

THE DENSEST LATTICES IN $\mathrm{PGL}_3(\mathbb{Q}_2)$

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ABSTRACT. We find the smallest possible covolume for lattices in $\mathrm{PGL}_3(\mathbb{Q}_2)$, show that there are exactly two lattices with this covolume, and describe them explicitly. They are commensurable, and one of them appeared in Mumford's construction of his fake projective plane. We also discuss a new 2-adic uniformization of another fake projective plane.

Correction added September 24, 2016: The proof of lemma 1.1 contains an error; even the statement of the lemma is wrong. But this does not affect the rest of the paper. Here are the details. It is not true that “central extensions of finite groups by torsion-free groups are trivial”; for example, $\beta_M \in \mathrm{GL}_3\mathbb{Q}_2$ from section 3 generates a subgroup $\mathbb{Z}/3$ of $\mathrm{PGL}_3\mathbb{Q}_2$, and this $\mathbb{Z}/3$ has no lift to $\mathrm{SL}_3\mathbb{Q}_2$. What is true is that $\mathrm{SL}_3\mathbb{Q}_2 \rightarrow \mathrm{PSL}_3\mathbb{Q}_2$ is an isomorphism whose image has index 3 in $\mathrm{PGL}_3\mathbb{Q}_2$. This follows from the structure of $\mathbb{Q}_2^* \cong \{\pm 1\} \times \mathbb{Z} \times \mathbb{Z}_2$. Furthermore, if an element of $\mathrm{GL}_3\mathbb{Q}_2$ represents an element of $\mathrm{PGL}_3\mathbb{Q}_2$ outside $\mathrm{PSL}_3\mathbb{Q}_2$, then its determinant is not a perfect cube in \mathbb{Q}_2 . By the lattice interpretation of the vertices of Bruhat-Tits building, it cannot fix any vertex. This establishes an analogue of lemma 1.1 that is sufficient for all applications in the paper: *a finite subgroup of $\mathrm{PGL}_3\mathbb{Q}_2$ is the isomorphic image of a unique subgroup of $\mathrm{SL}_3\mathbb{Q}_2$, if it fixes some vertex of the Bruhat-Tits building.*

Here is another way to fix lemma 1.1; we omit the proof since it is not needed to fix the paper. *Suppose G is a finite subgroup of $\mathrm{PGL}_3\mathbb{Q}_2$. Then either (i) it is the isomorphic image of a unique subgroup of $\mathrm{SL}_3\mathbb{Q}_2$, or (ii) $G \cong \mathbb{Z}/3$, or (iii) G is isomorphic to the alternating group A_4 and conjugate to the group generated by projectivizations of*

$$\begin{pmatrix} - & & \\ & - & \\ & & + \end{pmatrix}, \begin{pmatrix} + & & \\ & - & \\ & & - \end{pmatrix}, \text{ and } \begin{pmatrix} & \sigma & \\ & & \\ 1 & & 1 \end{pmatrix}$$

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where $\sigma \in \mathbb{Q}_2^*$ is not a perfect cube.

We now return you to the original paper.

The most famous lattice in the projective group $\mathrm{PGL}_3(\mathbb{Q}_2)$ over the 2-adic rational numbers \mathbb{Q}_2 is the one Mumford used to construct his fake projective plane [22]. Namely, he found an arithmetic group $P\Gamma_1$ (we call it $P\Gamma_M$) containing a torsion-free subgroup of index 21, such that the algebraic surface associated to it by the theory of p -adic uniformization [23, 24] is a fake projective plane. The full classification of fake projective planes has been obtained recently [26]. Fake projective planes play an important and interesting role, not only in the theory of algebraic surfaces and non-archimedean geometry, but also in the theory of algebraic cycles, especially in connection with the Bloch conjecture, cf. [1].

The second author and his collaborators have developed a diagrammatic calculus [8, 15] for working with algebraic curves (including orbifolds) arising from p -adic uniformization using lattices in PGL_2 over a nonarchimedean local field. It allows one to read off properties of the curves from the quotient of the Bruhat-Tits tree and to construct lattices with various properties, or prove they don't exist. We hope to develop a higher-dimensional analogue of this theory, although only glimpses of it are now visible. Pursuing these glimpses suggested the existence of another lattice $P\Gamma_L$ with the same covolume as Mumford's, and we were able to prove that it really does exist. We show that $P\Gamma_L$ and $P\Gamma_M$ have the smallest possible covolume in $\mathrm{PGL}_3(\mathbb{Q}_2)$, are the only lattices with this covolume, and meet each other in a common index 8 subgroup. We also give explicit generators and a geometric description of their actions on the Bruhat-Tits building.

