_0 and β_1 are any two local geodesics that are uniformly within $R/2$ of γ. By cutting the quadrilateral with vertices $\beta_0(0)$, $\beta_0(1)$, $\beta_1(0)$ and $\beta_1(1)$ into two long thin triangles one can show that $t \mapsto d(\beta_0(t), \beta_1(t))$ is convex. The uniqueness part of (a) follows. To prove (c) one just takes $\beta_i = \gamma_i$. The length-minimizing property of γ_0 can also be proven by using long thin triangles. □

A *geodesic-germ* at $x \in X$ means an equivalence class of geodesics [0, ε>0] $\to X$ with initial endpoint x, where two such are equivalent if they coincide as functions on a neighborhood of 0. The constant geodesic at x is allowed. The following function D on pairs of germs is symmetric and satisfies the triangle inequality: geodesic-germ D

$$D(\gamma, \gamma') := \limsup_{t \to 0} \frac{d\bigl(\gamma(t), \gamma'(t)\bigr)}{t} \in [0, \infty). \tag{2.2}$$

For example, if γ and γ' differ only by reparameterization, then their D-distance is the difference between their speeds. If we identify γ and γ' when $D(\gamma, \gamma') = 0$, then the set of equivalence classes forms a metric space, called the *tangent space* T_xX at x. The basic properties of tangent spaces are developed in [8, II3.18–22]. That formulation is slightly different, and uses the term tangent cone rather than tangent space. But it is easy to convert between our approach and theirs. tangent space T_xX

Positive real numbers act on T_xX by scaling the speeds of geodesic-germs; this scales the metric in the obvious way. If X is a Riemannian manifold then T_xX is the usual tangent space with its Euclidean metric and the standard scaling. We give some other interesting tangent spaces in examples 3.2 and 3.3.

3. The CAT(χ) extension theorem

Let $\chi \leq 0$ be fixed; see section 4 for the positive-curvature case.

Theorem 3.1. *Let X be a complete geodesic space and Δ a nonempty closed convex subset. Assume also:* X Δ

- (A) *every geodesic triangle with a vertex in Δ satisfies CAT(χ);*
- (B) *local geodesics in the metric completion $\overline{T_cX}$ of T_cX are unique, for every $c \in \Delta$; and*
- (C) *there exists $\lambda > 0$ such that for all $x \in X - \Delta$, the closed ball with center x and radius $\lambda \cdot d(x, \Delta)$ is complete and CAT(χ).* λ

Then X is CAT(χ).

Example 3.2. The non-obvious condition is (C), so we illustrate its role. Take X to be the surface of revolution of $y = x^2$, $x \geq 0$, around the x-axis and Δ to be the cusp point. With $\chi = 0$, every hypothesis except (C) holds, yet X is not CAT(0). The key point is that the tangent space at the cusp is a ray, but this forgets too much of the geometry near the cusp. An interesting twist on this example is to take X' equal to the metric completion of the universal cover of $X - \Delta$. The tangent space at the cusp is still just a ray, but now (C) holds and X' is CAT(0).

Example 3.3. This example cautions against weakening hypothesis (A). Let X be a Euclidean half-plane with its boundary crushed to a point. That is, $X = \{0\} \cup \{(x, y) \in \mathbb{R}^2 : x > 0\}$, with

$$d\big((x, y), 0\big) = x$$

$$d\big((x_1, y_1), (x_2, y_2)\big) = \min\Big\{x_1 + x_2,\ \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\Big\}$$

and $\Delta = \{0\}$. With $\chi = 0$, every hypothesis holds except (A), which holds in a weaker form: every triangle with an edge in Δ satisfies CAT(0). Yet X is not CAT(0). Here the tangent space at 0 is the cone on an uncountable discrete set. In a sense this example is the opposite of the previous one, because the metric topology on the space of geodesics emanating from 0 is much finer than the topology of uniform convergence, whereas it was much coarser in example 3.2.

The following lemma contains almost all the content of the theorem.

Lemma 3.4. *There is a unique local geodesic joining any two points of X.*

Proof of theorem 3.1, given lemma 3.4: We must show that every geodesic triangle T satisfies the CAT(χ) inequality. If T has a vertex in Δ then this is hypothesis (A). If an edge of T meets Δ then we use Alexandrov subdivision and reduce to the previous case. If $\overline{xy}$ meets Δ for some vertex x of T and point y of the opposite edge E, then we subdivide along $\overline{xy}$ and reduce to the previous case. In the remaining case we consider the set of geodesics $\overline{xy}$ from x to the points y of E. None of them meets Δ, and therefore (lemma 2.2) each may be varied through a continuous family of local geodesics from x to nearby points of E. By lemma 3.4, these local geodesics are the same as the geodesics we started with. That is, the geodesics $\overline{xy}$ vary continuously with y. Now the Alexandrov patchwork argument [8, fig. 4.2] proves that T satisfies CAT(χ). □

The rest of the section is devoted to proving lemma 3.4. Our first step is to improve hypothesis (A) to apply to all triangles with a vertex in Δ, not just geodesic ones. This also proves a special case of lemma 3.4.

Lemma 3.5. *Every triangle with a vertex in Δ admits a comparison triangle and satisfies CAT(χ). Also, there is a unique local geodesic from any given point of Δ to any given point of X.*

Proof. Suppose $c \in \Delta$ and $x, y \in X$. First we prove a special case of the first claim. If cxy is a triangle such that $\overline{cx}$ and $\overline{cy}$ are geodesics, then one can subdivide $\overline{xy}$ into geodesic segments and join the subdivision points to c by geodesics. Each triangle obtained by subdivision satisfies CAT(χ) by hypothesis (A), and repeated use of Alexandrov's lemma shows that cxy has a comparison triangle and satisfies CAT(χ).

Now we prove the second claim. We must show that if γ is a local geodesic from c to x then it equals $\overline{cx}$. We use the case just proven with $y = c$ and $\overline{cy}$ being the constant path at y. The comparison triangle is then a segment, and the CAT(χ) inequality forces $\gamma = \overline{cx}$. Finally, if cxy is any triangle with a vertex at c, then we have just seen that $\overline{cx}$ and $\overline{cy}$ are geodesics, and the lemma reduces to the special case. □

We will use lemma 3.5 many times, without specific reference. What remains to prove lemma 3.4 is the case $x, y \in X - \Delta$. Before the real work begins there is one more easy case: when x and y are joined by a local geodesic long enough to touch Δ. Define $B(x, y) = \inf_{c\in\Delta}\big(d(x, c) + d(c, y)\big)$, and call any c realizing this infimum a *center* for x and y. Standard arguments [8, prop. II.2.7] show that c exists. (It is also unique, but we won't need this.) $B(x, y)$ center

Lemma 3.6. *If x and y are* far apart*, meaning that they are joined by a local geodesic β of length $\geq B(x, y)$, then that is the unique local geodesic joining them.* far apart

Proof. Applying the CAT(χ) inequality to the triangle $\overline{xc}, \overline{cy}, \beta$ shows that β equals $\overline{xc} \cup \overline{cy}$. This argument applies to any local geodesic of length $\geq B(x, y)$, so β is the unique local geodesic of length $\geq B(x, y)$. The CAT(χ) inequality also implies that there cannot be a local geodesic β' from x to y of length $< B(x, y)$; otherwise $\overline{xc} \cup \overline{cy}$ would fail to be a local geodesic at c □

For the rest of the section we suppose x, y are not far apart and that $c \in \Delta$ is their center. The proof of lemma 3.4 in this case relies on several delicate rescaling arguments. c

For $z \in X$ and $t \in [0, 1]$ we define $t.z$ to be the unique point of $\overline{cz}$ at distance $t \cdot d(c, z)$ from c. For $w, z \in X$ and $t \in (0, 1]$ we define the possibly-degenerate metric $d_t(w, z) = \frac{1}{t}d(t.w, t.z)$. The CAT($\chi$) inequality for triangles with a vertex at c implies that for fixed w and z, $d_t(w, z)$ is nonincreasing as $t \to 0$, so it has a limit $d_0(w, z)$. In fact this limit is $D(\overline{cw}, \overline{cz})$, the distance in T_cX between $\overline{cw}$ and $\overline{cz}$, where both geodesics are parameterized by $[0, 1]$. So the d_t are the link between X and T_cX.

Suppose β is a local geodesic from x to y; ultimately we will show it is the only one. Since x and y are not far apart, β is too short to meet Δ, so it lies in $X - \Delta$, which is locally complete and locally CAT(χ). Therefore deforming β's endpoints along $\overline{cx}$ and $\overline{cy}$ yields a β

deformation of β through local geodesics in $X - \Delta$. Our first result is that this deformation meets no obstructions until the endpoints reach c.

Lemma 3.7. *There is a unique continuous map* $(0,1]\times[0,1] \to X-\Delta$, *which we write* $(t,s) \mapsto \beta_t(s)$, *with* $\beta_1 = \beta$ *and* β_t *a local geodesic from* $t.x$ *to* $t.y$. *Furthermore,* $\beta_t(s)$

(a) $\frac{1}{t}\ell(\beta_t)$ *is nonincreasing as* t *decreases; in particular* $\ell(\beta_t) \leq t\ell(\beta)$.

(b) *Every point of* β_t *lies at distance* $\geq \frac{t}{2}\big(B(x,y) - \ell(\beta)\big) > 0$ *from* Δ.

Proof. Certainly there is some $0 \leq t_0 < 1$ for which the first assertion holds with $(t_0, 1]$ in place of $(0,1]$. We will prove (a) and (b) for $t \in (t_0, 1]$ and then use them to show that we may take $t_0 = 0$.

(a) is essentially a standard fact, but since the triangle cxy is not in a (known) CAT(χ) space and the β_t's are not (known) geodesics, we indicate a proof. Given $t \in (t_0, 1]$, suppose t' is slightly smaller than t. By moving one endpoint of β_t, we obtain a local geodesic from $t'.x$ to $t.y$, and then by moving the other we obtain $\beta_{t'}$. This uses the uniqueness in lemma 2.2. The triangle with vertices c, $t'.x$ and $t'.y$ satisfies CAT(χ) by lemma 3.5. The triangle with vertices $t.x$, $t'.x$ and $t.y$ satisfies CAT(χ) by lemma 2.2, and similarly for the triangle with vertices $t'.x$, $t.y$ and $t'.y$. Assemble the comparison triangles in the obvious way. Then the interior angle at the point corresponding to $t'.x$ must be at least π, or else $\overline{cx}$ would fail to be a geodesic at $t'.x$. And similarly for $t'.y$. This implies $\ell(\beta_{t'}) \leq \frac{t'}{t}\ell(\beta_t)$.

(b) For $t \in (t_0, 1]$, consider the path along $\overline{cx}$ from x to $t.x$, then along β_t, and then along $\overline{cy}$ from $t.y$ to y. Using (a) and $B(x,y) = d(x,c) + d(c,y)$, its length is bounded above by

$$(1-t)d(x,c)+t\ell(\beta)+(1-t)d(y,c) = B(x,y)-t\big(B(x,y)-\ell(\beta)\big). \tag{3.1}$$

Since every path from x to y that meets Δ has length $\geq B(x,y)$, every point of β_t lies at distance at least $\frac{t}{2}\big(B(x,y)-\ell(\beta)\big)$ from Δ, as desired. Also, $B(x,y) - \ell(\beta) > 0$ since x and y are not far apart.

Now we show that we may take $t_0 = 0$; suppose $t_0 > 0$. The β_t are uniformly Cauchy (lemma 2.2), so they have a limit β_{t_0} in X. Since (3.1) holds for all $\beta_{t>t_0}$, it also holds for β_{t_0}. That is, β_{t_0} lies at distance at least $\frac{t_0}{2}\big(B(x,y) - \ell(\beta)\big) > 0$ from Δ, so it is covered by CAT(χ) neighborhoods. In each of these β_{t_0} is a limit of geodesics, so β_{t_0} is a local geodesic. So we can continue the deformation by deforming β_{t_0}. It follows that we may take $t_0 = 0$. □

Now we define scaled-up versions of the β_t. Because we haven't assumed any sort of extendibility of geodesics, we must do the scaling in T_cX. There is a natural projection $\pi : X \to T_cX$, defined by π assigning each $z \in X$ to the germ of the geodesic $[0,1] \to X$ from c to z. (Caution: π may be discontinuous, such as in example 3.3.) For $t \in (0,1]$ we define $\gamma_t : [0,1] \to T_cX$ to be β_t, followed by π, followed γ_t by scaling up by $1/t$. It follows from (A) that the γ_t are continuous, but we won't actually use this.