We discovered a new 2-adically uniformized fake projective plane by considering the subgroup of $P\Gamma_L$ corresponding to a Sylow 2-subgroup of $\mathrm{PGL}_2(\mathbb{F}_7)$ under a natural surjection $P\Gamma_L \rightarrow \mathrm{PGL}_2(\mathbb{F}_7)$. More precisely, let \mathcal{V} be the \mathbb{Z}_2 -scheme associated to the (torsion-free) kernel K of $P\Gamma_L \rightarrow \mathrm{PGL}_2(\mathbb{F}_7)$ by non-Archimedean uniformization. Unlike in Mumford's case, the Sylow 2-subgroup of $P\Gamma_L/K$ acts quite badly on \mathcal{V} : there are components of the central fiber \mathcal{V}_0 with nontrivial pointwise stabilizer. *But it acts freely on the generic fiber*, with quotient a fake projective plane defined over \mathbb{Q}_2 .

In particular, $P\Gamma_L$ led directly to a way to use discrete groups with torsion to construct interesting varieties that are genuinely smooth, not just smooth as orbifolds. Among other things, this opens up the possibility of p -adic uniformizations of fake projective planes with $p > 2$. This is unexpected because Mumford's calculations in [22] show that

such uniformization is possible only if the residue field is \mathbb{F}_2 . Since Mumford only considered torsion-free groups, our results are compatible with his.

Except for the presence of torsion, our construction parallels Mumford's. The algebraic surfaces associated to $P\Gamma_L$ and $P\Gamma_M$ are also surprisingly similar, for example they have the same configuration of singularities. Ishida [14] has described the surface in considerable detail in the $P\Gamma_M$ case. We intend to carry out the corresponding analysis for $P\Gamma_L$, as well as the treatment of the fake projective plane, in a future paper.

Finding densest-possible lattices in Lie groups has a long history, beginning with Siegel's treatment [27] of the unique densest lattice in $\mathrm{PSL}_2(\mathbb{R})$. Lubotzky [17] found the minimal covolume in $\mathrm{SL}_2(\mathbb{Q}_p)$ and $\mathrm{SL}_2(\mathbb{F}_q((t)))$, and some lattices realizing it. With Weigel [18] he obtained the complete classification of (isomorphism classes of) densest lattices in SL_2 over any finite extension of \mathbb{Q}_p . Golsefidy [11] identified the unique densest lattice in $G(\mathbb{F}_q((t)))$, where G is any simply connected Chevalley group of type E_6 or classical type $\neq A_1$ and q is neither 5 nor a power of 2 or 3. A generalization of Lubotzky's result in another direction regards $\mathrm{SL}_2(\mathbb{F}_q((t)))$ as the loop group of $\mathrm{SL}_2(\mathbb{F}_q)$. This is the simplest infinite-dimensional Kac-Moody group, having type \tilde{A}_1 over \mathbb{F}_q . The next-simplest Kac-Moody groups correspond to symmetric rank 2 Cartan matrices of hyperbolic type. The minimal covolumes of lattices in these Kac-Moody groups, and some lattices realizing them, have been found by Capdeboscq and Thomas [7].

Lattices in $\mathrm{PO}(n, 1)$ and $\mathrm{PU}(n, 1)$ present special challenges because not all of them are arithmetic. Meyerhoff [20, 21] identified the unique densest non-cocompact lattice in the identity component $\mathrm{PO}^\circ(3, 1)$ of $\mathrm{PO}(3, 1)$, and with Gabai and Milley he identified the densest cocompact lattice [9, 10]. Hild and Kellerhals [13] found the unique densest non-cocompact lattice in $\mathrm{PO}(4, 1)$, and Hild [12] extended this to $\mathrm{PO}(n, 1)$ for $n \leq 9$. Among arithmetic lattices, Belolipetsky [2, 3] found the unique densest lattice in $\mathrm{PO}^\circ(n, 1)$ for even n , in both the cocompact and non-cocompact cases. With Emery [4] he extended this to the case of odd $n \geq 5$. Stover [29] found the two densest non-cocompact arithmetic lattices in $\mathrm{PU}(2, 1)$.

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1. FINITE AND DISCRETE SUBGROUPS OF $\mathrm{PGL}_3(\mathbb{Q}_2)$

Throughout this section V is a 3-dimensional vector space over the 2-adic rational numbers \mathbb{Q}_2 . Our goal is to study the finite subgroups of $\mathrm{PGL}(V)$ and how they constrain the discrete subgroups.

We will write F_{21} for the Frobenius group of order 21 (the unique nonabelian group of this order), S_n for the symmetric group on n objects, and sometimes 2 for $\mathbb{Z}/2$ and 2^n for $(\mathbb{Z}/2)^n$. Also, $L_3(2)$ means the simple group $\mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$ of order 168. It has three conjugacy classes of maximal subgroups [5]: the stabilizers of points and lines in $\mathbb{P}^2\mathbb{F}_2$, isomorphic to S_4 , and the Borel subgroup of $\mathrm{PSL}_2(7)$, isomorphic to F_{21} .

The first step in studying the finite subgroups of $\mathrm{PGL}(V)$ is that passage between PGL and SL is free of complications:

Lemma 1.1. *Every finite subgroup of $\mathrm{SL}(V)$ maps isomorphically to its image in $\mathrm{PGL}(V)$, and every finite subgroup of $\mathrm{PGL}(V)$ is the image of a unique subgroup of $\mathrm{SL}(V)$.*

Proof. For the first claim, note that the only scalar in $\mathrm{SL}(V)$ is the identity, so $\mathrm{SL}(V) \rightarrow \mathrm{PGL}(V)$ is injective. Now let A be a complement to $\{\pm 1\}$ in $\mathbb{Q}_2^* \cong \{\pm 1\} \times \mathbb{Z} \times \mathbb{Z}_2$, and let $H \subseteq \mathrm{GL}(V)$ consist of the transformations with determinants in A . The center of H is torsion-free. Since central extensions of finite groups by torsion-free groups are trivial, a finite subgroup of $\mathrm{PGL}(V)$ has a unique lift to H . The determinant of any element of the lift is a finite-order element of A , hence equals 1. We have shown that every finite subgroup of $\mathrm{PGL}(V)$ has a unique lift to $\mathrm{SL}(V)$. \square

If A is an integral domain with fraction field k and W a k -vector space, then an A -lattice in W means an A -submodule L with $L \otimes_A k = W$. We write $\mathrm{GL}(L)$ for the group of A -module automorphisms of L , and $\mathrm{SL}(L)$ for its intersection with $\mathrm{SL}(W)$. In this paper A will be the ring of integers \mathcal{O} in $\mathbb{Q}(\sqrt{-7})$, or $\mathcal{O}[\frac{1}{2}]$, or the 2-adic integers \mathbb{Z}_2 . In this section it will always be \mathbb{Z}_2 , and L means a \mathbb{Z}_2 -lattice in V . The following lemma is well-known and holds much more generally, but we only give the case we need.

Lemma 1.2. *Suppose an involution $g \in \mathrm{GL}(L)$ acts trivially on $L/2L$. Then L is the direct sum of g 's eigenlattices (the intersections of L with g 's eigenspaces).*

Proof. Write x_{\pm} for the projection of any $x \in L$ to g 's ± 1 eigenspaces. By hypothesis $x \pm gx$ is 0 mod $2L$, which is to say $2x_{\pm} \in 2L$. So $x_{\pm} \in L$, proving the lemma. \square

Lemma 1.3. *Suppose G is a finite subgroup of $\mathrm{SL}(L)$. Then either*

- (i) G acts faithfully on $L/2L$, or
- (ii) $(G, L) \cong (S_4, \mathbb{Z}_2^3)$ where S_4 is the determinant 1 subgroup of the group of signed permutations, or
- (iii) $|G| \leq 12$.

Proof. Suppose G is not faithful on $L/2L$. The kernel of $\mathrm{SL}(L) \rightarrow \mathrm{SL}(L/2L)$ is a pro-2-group, so its intersection K with G is a 2-group. In fact it is elementary abelian because a modification of Siegel's argument [28, §39] shows that every element of K has order 1 or 2. (If $h \in K$ has order 4 then decompose L as a direct sum of h^2 's eigenlattices using lemma 1.2. Restricting to the -1 eigenlattice gives $h^2 = -I$ and $h = I + 2T$ for some matrix T over \mathbb{Z}_2 . It is easy to see that these are incompatible modulo 4.)