Our next goal is lemma 3.9, which yields a limit γ_0 of the γ_t's. We will need the following tool for showing that a family of curves are uniformly close when one of them is a local geodesic and the others are not much faster than it. This lemma is independent of the hypotheses of theorem 3.1.

Lemma 3.8. *Let $L \geq 0$, $R > 0$ and $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that the following holds. Suppose X is a metric space and $\alpha : [0,1] \to X$ is a local geodesic of length $\leq L$, such that the closed R-ball around every point of α is complete and CAT(0). If α_u is a continuous variation of α through paths $[0,1] \to X$ of speed $\leq \ell(\alpha) + \delta$, with the same endpoints, then each α_u is uniformly within ε of α.*

Furthermore, if one scales L, R and ε by a positive number then one may scale δ by the same factor.

Proof. Suppose L, R and ε are given and without loss of generality assume $\varepsilon < R/2$. It is easy to prove an analogue of the lemma for $X = \mathbb{R}^2$: there exists $\delta > 0$ such that for any geodesic $\alpha' : [0,1] \to \mathbb{R}^2$ of length $\leq L$ and any path α'_u of speed $\leq \ell(\alpha') + \delta$ having the same endpoints, α'_u and α' are uniformly within ε. The only care required is that one must know L before choosing δ. We will prove that this δ satisfies the lemma.

So suppose X, α, α_u are as stated, with u varying over $[0,1]$ and $\alpha_0 = \alpha$. Write a and c for the common endpoints of all these paths. Let I be the set of u for which α_u is uniformly within ε of α. This is open, so by connectedness it will suffice to show that $[0,u) \subseteq I$ implies $u \in I$. So let $s \in [0,1]$ and write b for $\alpha_u(s)$. We must show $d(b, \alpha(s)) < \varepsilon$.

Now, α_u is uniformly at most $\varepsilon < R/2$ from α, since it is a limit of paths with this property. Let $\overline{ab}$ (resp. $\overline{bc}$) be the unique geodesic from a to b (resp. b to c) that is uniformly within $R/2$ of $\alpha|_{[0,s]}$ (resp. $\alpha|_{[s,1]}$). These exist by lemma 2.2, which also tells us that $\overline{ab}$ and $\overline{bc}$ are no longer than the corresponding parts of α_u, and that the triangle $\overline{ab}$, $\overline{bc}$, α satisfies CAT(0). Let a', b', c' be the vertices of the comparison triangle. Note that

$$d(a',b') = \ell(\overline{ab}) \le \ell(\alpha_u|_{[0,s]}) \le s(\ell(\alpha)+\delta) = s(\ell(\alpha')+\delta)$$

and similarly $d(b',c') \le (1-s)(\ell(\alpha')+\delta)$. Therefore there is a path $\alpha'_u : [0,1] \to \mathbb{R}^2$ from a' to c' of speed $\le \ell(\alpha')+\delta$, with $\alpha'_u(s) = b'$. Then we have

$$d(b,\alpha(s)) \le d(b',\alpha'(s)) < \varepsilon,$$

the first step by the CAT(0) inequality and the second by the choice of δ.

We have proven that δ has the property in the statement of the lemma. The final claim of the lemma follows by scaling all distances. □

The conceptual content of the next lemma is its conclusion (b), that the γ_t have a limit γ_0. But the technical content is its conclusion (a), that the β_t are "uniformly Cauchy after rescaling". Several constants appear in the proof that we will need again later, so we define them here. We set $L = \ell(\beta)$ and $L_0 = \lim_{t\to 0} \frac{1}{t}\ell(\beta_t)$, which exists by lemma 3.7(a). Also, we define $R = \frac{\lambda}{2}\big(B(x,y) - \ell(\beta)\big)$, where λ is from hypothesis (C). The point of this definition is that any point of β_t lies at distance $\ge \frac{t}{2}\big(B(x,y)-\ell(\beta)\big)$ from Δ (lemma 3.7(b)), so the closed tR-ball centered there is complete and CAT(χ) by hypothesis (C).

L

L_0

R

Lemma 3.9.

(a) *If $\varepsilon > 0$ is given, then for all small enough $t > 0$ and all $\mu \in (0,1]$, $\mu.\beta_t$ and $\beta_{\mu t}$ are uniformly within $\mu\varepsilon t$.*

(b) *The functions γ_t converge uniformly to a function $\gamma_0 : [0,1] \to \overline{T_cX}$.*

γ_0

Proof. (a) Suppose $\varepsilon > 0$ is given. Take δ from lemma 3.8, using the values L and R given above. Now suppose t is small enough that

$$L_0 \le \frac{1}{t}\ell(\beta_t) < L_0 + \delta. \tag{3.2}$$

We will prove that for any fixed $\mu \in (0,1]$, $\mu.\beta_t$ is uniformly within $\varepsilon\mu t$ of $\beta_{\mu t}$. To do this we will apply lemma 3.8 to the family of paths $\nu.\beta_{\mu t/\nu}$, $\nu \in [\mu,1]$, which interpolate between $\mu.\beta_t$ ($\nu = \mu$) and $\beta_{\mu t}$ ($\nu = 1$). We regard this as a deformation of $\beta_{\mu t}$, and we will apply the "rescaled" version of lemma 3.8 (i.e., its last assertion).

To apply that lemma we verify (i) $\ell(\beta_{\mu t}) \le \mu t L$, (ii) the closed $\mu t R$-ball around every point of $\beta_{\mu t}$ is complete and CAT(0), and (iii) every path $\nu.\beta_{\mu t/\nu}$ has speed $\le \ell(\beta_{\mu t}) + \delta\mu t$. The first condition holds because $\ell(\beta_{\mu t}) \le \mu t \ell(\beta) = \mu t L$ by lemma 3.7(a). The second condition holds by choice of R, as explained above. To address the third condition, recall that $\beta_{\mu t/\nu}$ is a local geodesic parameterized by $[0,1]$. Therefore its speed is $\ell(\beta_{\mu t/\nu})$. Then the CAT(χ) inequality implies that $\nu.\beta_{\mu t/\nu}$ has speed at most

$$\nu\ell(\beta_{\mu t/\nu}) \leq \nu\frac{\mu}{\nu}\ell(\beta_t) = \mu t\frac{\ell(\beta_t)}{t} < \mu t\Big(\frac{\ell(\beta_{\mu t})}{\mu t} + \delta\Big) = \ell(\beta_{\mu t}) + \delta\mu t,$$

as desired. The first step uses lemma 3.7(a) and the third uses (3.2). We have verified the hypotheses of lemma 3.8, so we deduce that all members of the deformation, in particularly $\mu.\beta_t$, are uniformly within $\mu t\varepsilon$ of $\beta_{\mu t}$.

(b) We will prove the γ_t uniformly Cauchy as $t \to 0$. Suppose $\varepsilon > 0$ is given, that t is small enough to satisfy (3.2), and $\mu \in (0,1]$. Then for any $s \in [0,1]$,

$$\begin{aligned} D\big(\gamma_t(s), \gamma_{\mu t}(s)\big) &= D\big(\tfrac{1}{t}\,\pi \circ \beta_t(s), \tfrac{1}{\mu t}\,\pi \circ \beta_{\mu t}(s)\big) \\ &= D\big(\tfrac{1}{\mu t}\,\pi(\mu.\beta_t(s)), \tfrac{1}{\mu t}\,\pi \circ \beta_{\mu t}(s)\big) \\ &= \tfrac{1}{\mu t}D\big(\pi(\mu.\beta_t(s)), \pi \circ \beta_{\mu t}(s)\big) \\ &= \tfrac{1}{\mu t}d_0\big(\mu.\beta_t(s), \beta_{\mu t}(s)\big) \\ &\leq \tfrac{1}{\mu t}d\big(\mu.\beta_t(s), \beta_{\mu t}(s)\big) < \tfrac{1}{\mu t}\mu t\varepsilon = \varepsilon. \end{aligned}$$

□

Our next goal is that γ_0 is a local geodesic. Note that the $\gamma_{t>0}$ are usually not geodesics, as simple examples show.

Lemma 3.10. *γ_0 is a local geodesic.*

Proof. If $0 \leq s < s' \leq 1$ are within R/L of each other then we call $\beta_t|_{[s,s']}$ a short segment of β_t. Here R and L are as before lemma 3.9, and we assume $L \neq 0$ because the $L = 0$ case of the lemma is trivial. The importance of short segments is that the speed of β_t is at most tL, so a short segment of β_t lies in the tR-ball around each of its points, which is CAT(χ) by hypothesis. Since β_t is a local geodesic, its short segments are geodesics. We will show that each short segment of γ_0 is a geodesic, which proves the lemma.

So suppose $s' - s < R/L$. For $\varepsilon > 0$, choose $t > 0$ small enough that γ_t is uniformly within ε of γ_0. By shrinking t, we may also suppose that the conclusion of lemma 3.9(a) holds. In the following calculation we allow μ but not t to vary. Given $s = s_0 < \cdots < s_m = s'$, we first use the convergence $\gamma_{\mu t} \to \gamma_0$:

$$\sum_{i=1}^{m} D\big(\gamma_0(s_{i-1}), \gamma_0(s_i)\big) = \lim_{\mu\to 0} \sum_{i=1}^{m} D\big(\gamma_{\mu t}(s_{i-1}), \gamma_{\mu t}(s_i)\big)$$

then rescaling in T_cX:

$$= \lim_{\mu\to 0} \frac{1}{\mu t} \sum_{i=1}^{m} d_0\big(\beta_{\mu t}(s_{i-1}), \beta_{\mu t}(s_i)\big)$$

then $d_0 \leq d$:

$$\leq \lim_{\mu\to 0} \frac{1}{\mu t} \sum_{i=1}^{m} d\big(\beta_{\mu t}(s_{i-1}), \beta_{\mu t}(s_i)\big)$$

then the fact that $\beta_{\mu t}|_{[s,s']}$ is a geodesic:

$$= \lim_{\mu\to 0} \frac{1}{\mu t} d\big(\beta_{\mu t}(s), \beta_{\mu t}(s')\big)$$

and then lemma 3.9(a):

$$\begin{aligned} &\leq \lim_{\mu\to 0} \frac{1}{\mu t}\Big(d\big(\mu.\beta_t(s), \mu.\beta_t(s')\big) + 2\mu t\varepsilon\Big) \\ &= 2\varepsilon + \frac{1}{t} d_0\big(\beta_t(s), \beta_t(s')\big) \\ &= 2\varepsilon + D\big(\gamma_t(s), \gamma_t(s')\big) \\ &< 4\varepsilon + D\big(\gamma_0(s), \gamma_0(s')\big). \end{aligned}$$

Now, $\ell_D\big(\gamma_0|_{[s,s']}\big)$ is the supremum of the left hand side over all choices of $s_0, \ldots, s_m$, so it is bounded above by the right hand side. Since this holds for all $\varepsilon > 0$, $\ell_D\big(\gamma_0|_{[s,s']}\big)$ is bounded above by $D\big(\gamma_0(s), \gamma_0(s')\big)$.