So we may choose a basis for V in which K is diagonal. Obviously each involution in K negates two coordinates, and K is 2 or 2^2 . If $K \cong 2^2$ then V is the sum of three distinct 1-dimensional representations of K , and we consider the intersections of L with these subspaces. Using lemma 1.2 twice shows that L is their direct sum. Since G normalizes K , it permutes these summands. Because G is finite and the torsion subgroup of \mathbb{Q}_2^* is $\{\pm 1\}$, the G -stabilizer of a summand acts on it by $\{\pm 1\}$. It follows easily that $L \cong \mathbb{Z}_2^3$ in a manner identifying G with a subgroup of the S_4 in (ii). So either (ii) or (iii) holds.

On the other hand, if K has a single involution, then lemma 1.2 shows that L is the direct sum of its eigenlattices. Of course, G/K preserves the images of these sublattices in $L/2L$. The $L_3(2)$ -stabilizer of a point of $\mathbb{P}^2\mathbb{F}_2$ and a line not containing it is S_3 , so $|G| \leq 12$. \square

$\mathrm{PGL}(V)$ acts on its Bruhat-Tits building \mathcal{B} . We recall from [6] that this is the simplicial complex with one vertex for each lattice in V , up to scaling. Often we speak of "the" lattice associated to a vertex when the scale is unimportant. Two vertices are joined if and only if one of the lattices contains the other of index 2 (after scaling). Whenever three vertices are joined pairwise by edges, there is a 2-simplex spanning those edges. $\mathrm{PGL}(V)$ acts transitively on vertices, with $\mathrm{PGL}(L)$ being the stabilizer of the vertex corresponding to L . The link of this vertex is the incidence graph of the points and lines of $\mathbb{P}(L/2L) \cong \mathbb{P}^2\mathbb{F}_2$, on which $\mathrm{PGL}(L)$ acts as $\mathrm{GL}(L/2L)$.

This subgroup $\mathrm{PGL}(L)$ is a maximal compact subgroup, and we scale the Haar measure on $\mathrm{PGL}(V)$ so that this subgroup has mass 1. If $P\Gamma$ is any discrete subgroup of $\mathrm{PGL}(V)$ then it acts on \mathcal{B} with finite stabilizers. For each $P\Gamma$ -orbit Σ of vertices, let n_Σ be the order of the $P\Gamma$ -stabilizer of any member of Σ . One can express $P\Gamma$'s covolume (the

Haar measure of $\mathrm{PGL}(V)/P\Gamma$) as the sum of $1/n_\Sigma$ over all $P\Gamma$ -orbits Σ .

A lattice in $\mathrm{PGL}(V)$ means a discrete subgroup of finite covolume. This double use of “lattice” is a standard confusion; we hope context will make our meaning clear. Mumford [22] exhibited a lattice in $\mathrm{PGL}_3(\mathbb{Q}_2)$ that acts transitively on vertices of \mathcal{B} , with stabilizer isomorphic to F_{21} . Its covolume is $1/21$. The goal of this paper is to show that this is the smallest possible covolume, and that there is exactly one other lattice realizing it.

Lemma 1.4. *Suppose $P\Gamma$ is a lattice in $\mathrm{PGL}(V)$ of covolume $\leq 1/21$. Then either*

- (i) *every vertex stabilizer is isomorphic to $L_3(2)$ and there are ≤ 8 orbits, or*
- (ii) *$P\Gamma$ acts transitively on vertices, with stabilizer isomorphic to F_{21} or S_4 , or*
- (iii) *$P\Gamma$ acts with two orbits on vertices, with stabilizers isomorphic to $L_3(2)$ and S_4 .*

Proof. Obviously every vertex stabilizer has order ≥ 21 . We claim that every finite subgroup G of $\mathrm{PGL}(V)$ of order ≥ 21 is isomorphic to F_{21} , S_4 or $L_3(2)$. It suffices by lemma 1.1 to prove this with PGL replaced by SL . Obviously G preserves some 2-adic lattice L , and lemma 1.3 implies that either $G \cong S_4$ or else G acts faithfully on $L/2L$. In the latter case we use the fact that every subgroup of $L_3(2)$ of order ≥ 21 is isomorphic to F_{21} , S_4 or $L_3(2)$. Having constrained the vertex stabilizers in $P\Gamma$ to these three groups, one works out which sums of $1/21$, $1/24$ and $1/168$ are $\leq 1/21$. \square

Some of these possibilities cannot occur. The key is to understand the three possible S_4 -actions on a \mathbb{Z}_2 -lattice L :

Lemma 1.5. *Suppose $G \subseteq \mathrm{SL}(L)$ is isomorphic to S_4 . Then the pair (G, L) is isomorphic to exactly one of the following pairs; in each case S_4 acts by the determinant 1 subgroup of the group of signed permutations.*

- (i) $(S_4, L_0 := \mathbb{Z}_2^3)$
- (ii) $(S_4, L_l := \{x \in L_0 \mid x_1 + x_2 + x_3 \equiv 0 \pmod{2}\})$
- (iii) $(S_4, L_p := \{x \in L_0 \mid x_1 \equiv x_2 \equiv x_3 \pmod{2}\})$.

We refer to the three cases as types 0, l and p . The notation reflects the fact that L_l and L_p correspond to a line and point in $\mathbb{P}(L_0/2L_0) \cong \mathbb{P}^2\mathbb{F}_2$ respectively. We have already seen type 0 in lemma 1.3. If a group $G \subseteq \mathrm{PGL}(V)$ isomorphic to S_4 fixes a vertex v of \mathcal{B} , then we say that

v has type 0, p or l according to the type of the action of (the lift to $\mathrm{SL}(V)$ of) G on the lattice represented by v .

Proof. Consider the sublattice L_0 of L spanned by the fixed-point sublattices of the three involutions in the Klein 4-group $K_4 \subseteq G$. Obviously L_0 has type 0. Now, L lies between $\frac{1}{2}L_0$ and L_0 , so it corresponds to a G -invariant subspace of $L_0/2L_0$. Besides the 0 subspace there are only two, leading to (ii) and (iii). The three cases may be distinguished by $[L : L_0]$, which is 1, 2 or 4 respectively. \square

Lemma 1.6. *Suppose $G \subseteq \mathrm{PGL}(V)$ is isomorphic to S_4 and stabilizes a vertex v of \mathcal{B} .*

- (i) *If v has type 0 then G stabilizes exactly two neighbors of v , which have types p and l with respect to G .*
- (ii) *If v has type p or l then G stabilizes exactly one neighbor of v , which has type 0 with respect to G .*

Proof. Choosing a G -equivariant isomorphism of the lattice represented by v with one of the models in lemma 1.5 makes visible the claimed neighboring lattice(s). In the proof of that lemma we saw that a type 0 G -lattice has exactly two G -invariant neighbors. Similarly, if L is a G -lattice of type not 0, then G acts faithfully on $L/2L$ (by lemma 1.3). Then, since $G \cong S_4$, G is the stabilizer of a point or line in $\mathbb{P}(L/2L)$. This makes it easy to see that G fixes only one neighbor of L . \square

Lemma 1.7. *Suppose $P\Gamma$ is a subgroup of $\mathrm{PGL}(V)$ and that the $P\Gamma$ -stabilizer of some vertex of \mathcal{B} is isomorphic to $L_3(2)$. Then the $P\Gamma$ -stabilizer of any neighboring vertex is isomorphic to S_4 .*

Proof. Suppose v is a vertex with stabilizer $G \cong L_3(2)$, w is any neighboring vertex, and L and M are the associated lattices. Write H for the G -stabilizer of w ; it is the stabilizer of a point or line of $\mathbb{P}(L/2L)$, so it is isomorphic to S_4 . Also, v has type p or l with respect to H , since H acts faithfully on $L/2L$ (indeed all of G does). By lemma 1.6(ii), w has type 0 with respect to H .