This shows that $\gamma_0|_{[s,s']}$ is a local geodesic except perhaps for its parameterization. Since this holds for all short segments, it is easy to see that the parameterization has constant speed, so γ_0 is a local geodesic. □

Lemma 3.11. *Suppose β and β' are local geodesics from x to y. Given $\varepsilon > 0$, there exist $t, \mu \in (0, 1]$ such that $\beta_{\mu t}$ and $\beta'_{\mu t}$ are uniformly within $4\mu t\varepsilon$.*

Proof. Applying the constructions beginning with lemma 3.7 to β' as we did to β, we obtain another local geodesic $\gamma'_0 \subseteq \overline{T_cX}$ from $\pi(x)$ to $\pi(y)$. By hypothesis (B), this coincides with γ_0. Now suppose $\varepsilon > 0$. First we choose $t > 0$ small enough that γ_t and γ'_t are uniformly within ε, and such that the conclusion of lemma 3.9(a) holds for both β and β'. Then we choose μ small enough that d_μ is uniformly within $t\varepsilon$ of d_0 on $\beta_t \cup \beta'_t$. (The uniform convergence $d_\mu \to d_0$ on compact sets in elementary.) Now we suppose $s \in [0,1]$ and apply lemma 3.9(a):

$$d\big(\beta_{\mu t}(s), \beta'_{\mu t}(s)\big) < 2t\mu\varepsilon + d\big(\mu.\beta_t(s), \mu.\beta'_t(s)\big)$$

then the definition of d_μ:

$$= 2t\mu\varepsilon + \mu d_\mu\big(\beta_t(s), \beta'_t(s)\big)$$

and then $d_\mu \approx d_0$:

$$\begin{aligned}
&< 2t\mu\varepsilon + \mu\Big(t\varepsilon + d_0\big(\beta_t(s), \beta'_t(s')\big)\Big)\\
&= 3t\mu\varepsilon + \mu D\big(\pi \circ \beta_t(s), \pi \circ \beta'_t(s)\big)\\
&= 3t\mu\varepsilon + \mu t D\big(\gamma_t(s), \gamma'_t(s)\big)\\
&< 4t\mu\varepsilon.
\end{aligned}$$

□

Conclusion of the proof of lemma 3.4: Suppose $x, y \in X - \Delta$ are not far apart and that β, β' are local geodesics from x to y. We must show $\beta = \beta'$. We apply lemma 3.11 with $\varepsilon = R/8$, so $\beta_{\mu t}$ and $\beta'_{\mu t}$ are uniformly within $\mu t R/2$, for some $\mu, t \in (0,1]$. By our choice of R, the closed $\mu t R$-ball around each point of $\beta_{\mu t}$ is CAT(χ). So lemma 2.2 says that there is a unique local geodesic from $\mu t.x$ to $\mu t.y$ that is uniformly within $\mu t R/2$ of $\beta_{\mu t}$. This implies $\beta'_{\mu t} = \beta_{\mu t}$.

We obtained $\beta_{\mu t}$ from β by deforming β through a family of local geodesics in $X - \Delta$, and we can reverse this deformation to recover β from $\beta_{\mu t}$. And similarly for β'. Since the deformation is unique given the motion of the endpoints, $\beta'_{\mu t} = \beta_{\mu t}$ implies $\beta' = \beta$, finishing the proof. □

4. The Positive-Curvature Case

In this section, we give the positive-curvature analogue of theorem 3.1. Happily, it follows from the non-positive-curvature case.

If $\chi > 0$ then X_χ is the sphere of radius $1/\sqrt{\chi}$. If T is a geodesic triangle in a metric space, then it need not have a comparison triangle T' in X_χ, and if it does then T' need not be unique. However, if T has perimeter $<$ circum $X_\chi = 2\pi/\sqrt{\chi}$, then T' exists and is unique. In this case we say that T satisfies CAT(χ) if (2.1) holds. We call a geodesic space CAT(χ) if every geodesic triangle of perimeter $<$ circum X_χ satisfies CAT(χ).

Theorem 4.1. *Suppose $\chi > 0$ and that X and Δ are as in theorem 3.1. Assume also that all of X lies within $R < \frac{1}{4}$ circum X_χ of some fixed point of Δ. Then X is CAT(χ).*

Proof. By scaling, it suffices to treat the case $\chi = 1$ (in which case $R < \pi/2$). By the positive-curvature form of the Cartan-Hadamard theorem [2, thm. 4.3], it suffices to show that X is locally CAT(1). This uses our hypothesis about X lying in the R-ball around a point of Δ. (We need a point in Δ, not just X, so that our hypothesis (A) implies the radial uniqueness hypothesis of [2].)

We can convert this into a CAT(0) problem by defining CX as the Euclidean cone on X, with vertex say v, and $C\Delta \subseteq CX$ to be the cone on Δ. We regard X as a subset of CX, namely the unit sphere around v. The given metric on X may be recovered as the path metric induced on it by the restriction of CX's metric.

The basic properties of Euclidean cones appear in [8]. In particular, [8, theorem I.5.10] states that a geodesic in CX between $t_1.x_1$ and $t_2.x_2$ ($t_1, t_2 > 0$, $x_1, x_2 \in X$) misses v and may be projected radially to X, yielding a geodesic in X. (This uses $\operatorname{diam} X < \pi$.) This establishes a bijection between the geodesics of CX from $t_1.x_1$ to $t_2.x_2$ and the geodesics of X from x_1 to x_2. Also, suppose T is a geodesic triangle in $CX - \{v\}$ whose radial projection to X has perimeter $< 2\pi$. Then T satisfies CAT(0) if and only if its radial projection satisfies CAT(1). (See the proof of [8, thm II.3.18].) Therefore, proving X locally CAT(1) is equivalent to proving $CX - \{v\}$ locally CAT(0).

This lets us apply theorem 3.1. Clearly CX is locally CAT(0) away from $C\Delta$. Now, an element of $C\Delta - \{v\}$ has the form $t.c$ with $t > 0$ and $c \in \Delta$. We choose a small closed ball X_0 around $t.c$ in such a way that theorem 3.1 applies to X_0 and $\Delta_0 := X_0 \cap C\Delta$. Indeed, a closed ball of any radius $< t$ will do; we now check the hypotheses of theorem 3.1. The completeness of X_0 uses the completeness of CX (see [8, prop. I.5.9]). That X_0 is a geodesic space and Δ_0 is convex follow from the correspondences between geodesics in CX and in X. Hypotheses (A) and (C) follow immediately from the corresponding hypotheses on X. Hypothesis (B) follows from the corresponding hypothesis on X, together with the observation that each tangent space to $CX - \{v\}$ is $\mathbb{R}$ times a tangent space to X. So X_0 is CAT(0). □

5. Branched Covers of Riemannian Manifolds: Local Properties

Suppose M is a complete Riemannian manifold with sectional curvature $\leq \chi \in \mathbb{R}$ and $\Delta \subseteq M$ is locally the union of finitely many complete totally geodesic submanifolds of codimension 2. We equip any covering space M_0' of $M_0 := M - \Delta$, with its natural path metric and complete it to obtain a metric space M'. We call $\pi : M' \to M$ the

M Δ M_0' M_0 M' π

branched cover associated to $M_0' \to M_0$. In the special case that M is connected and M_0' is the universal cover of M_0 then we call M' the *universal branched cover* of M over Δ. Here is the main result of this section: branched cover universal branched cover

Theorem 5.1. *M' has curvature $\leq \chi$ if and only if each tangent space $T_{x'}M'$ is CAT(0).*

We remark that Δ could be more general, for example the branch loci considered by Charney and Davis [12]. The current generality is enough for our applications.

In this section and the next a prime indicates an object in the branched cover, for example, Δ' means $\pi^{-1}(\Delta)$. M is naturally stratified by Δ, and we write M_i for the stratum of codimension i. This extends our notation M_0. We write M_i' for $\pi^{-1}(M_i)$. It is easy to see that each $M_i' \to M_i$ is a covering map, which we will use implicitly whenever we lift paths from M to M'. M_i M_i'

Applying theorem 5.1 requires understanding the tangent spaces $T_{x'}M'$, and the obvious result holds: $T_{x'}M'$ is a branched cover of T_xM. This shows that the question of its CAT(0)-ness is essentially a problem in piecewise-Euclidean geometry. This is the point of the theorem: to reduce a local curvature condition to an infinitesimal one.

To formulate this precisely, let $x' \in M'$, set $x = \pi(x')$ and suppose $r > 0$ is small enough that the exponential map identifies $B_r(0) \subseteq T_xM$ with $B_r(x)$ and $B_r(0) \cap T_x\Delta$ with $B_r(x) \cap \Delta$. Then the covering map $B_r(x') - \Delta' \to B_r(x) - \Delta$ corresponds to a covering space of $B_r(0) - T_x\Delta$ and therefore to a covering space of $T_xM - T_x\Delta$. We call this last the covering space of $T_xM - T_x\Delta$ at x', and its metric completion the branched covering of T_xM at x'.

Lemma 5.2. *$T_{x'}M'$ is the branched covering of T_xM over $T_x\Delta$ at x'.* □

We omit the proof because is an easy application of the stratum-wise covering space property.

Proof of theorem 5.1: The "only if" assertion is just the fact that a metric space with curvature bounded above has CAT(0) tangent spaces [8, thm. II.3.19]. So we assume all tangent spaces are CAT(0) and prove M' locally CAT(χ). For $x \in M$ define $r(x)$ as the supremum of all r such that the following hold: $r(x)$

(i) The exponential map identifies $B_{3r}(0) \subseteq T_xM$ with $B_{3r}(x) \subseteq M$ and $B_{3r}(0) \cap T_x\Delta$ with $B_{3r}(x) \cap \Delta$;

(ii) $B_{3r}(x)$ is convex in M;

(iii) if $\chi > 0$ then $r < \frac{1}{5} \operatorname{circum} X_\chi$.

We define D_x as $\overline{B_{r(x)}(x)}$, and for $x' \in M'$ lying over x we define $D'_{x'}$ as $\overline{B_{r(x)}(x')}$. Obviously it will suffice to prove the following for all i.

Claim $\mathcal{C}_i$: for all $x' \in M'_i$, $D'_{x'}$ is CAT(χ). Claim $\mathcal{C}_i$

We prove this by induction of i. In the base case $i = 0$ we are asserting that $\overline{B_{r(x)}(x')}$ is CAT(χ) for any $x' \in M' - \Delta$, which holds because it projects isometrically to $\overline{B_{r(x)}(x)} \subseteq M$, which is CAT($\chi$) because M has sectional curvature $\leq \chi$. See [8, thm. II.1A.6].

For the inductive step, fix $x' \in M'_i$. The rest of the proof will address $D'_{x'}$, so we will write D' for it and D'_j for $D' \cap M'_j$ and similarly for D and D_j. We will prove D' is CAT(χ) by applying theorem 3.1 with $X = D'$ and $\Delta = D'_i$. To do this we must verify the assumptions of that theorem. (If $\chi > 0$ then we use theorem 4.1 in place of theorem 3.1. This is the reason for (iii) above, which implies $r(x) \leq \frac{1}{5} \operatorname{circum} X_\chi$.) D' D'_j D D_j

First, D' is complete because it is closed in M'. It is a geodesic space because M' is (theorem 6.1, whose proof is independent of the current theorem) and D' is convex in M'. To see this convexity, suppose two points y', z' of it are joined by a geodesic γ' in M'. Since $\ell(\gamma') \leq 2r(x)$, γ' lies entirely in $B_{3r}(x')$, so it projects into $B_{3r}(x)$. If $\pi(\gamma')$ leaves D then we can shorten it by homotoping it (rel endpoints) along geodesics toward x. Because of the correspondence between $T_x\Delta$ and Δ, this homotopy respects strata, so it lifts to a homotopy from γ' to a shorter path from y' to z'. This is absurd, so $\pi(\gamma')$ lies in D, so γ' lies in D', so D' is convex in M'. A similar argument shows that D'_i is convex in D'.

To prove that condition (A) of theorem 3.1 holds we follow [12, lemmas I.5.5–6]. Suppose T is a geodesic triangle in D' with one vertex in D'_i, and call the opposite edge E. If E lies entirely in one stratum, then we may subdivide it so that each triangle in the corresponding subdivision of T projects isometrically into M. Then Alexandrov's lemma shows that T satisfies CAT(χ). Taking limits treats the case in which E lies in one stratum except for its endpoints. Then another use of Alexandrov's lemma treats the general case. Condition (B) holds because all tangent spaces are complete (lemma 5.2) and we are assuming they are CAT(0).