Now, the full $P\Gamma$ -stabilizer of w is finite (otherwise the $P\Gamma$ -stabilizer of v would be infinite), so lemma 1.3 applies to it. Since it contains an S_4 acting nonfaithfully on $M/2M$, it can be no larger than S_4 . \square

Lemma 1.8. *If $P\Gamma$ is a lattice in $\mathrm{PGL}(V)$ of covolume $\leq 1/21$, then its covolume is exactly $1/21$, and either it acts transitively on the vertices of \mathcal{B} , with stabilizer isomorphic to F_{21} , or else it has two orbits, with stabilizers isomorphic to $L_3(2)$ and S_4 .*

Proof. This amounts to discarding some of the possibilities listed in lemma 1.4. The case that every vertex stabilizer is isomorphic to $L_3(2)$

is ruled out by lemma 1.7. And if every vertex stabilizer is isomorphic to S_4 then lemma 1.6 shows that there are at least 3 orbits, ruling out the S_4 case of lemma 1.4(ii). \square

In section 3 we will exhibit lattices realizing these two possibilities, and in section 4 we will show they are the only ones.

2. THE HERMITIAN \mathcal{O} -LATTICES L AND M

In this section we introduce Hermitian lattices L and M over the ring of algebraic integers \mathcal{O} in $\mathbb{Q}(\sqrt{-7})$. In the next section we will study their isometry groups over $\mathbb{Z}[\frac{1}{2}]$, which turn out to be the two densest possible lattices in $\mathrm{PGL}_3(\mathbb{Q}_2)$. The construction of M is due to Mumford [22], and a description of L appears without attribution in the ATLAS [5] entry for $L_3(2)$. L is unimodular and contains 8 copies of M , while M has determinant 7 and lies in exactly one copy of L (lemma 2.6).

Let λ and $\bar{\lambda}$ be the algebraic integers $(-1 \pm \sqrt{-7})/2$. Let \mathcal{O} be the ring of algebraic integers in $\mathbb{Q}(\lambda)$, namely $\mathbb{Z} + \lambda\mathbb{Z}$. Everything about λ and $\bar{\lambda}$ can be derived from the equations $\lambda + \bar{\lambda} = -1$ and $\lambda\bar{\lambda} = 2$. For example, multiplying the first by λ yields the minimal polynomial $\lambda^2 + \lambda + 2 = 0$. \mathcal{O} is a Euclidean domain, so its class group is trivial, so any \mathcal{O} -lattice is automatically a free \mathcal{O} -module. We often regard \mathcal{O} as embedded in \mathbb{Z}_2 . There are two embeddings, and we always choose the one with $\bar{\lambda}$ mapping to a unit and λ to twice a unit.

A Hermitian \mathcal{O} -lattice means an \mathcal{O} -lattice L equipped with an \mathcal{O} -sesquilinear pairing $\langle | \rangle : L \times L \rightarrow \mathbb{Q}(\lambda)$, linear in its first argument and anti-linear in its second, satisfying $\langle y|x \rangle = \overline{\langle x|y \rangle}$. It is common to omit ‘‘Hermitian’’, but we will be careful to include it, because the unadorned word ‘‘lattice’’ already has two meanings in this paper.

L is called integral if $\langle | \rangle$ is \mathcal{O} -valued. Its determinant $\det L$ is the determinant of the inner product matrix of any basis for L . This is a well-defined rational integer since $\mathcal{O}^* = \{\pm 1\}$. An isometry means an \mathcal{O} -module isomorphism preserving $\langle | \rangle$, and we write $\mathrm{Isom} L$ for the group of all isometries.

The central object of this paper is the Hermitian \mathcal{O} -lattice

$$L := \left\{ (x_1, x_2, x_3) \in \mathcal{O}^3 \mid \begin{array}{l} x_i \equiv x_j \pmod{\bar{\lambda}} \text{ for all } i, j \text{ and} \\ x_1 + x_2 + x_3 \equiv 0 \pmod{\lambda} \end{array} \right\}$$

using one-half the standard Hermitian form, $\langle x|y \rangle = \frac{1}{2} \sum x_i \bar{y}_i$. The norm x^2 of a vector means its inner product with itself. We call a norm 2 vector a root. $\mathrm{Isom} L$ contains the signed permutations, which

we call the obvious isometries. Using them simplifies every verification below to a few examples.