The real content of the proof is verifying hypothesis (C) of theorem 3.1. For convenience we write $|y'|$ for $d(y', D'_i)$ when $y' \in D'$, and similarly for $y \in D$. We must exhibit $\lambda > 0$ such that $\overline{B_{\lambda|y'|}(y')}$ is CAT(χ) for all $y' \in D' - D'_i$. By our induction hypothesis we know $\overline{B_{r(y)}(y')}$ is CAT(χ) for all $y \in D' - D'_i$, but this is not immediately useful. The difficulty is that y' can be far away from D'_i, yet very close $|y'|$

to a stratum of lower dimension than the one containing y'. Then $r(y)$ is much smaller than $|y|$. To deal with this situation we observe that such a y' is very close to a point z' of this lower-dimensional stratum, around which we will prove by induction on stratum dimension that there is a fairly large CAT(χ) ball. Since y' is so close to z', it follows that a smaller but still fairly large ball around y' is also CAT(χ). Our precise version of this idea is better stated and proven in D rather than D':

Claim: Given $j < i$ and $\lambda > 0$, there exists $\lambda' > 0$ such that every $y \in D_j$ either has $r(y) \geq \lambda'|y|$ or else lies in

$$U := \bigcup_{z \in Z} B_{\lambda|z|}(z) \quad \textit{where} \quad Z = \bigcup_{j<k<i} D_k.$$

This is easy to prove with T_xM and $T_x\Delta$ in place of D and $D \cap \Delta$, as follows. The function $y \mapsto r(y)/|y|$ is not continuous, but its restriction to each stratum is. Let K be the unit sphere in $(T_xM_i)^\perp \subseteq T_xM$, minus its intersection with U. Let $\lambda' > 0$ be a lower bound for the restriction of $r(y)/|y|$ to the j-dimensional stratum in K, which exists by continuity and compactness. So $r(y) \geq \lambda'|y|$ holds for all $y \in K$. It follows that $r(y) \geq \lambda'|y|$ for all y in the j-dimensional stratum of $T_xM - U$, because $r(y)/|y|$ is invariant under translation in the T_xM_i direction and scaling in the $(T_xM_i)^\perp$ direction. (U is also invariant under these transformations.) This proves the claim with T_xM and $T_x\Delta$ in place of D and $D \cap \Delta$. The actual claim follows because the exponential map $\overline{B_{r(x)}(0)} \to D_x$ is bilipschitz.

Now we are ready to prove that hypothesis (C) of theorem 3.1 holds. We use another induction, to be proven by descending induction on $j = i-1, \ldots, 0$. Since $j < i$ we may assume claim $\mathcal{C}_j$ is known. Claim $\mathcal{D}_0$ is exactly the hypothesis (C) we want to verify.

Claim $\mathcal{D}_j$ ($j = i-1, \ldots, 0$): there exists $\lambda_j > 0$ such that for all $y' \in D'_j \cup \cdots \cup D'_{i-1}$, $\overline{B_{\lambda_j|y'|}(y')}$ is CAT(χ).

The base case is the largest $j < i$ for which $D_j \neq \emptyset$. Then the claim just proven states that there is a positive lower bound for $r(y)/|y|$ on D_j, which we take for our λ_j. By claim $\mathcal{C}_j$, $\overline{B_{r(y)}(y')}$ is CAT(χ), and since $\lambda_j|y'| \leq r(y)$, we have proven $\mathcal{D}_j$.

Now for the inductive step; suppose $\mathcal{D}_{j+1}$ is known. Observe that if $y' \in D'_j$ lies within $\frac{1}{3}\lambda_{j+1}|z'|$ of some $z' \in D'_{j+1} \cup \cdots \cup D'_{i-1}$, then

$$|y'| < |z'| + \frac{\lambda_{j+1}}{3}|z'| \leq 2|z'|.$$

(Obviously we may take $\lambda_{j+1} \leq 3$ without loss.) Therefore

$$\overline{B_{\lambda_{j+1}|y'|/3}(y')} \subseteq \overline{B_{2\lambda_{j+1}|z'|/3}(y')} \subseteq \overline{B_{\lambda_{j+1}|z'|}(z')}.$$

The right side is CAT(χ) by the inductive hypothesis $\mathcal{D}_{j+1}$, so the left side is also. Now we apply the claim proven above with $\lambda = \lambda_{j+1}/3$, obtaining λ'. Arguing as in the base case shows that $\overline{B_{\lambda'|y'|}(y')}$ is CAT(χ) for any $y' \in D'_j$ that *doesn't* lie at distance $< \frac{1}{3}\lambda_{j+1}|z'|$ from some $z' \in D'_{j+1} \cup \cdots \cup D'_{i-1}$. So we have proven $\mathcal{D}_j$ with $\lambda_j := \min\{\lambda_{j+1}/3, \lambda'\}$. □

Remark. Theorem 5.3 of [12] is similar to our theorem 5.1. However, there is a difficulty with the proof, which we alluded to in [3] and understand better now. In their notation, they first show in lemmas I.5.5–6 that every $\tilde{x}$ in the branched cover has a neighborhood $\widetilde{U}$ in which every geodesic hinge based at $\tilde{x}$ spreads out. This is equivalent to geodesic triangles in $\widetilde{U}$ with a vertex at $\tilde{x}$ satisfying CAT(χ). Then they consider a geodesic triangle in $\widetilde{U}$, use the fact that the hinges at each of its corners spread out, and appeal to the equivalence of the hinge-spreading condition with the CAT(χ) condition.

But while the edges at a given corner do diverge in a neighborhood of that corner, they might begin to reconverge while still within $\widetilde{U}$. Another way to say this is that the neighborhoods of the vertices admitting good local descriptions can be much smaller than $\widetilde{U}$. Most of theorem 5.1's proof amounts to wrestling with this issue. No argument using only the spreading of every hinge in a neighborhood of its basepoint can prove the local CAT(χ) property, because it would also prove that our example 3.2 is CAT(0).

6. Branched Covers of Riemannian Manifolds: Global Properties

We continue to use the notation of the previous section, including M_i and M'_i for strata in M and M'. After proving that M' is a geodesic space (theorem 6.1) we develop our approach of studying the universal cover of M_0 by relating it to the universal branched cover. Lemma 6.2 shows that $M'_0 \to M'$ is a homotopy-equivalence under some conditions on the local topology of the branching. The main case of interest is when M'_0 is the universal cover, but really all that is required is that M'_0 be "locally universal". After that we prove our main result on branched covers of Riemannian manifolds, that M'_0 is contractible and M' is CAT(χ) under reasonable hypotheses (theorem 6.3).

Theorem 6.1. *Each component of M' is a geodesic space.*

Proof Sketch. Let $y', z' \in M'$ and let γ'_n be a sequence of paths joining them in M', lying in M'_0 except perhaps for their endpoints, with $\ell(\gamma'_n) \to d(y', z')$. We may suppose they are parameterized proportionally to arclength. Project them to paths γ_n in M, use the Arzelà-Ascoli theorem to pass to a uniformly convergent subsequence, and write γ for the limit. If γ maps some interval entirely into one stratum, then it is a local geodesic in that stratum and remains there until it hits a lower-dimensional stratum. Otherwise some γ_n sufficiently close to γ could be shortened in a way that lifts to γ'_n, leading to an impossibly short path from y' to z'. One can use this to show that every point of γ has a neighborhood in γ that meets ≤ 3 strata. Therefore the domain of γ is covered by finitely many intervals, on the interior of each of which it is a local geodesic and lies entirely in one stratum. Then one can see that for γ_n sufficiently close to γ, we may homotope γ_n to γ, rel endpoints, such that the homotopy maps into M_0 except perhaps for the endpoints of the paths and the final path γ. Using the covering map $M'_0 \to M_0$ and the metric completion $M'_0 \to M'$, we may lift this to a homotopy rel endpoints from γ'_n to some path γ' lying over γ. This is the desired geodesic $\overline{y'z'}$. □

Lemma 6.2. *Suppose that for all $x \in M$,*

(a) *$T_xM - T_x\Delta$ is aspherical, and*
(b) *the covering space of $T_xM - T_x\Delta$ at any preimage of x is a universal covering.*

Then the inclusion $M'_0 \to M'$ is a homotopy equivalence.

Proof. This is a more sophisticated version of the argument for [3, lemma 3.3]. We assume inductively that the lemma is known for manifolds of dimension smaller than $n := \dim M$. The base case $n = 1$ is trivial since Δ is empty. Write Σ_i for $M_0 \cup \cdots \cup M_i$ and similarly for Σ'_i. We claim that each inclusion $\Sigma'_{i-1} \to \Sigma'_i$ is a homotopy equivalence. This proves the lemma because of the chain

$$M' = \Sigma'_n \simeq \Sigma'_{n-1} \simeq \cdots \simeq \Sigma'_0 = M'_0.$$

To prove the claim we must remove M'_i from Σ'_i without changing homotopy type. So let $x' \in M'_i$ lie over $x \in M_i$. We regard $T_x\Delta$ as a subset (a union of linear subspaces) of T_xM, and by restricting to directions orthogonal to T_xM_i we obtain the normal bundle $N_\Delta M_i \subseteq N_M M_i$.

For a smooth function $r : M_i \to (0, \infty)$ we consider the closed-ball-bundle "$\log B$" whose fiber over $x \in M_i$ is the closed ball of radius $r(x)$ in $N_M M_i$. Standard Riemannian geometry shows that we may choose r such that the exponential map sends $\log B$ diffeomorphically to its image $B \subseteq M$ and identifies $(N_\Delta M_i) \cap (\log B)$ with $\Delta \cap B$. For $0 < \alpha \leq 1$ we write αS for the image in M of the sphere-bundle in $N_M M_i$ whose radius at x is $\alpha r(x)$. In particular, $S := 1S$ is the boundary of B in Σ_i.

We write B' and $\alpha S'$ for the preimages of B and αS in Σ_i', which have obvious projection maps to M_i'. Locally this realizes S' as a fiber bundle over M_i'. (The fibers over different components of M_i' may be different.) We claim that every fiber is contractible. To see this, let $S'_{x'}$ be a fiber and S_x its image in M. By hypothesis (b), $S'_{x'} - \Delta'$ is a copy of the universal cover of $S_x - \Delta$, so it is contractible by hypothesis (a). Furthermore, both hypotheses (a) and (b) apply with M and Δ replaced by S_x and $S_x \cap \Delta$, essentially because they are the "restrictions" of these hypotheses on M to Σ_{i-1}. By induction on dimension, $S'_{x'} - \Delta' \to S'_{x'}$ is a homotopy-equivalence, so $S'_{x'}$ is also contractible.

A standard paracompactness argument shows that we may contract all the fibers simultaneously, that is, there is a fiber-preserving homotopy H of S' which contracts each fiber to a point in that fiber. By coning off we regard H also as self-homotopy of B'. From this we derive a new homotopy J of B'. The points of $\alpha S'$ move in $\alpha S'$ along some initial segment of their tracks under H. This initial segment is the whole segment for $\alpha \in [0, 1/2]$ and then decreases to the trivial segment (i.e., the constant homotopy) at $\alpha = 1$. By extending this to the constant homotopy on $\Sigma_i' - B'$ we regard J as a homotopy of Σ_i'.

Note that J collapses a neighborhood of M_i' to a radial interval-bundle over M_i'. Retracting along these radial intervals homotopes M_i' into Σ_{i-1}'. It is easy to see that this gives a homotopy-inverse to the inclusion $\Sigma_{i-1}' \to \Sigma_i'$. □

Theorem 6.3. *Suppose M is connected, $\chi \leq 0$ and that for all $x \in M$,*

(a) *$T_x M - T_x \Delta$ is aspherical, and*

(b) *the universal branched cover of $T_x M$ over $T_x \Delta$ is CAT(0).*

Then the universal cover M_0' of $M - \Delta$ is contractible and its metric completion M' is CAT(χ).

Proof. Without loss of generality we may replace M by its universal cover and Δ by its preimage. Then $M \cong T_x M$ by the exponential map and it is easy to see that $\pi_1(T_x M - T_x \Delta) \to \pi_1(M - \Delta)$ is injective. This verifies hypothesis (b) of lemma 6.2, and the hypothesis (a) of that theorem is the current hypothesis (a). So $M_0' \to M'$ is a homotopy equivalence. Since M_0' is simply connected, so is M'. Theorem 3.1 also applies, so M' is locally CAT(χ). Then the Cartan-Hadamard theorem shows M' is CAT(χ). In particular, it is contractible, and by the homotopy equivalence the same is true of M'_0. □

Remarks. The only reason we assume connectedness is so the universal cover is defined. Also, this proof shows that the "locally a universal cover" hypothesis of lemma 6.2 is automatic if M is nonpositively curved. It is also automatic if $M = S^n$ and Δ is a union of great S^{n-2}'s; one uses the same argument, together with the fact that if a point of S^n lies in Δ then so does its antipode.

We close this section with a result that simplifies the use of theorem 6.3. It immediately implies the main result of [3], namely the case of theorem 6.3 with $T_x\Delta$ locally modeled on the coordinate hyperplanes of $\mathbb{C}^n$ with its usual metric. We will use it in a more substantial way in our treatment of exceptional singularities (see the proof of lemma 8.2).

Lemma 6.4. *Suppose M is a complex manifold equipped with a Hermitian Riemannian metric and $\Delta_1, \ldots, \Delta_k$ are locally finite arrangements of totally geodesic complex hypersurfaces. Suppose also that every intersection of a component of Δ_i with a component of $\Delta_{j\neq i}$ is Hermitian-orthogonal. Then $\Delta = \cup_{i=1}^k \Delta_i$ satisfies hypotheses* (a) *and* (b) *of theorem 6.3 if each Δ_i does.*

Proof. By induction it suffices to treat the case $k = 2$; let $x \in M$. Because of the orthogonality, we have

$$(T_xM - T_x\Delta_1) \times (T_xM - T_x\Delta_2) \cong (T_xM - T_x\Delta) \times \mathbb{C}^{\dim M}.$$

The left side is aspherical by hypothesis so $T_xM - T_x\Delta$ is too. Similarly, the product of the universal branched covers of T_xM over $T_x\Delta_1$ and $T_x\Delta_2$ is isometric to the universal branched cover over $T_xM - T_x\Delta$, again times a trivial factor $\mathbb{C}^{\dim M}$. □

7. Coxeter Arrangements

In this section and the next we apply the machinery we have developed. Although our results are conditional on the following conjecture about finite Coxeter groups, we feel the unification of the problems we address and their reduction to the conjecture is progress in itself. Our main applications are the Arnol′d-Pham-Thom conjecture about $K(\pi, 1)$ spaces for Artin groups (theorem 7.3), the asphericity of the moduli spaces of amply lattice-polarized K3 surfaces (theorem 7.4), and the asphericity of discriminant complements for the 3 kinds of unimodal hypersurface singularities (section 8). Our methods allow a unified attack on all these problems. At the end of this section we make a few remarks on Bridgeland stability conditions for K3s.

Conjecture 7.1. *Let W be a finite Coxeter group acting isometrically on $\mathbb{C}^n$ and let Δ be the union of the hyperplanes fixed by the reflections in W. Then the metric completion of the universal cover of $\mathbb{C}^n - \Delta$ is CAT(0).*

Remarks. (1) Most of the applications require only the ADE cases.

(2) The case of a finite complex reflection group is also interesting, with some applications we omit here.

(3) The case $n = 2$ seems accessible with tools developed by Charney and Davis [12, theorem 9.1] and Panov [36] to address similar problems. Charney and Davis study finite-sheeted branched covers of S^3 over a union of 3 disjoint great circles. Panov studies complex surfaces with singular Euclidean metrics. Both papers use the pullback of the Hopf fibration to the branched cover and convert the problem into one about branched covers of the Hopf fibration's base S^2.

(4) This conjecture is very close to conjecture 3 of Charney and Davis [13]. In particular, we will see in the proof of theorem 7.3 that ours implies theirs. I don't know about the other direction.

Corollary 7.2. *Assume conjecture 7.1. Suppose M is a complete connected Riemannian manifold of nonpositive sectional curvature, and Δ is a union of totally geodesic submanifolds, whose tangent space at any point is isomorphic to the mirror arrangement of some complexified finite Coxeter group. Then the universal cover of $M - \Delta$ is contractible.*

Proof. This is an application of theorem 6.3. Verifying the asphericity of $T_xM - T_x\Delta$ for every $x \in \Delta$ amounts to the asphericity of $\mathbb{C}^n$ minus the mirrors of a finite Coxeter group group. This is a result of Deligne [17]. The CAT(0) hypothesis on the universal branched cover of $T_xM - T_x\Delta$ is exactly the conjecture. (For finite complex reflection groups one could replace Deligne's theorem with one of Bessis [6].) □

Theorem 7.3. *Assume conjecture 7.1. Let W be any Coxeter group, acting on its open Tits cone $C \subseteq \mathbb{R}^n$, and let M be the tangent bundle TC. Let Δ be the union of the tangent bundles to the mirrors of the reflections of W. Then $M - \Delta$ has contractible universal cover.*

The theorem applies to many cones besides the Tits cone, but to state the result in its natural generality one must discuss discrete linear reflection groups á la Vinberg [40]. We refer to Charney and Davis [13] for the more general formulation; we also assume familiarity with this paper in the following proof.

Proof. Charney and Davis [13, p. 601] show that the theorem follows from the claim: the Deligne complex of a finite-type Artin group is

CAT(1). To explain and prove this statement, suppose W is a finite Coxeter group, acting in the usual way on $\mathbb{R}^n$, and set $M := \mathbb{C}^n$ and Δ to be the union of the (complex) mirrors of W. We write M_0 for $M - \Delta$, M_0' for its universal cover, and M' for the metric completion of M_0'. The Artin group associated to W is defined as $\pi_1(M_0/W)$, and is called finite type to reflect the fact that $|W| < \infty$.

Formally, its Deligne complex is a metrized simplicial complex defined in terms of inclusions of cosets of Artin subgroups corresponding to subdiagrams of the Coxeter diagram. More convenient for our purposes is the following: it is the preimage of $S^{n-1} \subseteq \mathbb{R}^n$ in M', with the induced path-metric. To see that this is the same complex one need only check that face stabilizers are the same as in the Deligne complex (which is obvious since the Artin subgroups correspond to the strata of the branch locus). The metric in both cases makes each simplex into a copy of the usual fundamental domain for W in S^{n-1}.

The Euclidean cone on the Deligne complex is clearly the preimage $Y \subseteq M'$ of $\mathbb{R}^n \subseteq M$. By [8, thm. II.3.14], a space is CAT(1) if and only if the cone on it is CAT(0), so it suffices to show that Y is CAT(0).

To prove this we define a distance-nonincreasing retraction $M' \to Y$. Assuming conjecture 7.1, M' is CAT(0), and it then follows that Y is convex in M', hence CAT(0), proving the theorem. To define the retraction, consider the homotopy from $\mathbb{C}^n$ to $\mathbb{R}^n$ given by shrinking the imaginary parts of vectors toward 0. This defines an stratum-preserving "open homotopy" from M into itself, i.e., a continuous map $[0,1) \times M \to M$. Using covering spaces, we can lift this to an open homotopy $[0,1) \times M' \to M'$ with $\{0\} \times M' \to M'$ the identity map. (Properly speaking, one lifts the homotopy on each stratum separately and checks that they fit together to give a homotopy of M'. This is easiest to see by thinking of M_0 as the set of tangent vectors to $\mathbb{R}^n$ that are not tangent to any mirror. Our homotopy shrinks all vectors without moving their basepoints.)

Using metric completeness allows us to to extend this to a homotopy $[0,1] \times M' \to M'$. The result is a deformation retraction from M' to Y. It is distance-nonincreasing because the original homotopy is. □

The next theorem uses the global Torelli theorem for K3 surfaces, together with our corollary 7.2, to show that various moduli spaces of K3 surfaces have contractible orbifold universal covers. Lattice-polarized K3 surfaces were introduced by Nikulin [34] to generalize the classical case of K3 surfaces equipped with a single ample or semi-ample line bundle. They were developed further by Dolgachev [18], to which we refer the reader for details. The background we will need is the following.

Sometimes singular surfaces are called K3s if their minimal resolutions are K3s, but we include smoothness in the definition of a K3 surface. If X is one, then the isometry type of the intersection pairing on $H^2(X;\mathbb{Z})$ is independent of X and isomorphic to the "K3 lattice" $K := E_8^2 \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^3$. The Picard group $\operatorname{Pic} X$ is the sublattice spanned by (the Poincaré duals of) algebraic cycles. Now suppose M is a lattice (=integer bilinear form) of signature $(1, t)$ equipped with a choice of Weyl chamber for the subgroup of $\operatorname{Aut} M$ generated by reflections in the norm -2 vectors of M. An M-polarization of X means a primitive embedding $j : M \to \operatorname{Pic} X$, and j is called ample if the j-image of this chamber contains an ample class. The M-polarized K3s fall into families parameterized by the isometry classes of primitive embeddings $M \to K$, so fix one such embedding.

As concrete examples, when M is spanned by a vector of norm -4 (resp. -2), there is only one embedding $M \to K$ up to isometry. Then the moduli space of amply M-polarized K3s is the same as the moduli space of smooth quartic surfaces in $\mathbb{C}P^3$ (resp. that of the double covers of $\mathbb{C}P^2$ over smooth sextic curves).

Theorem 7.4. *Assume conjecture 7.1. Suppose M is an integer quadratic form of signature $(1, t)$ with a fixed embedding in K. Then the moduli space of amply M-polarized K3 surfaces (X, j), for which the composition $M \to \operatorname{Pic} X \to H^2(X)$ is isomorphic to $M \to K$, has contractible (orbifold) universal cover.*

The proof involves a symmetric space that will also play an important role in the following section. If L is a lattice of signature $(2, n)$ then

$$(7.1)\quad \Omega(L) := \text{a component of } \left\{ x \in L \otimes \mathbb{C} \;\middle|\; x \cdot x = 0 \text{ and } x \cdot \bar{x} > 0 \right\}.$$

$P\Omega(L)$ is the symmetric space for $\mathrm{O}(L \otimes \mathbb{R}) \cong \mathrm{O}(2, n)$. (One can check that $P\Omega(L)$ is the same as the set of positive-definite 2-planes in $L \otimes \mathbb{R}$.) As a symmetric space of noncompact type, its natural Riemannian metric is complete and has nonpositive sectional curvature.

Proof. The global Torelli theorem for lattice-polarized K3 surfaces [18, theorem 3.1] says that the moduli space is covered (as an orbifold) by a certain hyperplane complement in the symmetric space $P\Omega(M^\perp)$ where $M^\perp$ refers to the complement in K. Namely, it is $P\Omega(M^\perp) - \Delta$ where Δ is the union of the orthogonal complements of the norm -2 vectors in $M^\perp \subseteq K$. Because the orthogonal complement of a norm -2 vector is the mirror for the reflection in that vector, and this reflection preserves

$M^\perp$, we see that Δ is locally modeled on the hyperplane arrangements for ADE Coxeter groups. So we can apply corollary 7.2 (which assumes conjecture 7.1). □

I am grateful to D. Huybrechts for pointing me toward a similar situation in Bridgeland's work on stability conditions on K3 surfaces. Bridgeland defined the notion of a stability condition on a triangulated category, and described one component $\mathrm{Stab}^\dagger(X)$ of the space of locally finite numerical stability conditions on the derived category of an algebraic K3 surface X [9, thm. 1.1]. It turns out to be a covering space of $\Omega\bigl(\mathrm{Pic}\,X \oplus \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\bigr) - \mathcal{H}$ where $\mathcal{H}$ is the union of the orthogonal complements of the norm -2 vectors of $\mathrm{Pic}\,X \oplus \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. This situation is exactly like that of theorem 7.4. In particular, our conjecture 7.1 implies that $\mathrm{Stab}^\dagger(X)$ is aspherical. Furthermore, Bridgeland conjectured that this covering is a universal covering [9, conj. 1.2]. If both this and our conjecture 7.1 hold then $\mathrm{Stab}^\dagger(X)$ is contractible.