Lemma 2.1. *L is integral and unimodular, its minimal norm is 2, and has basis $(2, 0, 0)$, $(\bar{\lambda}, \bar{\lambda}, 0)$ and $(\lambda, 1, 1)$. The full set of roots consists of their 42 images under obvious isometries.*

Proof. It's easy to see that the listed vectors lie in L . To see they generate it, consider an arbitrary element of L . By adding a multiple of $(\lambda, 1, 1)$ we may suppose the last coordinate is 0. Then the $x_i \equiv x_j \pmod{\bar{\lambda}}$ condition shows that $\bar{\lambda}$ divides the remaining coordinates. By adding a multiple of $(\bar{\lambda}, \bar{\lambda}, 0)$ we may suppose the second coordinate is also 0. Then $x_1 + x_2 + x_3 \equiv 0 \pmod{\lambda}$ says that the first coordinate is divisible by λ as well as $\bar{\lambda}$, hence by 2. So it lies in the \mathcal{O} -span of $(2, 0, 0)$. We have shown that the three given roots form a basis for L , and computing their inner product matrix shows that L is integral with determinant 1.

All that remains is to enumerate the vectors of norm ≤ 2 and see that they are as claimed. This is easy because the only possibilities for the components are $0, \pm 1, \pm \lambda, \pm \bar{\lambda}$ and ± 2 . \square

The word ‘‘root’’ is usually reserved for vectors negated by reflections, and our next result justifies our use of the term.

Lemma 2.2. *Isom L is transitive on roots and contains the reflection in any root r , i.e., the map*

$$x \mapsto x - 2 \frac{\langle x|r \rangle}{r^2} r.$$

Proof. The reflection negates r and fixes r^\perp pointwise. It also preserves L , since $\langle x|r \rangle \in \mathcal{O}$ and $r^2 = 2$. Therefore it is an isometry of L . To show transitivity on roots, note that $(\lambda, 1, 1)$ has inner product -1 with $(\bar{\lambda}, \bar{\lambda}, 0)$. It follows that they are simple roots for a copy of the A_2 root system. Since the Weyl group $W(A_2) \cong S_3$ generated by their reflections acts transitively on the 6 roots of A_2 , these two roots are equivalent. The same argument shows that $(\lambda, 1, 1)$ is equivalent to $(0, 0, -2)$. Together with obvious isometries, this proves transitivity. \square

Corollary 2.3. *Isom L is isomorphic to $L_3(2) \times \{\pm 1\}$ and acts on $L/\lambda L$ as $\mathrm{GL}(L/\lambda L)$.*

Proof. Since $\mathcal{O}^* = \{\pm 1\}$, Isom L is the product of its determinant 1 subgroup Isom $^+ L$ and $\{\pm 1\}$. Now, Isom $^+ L$ has order divisible by 7 by transitivity on the 42 roots (lemma 2.2), and also contains 24 obvious

isometries. So it has order at least 168. Since $L/\lambda L = (L \otimes \mathbb{Z}_2)/(2L \otimes \mathbb{Z}_2)$, lemma 1.3 shows that $\text{Isom}^+ L$ injects into $\text{GL}(L/\lambda L) \cong L_3(2)$ (since cases (ii) and (iii) cannot apply). Since $L_3(2)$ has order 168, the injection is an isomorphism. \square

Now we turn to defining a Hermitian \mathcal{O} -lattice M which we will recognize after lemma 2.5 as a copy of Mumford's (see [22, p. 240]). It turns out that everything about it is best understood by embedding it in L . So even though our aim is to understand M , we will develop some further properties of L .

We use the term "frame" to refer to either a nonzero element of $L/\lambda L$ or the set of roots mapping to it modulo λ . Since $\text{Isom} L$ acts on $L/\lambda L \cong \mathbb{F}_2^3$ as $\text{GL}(L/\lambda L)$, it acts transitively on frames, so each frame has $42/7 = 6$ roots. The standard frame means $\{(\pm 2, 0, 0), (0, \pm 2, 0), (0, 0, \pm 2)\}$. The language "frame" reflects the fact that each frame consists of 3 mutually orthogonal pairs of antipodal vectors. The stabilizer of the standard frame is exactly the group of obvious isometries.