8. Singularity Theory

For us, a singularity means the germ (X_0, x_0) of a complex space at a singular point, and conceptually its semiuniversal deformation (SUD) is the space of all its smoothings and partial smoothings. The discriminant complement means the set of those that are actually smooth. Assuming conjecture 7.1 we will prove that this is aspherical for all the unimodal hypersurface singularities in Arnol′d's hierarchy.

The precise definitions are as follows. A deformation of (X_0, x_0) means a flat holomorphic map $F : (\mathfrak{X}, x_0) \to (S, s_0)$ of germs of complex spaces, together with an isomorphism of (X_0, x_0) with the fiber over s_0. F is versal if any deformation $F' : (\mathfrak{X}', x_0) \to (S', s_0')$ can be pulled back from it, i.e., if there exist holomorphic maps $\phi : S' \to S$ and $\Phi : \mathfrak{X}' \to \mathfrak{X}$ where $F \circ \Phi = F' \circ \phi$ and Φ respects the identification of the fibers over s_0 and s_0' with X_0. F is called semi-universal if in these circumstances the derivative of ϕ at s_0 is always uniquely determined. Grauert [20] proved that an isolated singularity always admits a SUD and that it is unique up to non-unique isomorphism. So we fix a SUD $F : (\mathfrak{X}, x_0) \to (S, s_0)$ and refer to it as "the" SUD. We will sometimes speak of spaces when we really mean germs.

The discriminant Δ means the subspace of S over which the fibers of $\mathfrak{X}$ are singular. Studying the inclusion $\Delta \to S$ and the topology of $S - \Delta$ has been at the forefront of singularity theory for decades, beginning with Brieskorn's famous result [10]. He proved that for the A_n, D_n and E_n singularities, $\Delta \to S$ is the inclusion of the mirrors of the corresponding Weyl group W into $\mathbb{C}^n$, modulo the action of W. In particular, $S-\Delta$ has $\mathbb{C}^n-$(the mirrors of W) as an unramified covering space. This connection to singularity theory helped move Deligne to prove his theorem [17] on the asphericity of hyperplane complements like this: it implies that $S-\Delta$ is aspherical.

Our goal is to extend Deligne's result to the next level of complexity in Arnol′d's hierarchy of isolated hypersurface singularities: the "unimodal" singularities. Unfortunately, our results are conditional on conjecture 7.1. But we do succeed in unifying the different problems and reducing them to a question about finite Coxeter groups. Our arguments also apply to some other singularities, but we have restricted to the unimodal hypersurface case to avoid complicated statements. There are three flavors of these singularities, all of which occur already for surfaces in $\mathbb{C}^3$. It is standard that we lose nothing by restricting to this dimension [5, p. 184].

First come the "simply elliptic" singularities $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$, using language due to K. Saito [38]. These are

$$\begin{array}{ll} \tilde{E}_6 & y(y-x)(y-\lambda x)+xz^2 \\ \tilde{E}_7 & yx(y-x)(y-\lambda x)+z^2 \\ \tilde{E}_8 & y(y-x^2)(y-\lambda x^2)+z^2 \end{array}$$

which are quasihomogeneous of degrees 3, 4 and 6 with respect to the weights $(1,1,1)$, $(1,1,2)$ and $(1,2,3)$. Here λ must be such that the singularity is isolated. For almost all such values of λ one can change coordinates to obtain the more memorable forms

$$\begin{array}{ll} \tilde{E}_6 & x^3+y^3+z^3+\lambda' xyz \\ \tilde{E}_7 & x^4+y^4+z^2+\lambda' xyz \\ \tilde{E}_8 & x^6+y^3+z^2+\lambda' xyz \end{array}$$

Then come the "cusp singularities"

$$T_{p,q,r} \qquad x^p+y^q+z^r+xyz \tag{8.1}$$

for $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$. Finally there are the "exceptional" singularities in table 1. These represent 28 singularities because the cases $\lambda=0$, $\lambda\neq 0$ are essentially different, while rescaling variables allows one to replace any $\lambda\neq 0$ by $\lambda=1$. When $\lambda=0$, the singularities are quasihomogeneous with the listed weights and degrees. (Note: "unimodal" refers to the existence of 1-parameter families of isomorphism classes of fibers in the SUD, rather than the number of moduli occurring in the description of the singularity itself.)

Now let $f(x,y,z)$ be one of the functions above, and $(X,0)\subseteq(\mathbb{C}^3,0)$ the singularity defined by $f=0$. We refer to [24] for the following standard model of the SUD. Let $\mathcal{O}$ be the ring of germs of convergent

Type	f	weights	degree	Dolgachev numbers
Q_{10}	$x^2z + y^3 + z^4 + \lambda yz^3$	$9, 8, 6$	24	2,3,9
Q_{11}	$x^2z + y^3 + yz^3 + \lambda z^5$	$7, 6, 4$	18	2,4,7
Q_{12}	$x^2z + y^3 + z^5 + \lambda yz^4$	$6, 5, 3$	15	3,3,6
S_{11}	$x^2z + yz^2 + y^4 + \lambda z^3$	$5, 4, 6$	16	2,5,6
S_{12}	$x^2z + yz^2 + xy^3 + \lambda y^5$	$4, 3, 5$	13	3,4,5
U_{12}	$x^3 + y^3 + z^4 + \lambda xyz^2$	$4, 4, 3$	12	4,4,4
Z_{11}	$x^3y + y^5 + z^2 + \lambda xy^4$	$8, 6, 15$	30	2,3,8
Z_{12}	$x^3y + xy^4 + z^2 + \lambda x^4$	$6, 4, 11$	22	2,4,6
Z_{13}	$x^3y + y^6 + z^2 + \lambda x^4$	$5, 3, 9$	18	3,3,5
W_{12}	$x^4 + y^5 + z^2 + \lambda x^2y^3$	$5, 4, 10$	20	2,5,5
W_{13}	$x^4 + xy^4 + z^2 + \lambda x^3y^2$	$4, 3, 8$	16	3,4,4
E_{12}	$x^3 + y^7 + z^2 + \lambda xy^5$	$14, 6, 21$	42	2,3,7
E_{13}	$x^3 + xy^5 + z^2 + \lambda y^8$	$10, 4, 15$	30	2,4,5
E_{14}	$x^3 + y^8 + z^2 + \lambda xy^6$	$8, 3, 12$	24	3,3,4

TABLE 1. Arnol′d's 14 exceptional singularities.

power series on $\mathbb{C}^3$ at 0 and $\mathcal{I}$ be the ideal generated by f and its first partial derivatives. (When f is quasihomogeneous, it lies in the ideal generated by its partial derivatives, so $\mathcal{I}$ is the Jacobian ideal.) Suppose $p_1, \ldots, p_\tau \in \mathcal{O}$ project to a $\mathbb{C}$-basis for $\mathcal{O}/\mathcal{I}$. Then we define (S, s_0) as $(\mathbb{C}^\tau, 0)$, $(\mathfrak{X}, 0)$ as the subspace of $(\mathbb{C}^{\tau+3}, 0)$ defined by

$$f(x, y, z) + \sum_{i=1}^{\tau} t_i\, p_i(x, y, z) = 0 \tag{8.2}$$

and F as the projection that forgets x, y, z.

Theorem 8.1. *Assuming conjecture 7.1, the discriminant complement of a simply elliptic or exceptional hypersurface singularity is aspherical.*

Since the discriminant complement is the complement of one germ inside another, we clarify: we are asserting that $s_0 \in S$ has a basis of neighborhoods, such that the discriminant complement in each is aspherical, and any inclusion of one of these discriminant complements into another is a homotopy equivalence.

Proof. We postpone the case of non-quasihomogeneous exceptional singularities to the end of the proof. Weights and degrees refer to the weighting of variables given above. The central object of the proof is the restriction of F to a certain hypersurface T in S. In every case one can choose $p_1, \ldots, p_\tau$ to be quasihomogeneous, with all but one of them having degree less than that of f. We take the exceptional one to be $p_\tau :=$ the part of f involving λ. We define $T \subseteq S$ by $t_\tau = 0$. It turns out that p_τ is the Hessian of f, up to quotienting by the Jacobian ideal and scaling. Then a theorem of Wirthmüller [41, Satz 3.6] says that F is topologically trivial in the τ'th direction. Precisely: there are self-homeomorphisms Φ of $(\mathbb{C}^{3+\tau}, 0)$ and ϕ of (S, s_0) such that $\phi \circ F = F \circ \Phi$ and $\Phi(\mathfrak{X}) = \mathfrak{X}|_T \times (\mathbb{C}, 0)$. From this we can deduce that $\phi(\Delta) = (\Delta \cap T) \times (\mathbb{C}, 0)$. To see this, note that Δ is the closure of the locus in S over which the fibers of $\mathfrak{X}$ are not topological manifolds. (This formulation circumvents any worries about some fibers being singular as complex hypersurfaces but nonsingular topologically.) This shows that Δ and $\phi(\Delta)$ are determined by the topology of the fibers of F and $F \circ \Phi$, so $\phi(\Delta) = (\Delta \cap T) \times (\mathbb{C}, 0)$. What remains is to prove $T - \Delta$ aspherical.

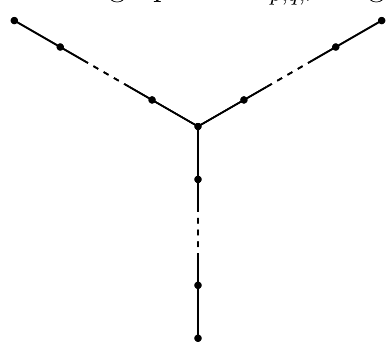
Looijenga analyzed $T - \Delta$ in the simply-elliptic case in [26]. Take L to be the E_6, E_7 or E_8 root lattice, according to the type of the singularity, and W the corresponding finite Weyl group. Let $\Lambda \subseteq \mathbb{C}$ be a lattice for which $\mathbb{C}/\Lambda$ is isomorphic to the exceptional divisor in the minimal resolution of X (which is an elliptic curve, so Λ exists). Both $L \otimes \Lambda \cong \mathbb{Z}^{12,\,14,\,\text{or}\,16}$ and W act on $H := L \otimes \mathbb{C}$, the former by translations and the latter via its action on L. Looijenga defines the "double affine Weyl group" $M := (L \otimes \Lambda) \rtimes W$, and Δ_H as the union of the mirrors of the complex reflections of M. These are just the $L \otimes \Lambda$-translates of W's mirrors. In [27, Rk. 7.10], he shows that $T - \Delta$ has an unramified covering space biholomorphic to $(H - \Delta_H) \times \mathbb{C}^*$.

That is, (T, s_0) has a representative with this property. By Looijenga's description, there is a basis of neighborhoods of s_0 in T whose preimages in $H \times \mathbb{C}^*$ have the form $H \times$ (exterior of a disk centered at 0). So the asphericity of $(T, s_0) - (\Delta, s_0)$ reduces to that of $H - \Delta_H$. We equip H with the standard Euclidean metric. Essentially by definition, Δ_H is locally modeled on finite Coxeter arrangements, so we can apply corollary 7.2, which we recall assumes conjecture 7.1. This finishes the proof in the simply-elliptic case.

In the (still quasihomogeneous) exceptional case, there is again an unramified cover of $T - \Delta$ which is a $\mathbb{C}^*$-bundle over a hyperplane arrangement complement. The details are much more complex, and we need to present certain of them in order to describe the arrangements well enough to apply theorem 6.3. We follow Looijenga [28] for an overview of the construction, which depends essentially on a method of Pinkham [37]. The whole theory is laid out in much more detail and generality in [29], and we will use its description of the hyperplane arrangement rather than the one in [28]. See also Brieskorn's lovely paper [11] for a tour of the ideas. In the notation of [29], the (quasi-homogeneous) exceptional singularities are $D_{p,q,r}$ triangle singularities, where p, q, r are the Dolgachev numbers given in table 1. Note that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ in every case.

Here is as much detail as we will need. Pinkham found a representative for $\mathfrak{X}|_T$ which is an algebraic family over $\mathbb{C}^{\tau-1}$. This is constructed from $\mathfrak{X}|_T$ by a quasihomogeneous scaling process, so the discriminant complement in (T, s_0) is homotopy-equivalent to that in the algebraic family. So henceforth T will refer to the base of this algebraic family and $\mathfrak{X}|_T$ to the total space. Pinkham also showed that this family of algebraic surfaces may be simultaneously compactified by adjoining a suitable divisor.

That is, there is a proper flat family over T and a divisor therein whose complement is $\mathfrak{X}|_T$. This divisor has $p + q + r - 2$ components, each of which meets every fiber in a smooth rational curve of self-intersection -2. The rational curves in each fiber meet each other transversally with incidence graph the $Y_{p,q,r}$ diagram



(8.3)

where the arms have p, q and r vertices including the central vertex. Every smooth fiber is a K3 surface. We write L for H^2(generic fiber; $\mathbb{Z}$) $\cong E_8^2 \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^3$, Q for the $\mathbb{Z}$-span of these (-2)-curves, and V for $Q^\perp \subseteq L$, which turns out to have signature $(2, 22 - p - q - r)$. The fundamental group of $T - \Delta$ acts on L via its monodromy representation, fixing each of the (-2)-curves, hence acting on V. Write M for its image. The unramified cover T_0' of $T_0 := T - \Delta$ associated to the kernel of the monodromy representation turns out to be $\Omega := \Omega(V)$, minus the subset lying over a hyperplane arrangement in the symmetric space $P\Omega$. ($\Omega(V)$ was defined in the previous section.)

To quote Looijenga's theorem we must do a little more preparation. Let T_f be the set of $t \in T$ which admit a neighborhood $U \subseteq T$ such that the monodromy of $\pi_1(U - \Delta)$ is a finite group. The unramified covering $T_0' \to T_0$ extends naturally to a branched cover $T_f' \to T_f$. (This is the normalization of T_0' over T_f, not the current paper's notion of branched cover, although in the end they are the same.) Looijenga shows ([28, §5] or [29, III(6.4)]) that T_f is M-equivariantly diffeomorphic to $\Omega \subseteq V \otimes \mathbb{C}$, minus the part of Ω that lies over a certain hyperplane arrangement $\mathcal{H}_\infty \subseteq P\Omega$, described in detail in the proof of lemma 8.2 below. Identifying T_f' with its image in Ω, it turns out that one of its points lies outside T_0' just if it is orthogonal to a norm -2 vector of V. We write $\mathcal{H}_r$ for the corresponding hyperplane arrangement in $P\Omega$; the r subscript is for "reflection", since M contains the reflections in these vectors. We summarize the development so far: T_0 has an unramified cover T_0' that is a $\mathbb{C}^*$-bundle over $P\Omega - \mathcal{H}$, where $\mathcal{H} := \mathcal{H}_\infty \cup \mathcal{H}_r$. So the asphericity of T_0 is reduced to that of $P\Omega - \mathcal{H}$, which we establish in lemma 8.2 below (which uses conjecture 7.1). This finishes the proof in the quasihomogeneous case.

In the non-quasihomogeneous case, say with $\lambda = 1$, the versal deformation is got from (8.2) by restricting to the $t_\tau = 1$ subspace of the S we used for the corresponding quasihomogeneous case, call it Σ. It turns out that the inclusion $\Delta \cap \Sigma \subseteq \Sigma$ is topologically equivalent to the inclusion $\Delta \cap T \subseteq T$ in the corresponding quasihomogeneous case, so the asphericity of the discriminant complement reduces to what we have already proved. The topological equivalence follows from a slight strengthening of the topological triviality we used above: the homeomorphism $\phi : S \to S$ may be taken to preserve the family of hypersurfaces $t_\tau =$ constant. For this we refer to Wirthmüller [41]. □

Remark. It takes some work to extract the required topological triviality results from the literature. In the simply-elliptic case Looijenga [26] explicitly states everything we need. Wirthmüller's Satz 3.6 doesn't quite state what we need, merely that the SUD is a topological product of the Hessian direction with some analytic hypersurface. However, his proof [41, p. 63] shows that our T can play this role. Wirthmüller's thesis remains unpublished, though a statement appears in [14]. His work was later absorbed into a large machine of Damon. To extract the results we need from Damon's work, apply [16, 6.7(ii)] to f (there called f_0) to deduce that $F|_T$ (there called f) is "finitely $\mathcal{A}$-determined", and then apply [15, Cor. 3] to deduce that F is a topologically trivial unfolding of $F|_T$.

Lemma 8.2. *In the notation of the previous proof, and assuming conjecture 7.1, $P\Omega - \mathcal{H}$ is aspherical.*

Proof. We begin by describing $\mathcal{H}_\infty \subseteq P\Omega$ where $\Omega = \Omega(V)$ was defined in section 7. We use the description from [29, II§6]. Recall that L contains the classes of curves intersecting in the $Y_{p,q,r}$ pattern, Q is their span, and $V = Q^\perp \subseteq L$ has signature $(2, 22-p-q-r)$. We attach subscripts Q and V to vectors to indicate their projections to (the rational spans of) these lattices. We define the *core* of $Y_{p,q,r}$ to be its $\tilde{E}_6$ diagram if it contains one, otherwise its $\tilde{E}_7$ diagram if it contains one, and otherwise its $\tilde{E}_8$ diagram. Let $\mathcal{E}$ be the set of $y \in L$ such that $y^2 = -2$, $y \cdot e = 1$ for one end e of $Y_{p,q,r}$ that is not in the core, $y \cdot e' = -1$ for another end e' also not in the core, and y is orthogonal to all other roots of $Y_{p,q,r}$. We will see that y_V has negative norm, so $y_V^\perp$ defines a hyperplane in $P\Omega$. Then $\mathcal{H}_\infty = \cup_{y\in\mathcal{E}}\, y_V^\perp$. We need to study how these hyperplanes can meet. The method is simple: the hyperplanes of $y, y' \in \mathcal{E}$ meet just if the inner product matrix of y_V, y'_V is negative-definite. We can understand this matrix in terms of $y \cdot y'$ and the inner product matrix of y_Q, y'_Q.

The following model of Q is convenient for calculations. We use $1+p+q+r$ variables, separated into blocks of sizes 1, p, q and r. The inner product matrix is

$$\operatorname{diag}[1; -1, \ldots, -1; -1, \ldots, -1; -1, \ldots, -1]$$

and the central root has components

$$(1; 1, 0, \ldots, 0; 1, 0, \ldots, 0; 1, 0, \ldots, 0).$$

The roots along the p-arm have components $(-1, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, -1, 1)$ in the p-block and all other coordinates zero, and similarly for the q- and r-arms. We write e_p, e_q and e_r for the roots corresponding to the end nodes. If $t \in \mathcal{E}$ has nonzero inner products with e_p and e_q then we say it has type $\{e_p, e_q\}$. All vectors in Q satisfy the 3 linear relations that the sum of the coordinates in a block is independent of the block. If $y \in \mathcal{E}$ has $y \cdot e_p = 1$ and $y \cdot e_q = -1$, then one can compute

$$y_Q = (a; b, \ldots, b, b-1; c, \ldots, c, c+1; d, \ldots, d) \tag{8.4}$$

where

$$a = \frac{\frac{1}{p} - \frac{1}{q}}{1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}} \qquad b = \frac{a+1}{p} \qquad c = \frac{a-1}{q} \quad \text{and} \quad d = \frac{a}{r}.$$

We write N for its norm. Then $y_V^2 = -2-N$, which is negative because calculation reveals

$$2 + N = \frac{\left(\frac{1}{p} - \frac{1}{q}\right)^2}{1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}} + \frac{1}{p} + \frac{1}{q} > 0. \tag{8.5}$$

Our first claim is that if $y, y' \in \mathcal{E}$ have the same type then their hyperplanes in $P\Omega$ are disjoint. For suppose to the contrary and that both have type $\{e_p, e_q\}$, so that y_Q and y'_Q both equal (8.4), after negating y, y' if needed. Setting $\alpha = y \cdot y'$, the inner product matrix of y_V, y'_V is

$$\begin{pmatrix} -2-N & \alpha - N \\ \alpha - N & -2-N \end{pmatrix}.$$

Because the hyperplanes meet, this matrix is negative-definite. Since its diagonal entries are negative, this is the same as the determinant being positive, which boils down to $|\alpha - N| < 2 + N$. That is,

$$-2 < \alpha < 2 + 2N.$$

This is a contradiction because $\alpha \in \mathbb{Z}$ and the right side is ≤ -1. This inequality is the same as $2 + N \leq \frac{1}{2}$, which is easy to verify in the cases $p = q$ and $(p, q, r) = (4, 5, 2)$, where (8.5) reduces to $2/p$ and $1/2$. (Note: $p, q \geq 4$ because neither e_p nor e_q is in the core.) The general case follows from this, $\partial N / \partial q \leq 0$ and symmetry in p and q. We have proven our disjointness claim.

It follows that if a component of $\mathcal{H}_\infty$ and a component of $\mathcal{H}_r$ meet, then they meet orthogonally. Otherwise, the reflection across the latter component would carry the former to another component of $\mathcal{H}_\infty$ of same type, meeting it nontrivially. We use this orthogonality to break our problem into two simpler problems. Recall that $P\Omega$ is complete and nonpositively curved. So to finish the proof it suffices to verify that $\mathcal{H}$ satisfies hypotheses (a) and (b) of theorem 6.3. By lemma 6.4 it suffices to do this for $\mathcal{H}_r$ and $\mathcal{H}_\infty$ separately. The case of $\mathcal{H}_r$ is corollary 7.2 (which uses conjecture 7.1). So it suffices to show that for every $x \in P\Omega$, $T_x P\Omega - T_x \mathcal{H}_\infty$ is aspherical and that the universal branched cover of $T_x P\Omega$ over $T_x \mathcal{H}_\infty$ is CAT(0).

We have seen that two components of $\mathcal{H}_\infty$ can meet only if they have different types. So there is nothing to prove unless two types are present, which requires that none of the three end nodes of $Y_{p,q,r}$ lies in the core. Inspecting the list of Dolgachev numbers, we see that $p = q = r = 4$ is the only case remaining.

So suppose $y \in \mathcal{E}$ projects to (8.4), and y' similarly with p, q, r cyclically permuted. The formulas simplify dramatically since $p = q = r$, and one finds that y_Q and y'_Q have norm $-3/2$ and inner product $3/4$. It follows that y_V and y'_V have inner product matrix

$$\begin{pmatrix} -1/2 & \alpha - 3/4 \\ \alpha - 3/4 & -1/2 \end{pmatrix},$$

and from negative-definiteness that $\alpha = 1$. Therefore $y'' = -y - y'$ lies in $\mathcal{E}$ and has the third type. We conclude that where two components of $\mathcal{H}_\infty$ meet, a third does also. And this intersection meets no other components of $\mathcal{H}_\infty$ because it already lies in one of each type. So $\mathcal{H}_\infty$

is locally modeled on $y_V^\perp \cup y_V'^\perp \cup y_V''^\perp$ where y_V, y_V' and y_V'' are three norm $-1/2$ vectors with sum 0. This is the hyperplane arrangement for the Coxeter group $W(A_2)$, so we can complete the proof by appealing to corollary 7.2. (This part of the proof doesn't really need conjecture 7.1; see the next remark.) □

Remark (Triangle singularities). The appearance of the A_2 arrangement is a coincidence arising from $p = q = r$, but the use of conjecture 7.1 to treat it is somewhat artificial. Looijenga [29] treats $D_{p,q,r}$ for general p, q, r, not just the ones in table 1. All our arguments go through, the only differences being that one addresses one smoothing component of the singularity at a time, and the calculations on the intersection of components of $\mathcal{H}_\infty$ are messier. There are only finitely many cases, because $D_{p,q,r}$ is non-smoothable for $p+q+r > 22$. A short computer calculation shows $\alpha = 1$ in all cases. It follows that $\mathcal{H}_\infty$ is locally of the form $y_V^\perp \cup {y_V'}^\perp \cup {y_V''}^\perp$ where $y_V + y_V' + y_V'' = 0$, i.e., three hyperplanes whose intersection has codimension 1 in each of them. Local asphericity is easy, and the question of CAT(0)-ness of tangent spaces boils down to a study of the universal cover of $\mathbb{C}^2$ branched over three lines whose corresponding points in $\mathbb{C}P^1 \cong S^2$ lie in no open hemisphere. This should be accessible by adapting the methods of Charney and Davis [12, theorem 9.1] and Panov [36] referred to after conjecture 7.1.

Remark (The N_{16} singularity). An N_{16} singularity is the 2-fold cover of $\mathbb{C}^2$ branched over a union of 5 lines through the origin, and is a trimodal singularity. Laza has shown [25, Thm. 5.6] that after discarding a 1-dimensional topologically trivial factor, its discriminant complement has the same form as a triangle singularity. Namely, it is a $\mathbb{C}^*$-bundle over $P\Omega(T) - (\mathcal{H}_r \cup \mathcal{H}_\infty)$ for a suitable lattice T of signature $(2, 14)$. Here $\mathcal{H}_r$ is as above and $\mathcal{H}_\infty$ is similar to the above. His $\mathcal{H}_\infty$ is more complicated than for a triangle singularity and I have not studied it in any detail. He has obtained similar but unpublished results for the O_{16} singularity (the cone on a cubic surface).

Theorem 8.3. *Assuming conjecture 7.1, the cusp singularities* (8.1) *have aspherical discriminant complements.*

Proof. Looijenga [30] found a very beautiful description of (S, Δ). The brief version is that $S - \Delta$ is $(\Omega_d - \mathcal{H})/\widetilde{W}$, where Ω_d is the complexified open Tits cone of the Weyl group W with diagram $Y_{p,q,r}$, an action of $\widetilde{W} \cong \mathbb{Z}^{p+q+r-2} \rtimes W$ on Ω_d is given, and $\mathcal{H}$ is the locus of points with nontrivial $\widetilde{W}$-stabilizer.

p,q,r	$c_0,\dots,c_{m-1}$	Zykel $c'_0,\dots,c'_{m-1}$	Zykel* $d'_0,\dots,d'_{s-1}$	$d_0,\dots,d_{s-1}$
$2,3,7$	1	3	3	1
$2,4,5$	2	4	2,3	←
$3,3,4$	3	5	$2,2,3$	←
$2,3,r$	$3,2^{r-7}$	←	$r-4$	$r-6$
$2,4,r$	$4,2^{r-5}$	←	$2,r-2$	←
$3,3,r$	$5,2^{r-4}$	←	$2,2,r-1$	←
$2,q,r$	$3,2^{q-5},3,2^{r-5}$	←	$q-2,r-2$	←
$3,q,r$	$3,2^{q-4},4,2^{r-4}$	←	$q-1,2,r-1$	←
p,q,r	$3,2^{p-4},3,2^{q-4},3,2^{r-4}$	←	$p-1,q-1,r-1$	←

TABLE 2. Data concerning the cusp singularities needed in the proof of theorem 8.3. The notation 2^n means a string $2,\dots,2$ of n many 2's, and "←" means "the same as in the column to the left". We assume $p \le q \le r$, and earlier lines take precedence, for example the last line applies only when $4 \le p \le q \le r$.

Suppose $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$ and consider the singularity (8.1). One can work out a minimal resolution, and the exceptional divisor turns out to be a union of rational curves. These are smooth and meet each other transversely to form a cycle, except when there is only one component, when it meets itself transversely at one point. The negated self-intersection numbers $c_0,\dots,c_{m-1}$ of these curves are given in table 2 (cf. [32, (1.3)]). The notation 2^n means a string $2,\dots,2$ of n many 2's. From these one constructs what Nakamura [32] calls the first cycle numbers Zykel$(T_{p,q,r})$, for which we write $(c'_0,\dots,c'_{m-1})$. They are the same as $(c_0,\dots,c_m)$ except when there is only one component, in which case $c'_0=c_0+2$. Then one works out the "dual cycle numbers" Zykel$^*(T_{p,q,r})$ by an algorithm due to Hirzebruch and Zagier [30, p. 311] and Nakamura [32], for which we write $d'_0,\dots,d'_{s-1}$. Then we define $d:=(d_0,\dots,d_{s-1})$ to be the same sequence, except when there is only one term, when we set $d_0=d'_0-2$. ([30] omits mention of the special treatment of the one-component case.)

One can construct a remarkable surface from these data, a (singular) hyperbolic Inoue surface [23][30, p. 307]. It is a normal but non-algebraic complex surface with two singularities. One is a $T_{p,q,r}$ singularity and the other is its "dual cusp", the exceptional divisor D of whose minimal resolution is a cycle of s rational curves with negated self-intersection numbers $d = (d_0, \dots, d_{s-1})$. We let Y be the surface obtained by minimally resolving this dual cusp. Looijenga explains [30, III(2.7)] that a universal deformation $\mathfrak{Y} \to S$ of Y preserving D, restricted to the $T_{p,q,r}$ singularity, gives a semiuniversal deformation of it. Write Y_t for the fiber over $t \in S$ and $\Delta \subseteq S$ for the discriminant locus $\{t \in S \mid Y_t \text{ is singular}\}$. So our goal is to show the asphericity of $S - \Delta$. We will also need to fuss over the difference between $S - \Delta$ and $(S, s_0) - (\Delta, s_0)$.

Looijenga works with a larger space S_f, the set of $t \in S$ for which Y_t has only ADE singularities. He proves [30, III(2.8iii)] that for every $t \in S_f$, Y_t is a rational surface. Each such Y_t comes equipped with a copy D_t of D in it. In the last paragraph of [30, II(3.6)] he defines a complex manifold M_d and a subset D_d, and in [30, II(3.7)] he constructs a family of pairs (rational surface, divisor in it) over M_d. In [30, II(3.10)] he describes the singular fibers of this family; in particular the discriminant is exactly D_d. Using [30, II(3.8 and 3.10)] he shows there is a unique holomorphic map $\Phi_f : S_f \to M_d$ such that $\mathfrak{Y}|_{S_f}$ is the pullback of the family over M_d (up to a possible minor alteration which in the end doesn't occur; see [30, p. 316]). With Φ_f in hand, he extends it [30, III(3.5)] to a holomorphic map Φ from S to a certain completion $\widehat{M}_d$ of M_d, and shows that this is an isomorphism [30, III(3.6)]. As a consequence Φ_f is an isomorphism, so $S - \Delta \cong M_d - D_d$. So we want to prove that $M_d - D_d$ is aspherical.

Here are the definitions of M_d and D_d. Following [30, II(3.3)], let Y^0 be a fixed rational surface with an anticanonical divisor D^0 which is a cycle of type d. Following [30, I§2], let Q be the subspace of $H_2(Y^0; \mathbb{Z})$ orthogonal to every component of D, and B a certain explicit basis of Q. One follows Looijenga's recipe for B and checks that in every case it is a set of simple roots for the Dynkin diagram $Y_{p,q,r}$ we saw in (8.3). It follows that Q has signature $(1, p + q + r - 3)$. Following [30, I§3] let W be the Weyl group of B and $I \subseteq Q \otimes \mathbb{R}$ its Tits cone. Following [31, p. 1], define I° as the interior of I and

$$\Omega_d := \big\{x + iy \in Q \otimes \mathbb{C} \bigm| y \in I^\circ\big\}.$$

(Note: this coincides with [31, p. 16] for X the empty subset of B. Also, Looijenga writes Ω_d in [30] and Ω in [31].) Now, Q acts on Ω_d by translations in the real directions and W acts by the complexification of its action on Q. Then $M_d := \Omega_d/\widetilde{W}$ where $\widetilde{W} := Q \rtimes W$, and D_d is defined as the image of the locus $\mathcal{H} \subseteq \Omega_d$ of points with nontrivial $\widetilde{W}$-stabilizer. Since $M_d - D_d$ has $\Omega_d - \mathcal{H}$ as an unramified cover, what remains to prove is the asphericity of $\Omega_d - \mathcal{H}$.

(Properly speaking, Ω_d is defined in [30, II(3.6)] and some unwinding of definitions in [30, II(3.2)] is required to obtain the description above. The key points are that Q is nondegenerate, so that I can be identified with the dual Tits cone J, and that choosing a point of the "affine lattice" A identifies it with Q.)

Having described Ω_d we can now address the difference between S and its germ (S, s_0). It turns out that for any W-invariant neighborhood $V \subseteq I$ of 0, there is a neighborhood U_V of $s_0 \in S = \widehat{M}_d$ whose preimage in Ω_d is $\widetilde{U}_V = \{x + iy \in \Omega_d \mid y \in V\}$. Furthermore, such U_V give a basis for the topology of $\widehat{M}_d$ at s_0. (Refer to [31, (2.18)] and use the fact that $\{s_0\}$ is the stratum of $\widehat{M}_d$ corresponding to the full $Y_{p,q,r}$ diagram, regarded as a subdiagram of itself.) Obviously we may restrict to those V that are starshaped around 0. For such V, $\widetilde{U}_V - \mathcal{H} \to \Omega_d - \mathcal{H}$ is a homotopy-equivalence. Being $\widetilde{W}$-equivariant, it descends to a homotopy-equivalence $U_V - \Delta \to S - \Delta$. Therefore the asphericity of $(S, s_0) - (\Delta, s_0)$ is equivalent to that of $S - \Delta$, which we have already reduced to the asphericity of $\Omega_d - \mathcal{H}$.

At this point the algebraic geometry vanishes into the background, because Ω_d and $\mathcal{H}$ are described in terms of $Y_{p,q,r}$. By [31, (2.17)], point stabilizers in Ω_d are finite and generated by reflections of $\widetilde{W}$, so $\mathcal{H}$ is locally modeled on finite Coxeter arrangements. An obstruction to applying theorem 6.3 is the absence of a nonpositively curved metric on Ω_d. We can remedy this as follows. W is a hyperbolic reflection group, and I° contains one of the two cones of positive-norm vectors in $Q \otimes \mathbb{R}$, say I'°. Now,

$$\Omega'_d := \bigl\{x + iy \in Q \otimes \mathbb{C} \bigm| y \in I'^\circ\bigr\} \subseteq \Omega_d$$

does admit a complete nonpositively curved metric. In the notation of section 7, it is a guise of the symmetric space $P\Omega\bigl(Q \oplus \bigl(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\bigr)\bigr)$ for $\mathrm{O}(2, p + q + r - 2)$. So corollary 7.2 (which assumes conjecture 7.1) says that $\Omega'_d - \mathcal{H}$ is aspherical. To finish the proof we observe that $\Omega'_d - \mathcal{H} \to \Omega_d - \mathcal{H}$ is a homotopy-equivalence. To see this, find a W-equivariant deformation retraction of I° into I'° and apply it to the imaginary parts of points of Ω_d, leaving their real parts fixed. This is a $\widetilde{W}$-equivariant deformation retraction of Ω_d into Ω'_d, so $\Omega_d - \mathcal{H}$ and $\Omega'_d - \mathcal{H}$ are homotopy-equivalent. □

Remark. In the context of [30] a "cusp singularity" means one whose minimal resolution has exceptional divisor a cycle of rational curves. If the cusp is smoothable and the dual cusp has ≤ 5 components, then Looijenga obtained a similar description of S and Δ. Our retraction-to-Ω'_d trick always works because the Picard group of Y^0 (hence Q) still has hyperbolic signature. Referring to [30, p. 307] we see that conjecture 7.1 implies the asphericity of the discriminant complement for any 2-dimensional smoothable cusp singularity whose embedding dimension is ≤ 5. Recently Gross, Hacking and Keel [22] have generalized part of [30], so our methods may apply even more generally.

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