Lemma 2.4. *L has 56 norm 3 and 336 norm 7 vectors, and $\text{Isom} L$ is transitive on each set.*

Proof. Every element of λL has even norm, so the norm 3 and 7 vectors lie outside λL . We will find all $x \in L$ of norms 3 and 7 that represent the standard frame. Representing the standard frame implies that λ divides x 's coordinates. Now, either $\bar{\lambda}$ divides all the coordinates or none; if it divides all then so does 2 and $x^2 = \text{odd}$ is impossible. So $\bar{\lambda}$ divides none. Writing $x = \lambda(a, b, c)$ we have $a, b, c \in \mathcal{O} - \bar{\lambda}\mathcal{O}$ with $|a|^2 + |b|^2 + |c|^2 = 3$ or 7. The possibilities for (a, b, c) are very easy to work out, using the fact that the only elements of $\mathcal{O} - \bar{\lambda}\mathcal{O}$ of norm < 7 are $\pm 1, \pm \lambda, \pm \lambda^2$. The result is that there are 8 (resp. 48) vectors of norm 3 (resp. 7) that represent the standard frame, all of them equivalent under obvious isometries. By transitivity on frames, L contains $7 \cdot 8$ (resp. $7 \cdot 48$) norm 3 (resp. 7) vectors and $\text{Isom} L$ is transitive on them. \square

Remarks. The proof shows that each norm 3 resp. 7 vector x has a unique description as $\bar{\lambda}^{-1}(e + e' + e'')$ resp. $\bar{\lambda}^{-1}(e + \lambda e' + \lambda^2 e'')$, where e, e', e'' are mutually orthogonal roots of the frame represented by x . The same argument proves transitivity on the 168 norm 5 vectors, each of which has a unique description as $\bar{\lambda}^{-1}(e + \lambda e' + \lambda e'')$. One can also check that every norm 4 (resp. 6) vector lies in just one of λL and $\bar{\lambda} L$, so there are $42 + 42$ (resp. $56 + 56$) of them. It turns out that L admits an anti-linear isometry, namely the map β_L from section 3 followed by complex

conjugation. Enlarging $\mathrm{Isom} L$ to include anti-linear isometries gives a group which is transitive on the vectors of each norm $2, \dots, 7$.

We write θ for $\lambda - \bar{\lambda} = \sqrt{-7}$. We define s to be the norm seven vector $(\lambda, -\lambda^2, \lambda^3)$ and $M := \{x \in L \mid \langle x|s \rangle \equiv 0 \pmod{\theta}\}$. By lemma 2.4, using any other norm 7 vector in place of s would yield an isometric Hermitian lattice. To understand M we use the fact that it is the preimage of a hyperplane in $L/\theta L \cong \mathbb{F}_7^3$. Since L is unimodular, $\langle | \rangle$ reduces to a nondegenerate symmetric bilinear form on $L/\theta L$. Since vectors of norm ≤ 3 are too close together to be congruent mod θ , the roots (resp. norm 3 vectors) represent 42 (resp. 56) distinct elements of $L/\theta L$. These correspond to the 21 “minus” points (resp. 28 “plus” points) of $\mathbb{P}(L/\theta L)$, which in ATLAS terminology [5, p. xii] means the nonisotropic points orthogonal to no (resp. some) isotropic point. Lurking behind the scenes here is that $\mathrm{Isom} L \cong 2 \times L_3(2)$ has index 2 in the full isometry group $2 \times \mathrm{PGL}_2(7)$ of $L/\theta L$. By the transitivity of $\mathrm{Isom} L$ on the $|\mathbb{P}^1\mathbb{F}_7| \cdot |\mathbb{F}_7^*| = 48$ isotropic vectors, each is represented by $336/48 = 7$ norm 7 vectors.

Now, M is the preimage of $\hat{s}^\perp \subseteq L/\theta L$, where the hat means the image mod θ . It follows that the subgroup of $\mathrm{Isom} L$ preserving M is $2 \times F_{21}$. The reason we chose the sign on the second coordinate of s is so that M is preserved by the cyclic permutation of coordinates, rather than some more complicated isometry of order 3. To check this, one just computes

$$\langle (\lambda, -\lambda^2, \lambda^3) \mid (\lambda^3, \lambda, -\lambda^2) \rangle \equiv 0 \pmod{\theta}$$

and uses the fact that in a 3-dimensional nondegenerate inner product space, isotropic vectors are orthogonal if and only if they are proportional.

Lemma 2.5. *M contains no roots of L and exactly 42 norm 7 vectors. It contains exactly 14 norm 3 vectors, namely the images of*

$$e_1 = (-\bar{\lambda}^2, -\bar{\lambda}, 0) \quad e_2 = (\lambda, \lambda, \lambda) \quad e_3 = (1, \lambda^2, 1)$$

under $\langle \text{cyclic permutation}, -1 \rangle \cong \mathbb{Z}/6$. These three vectors form a basis for M , with inner product matrix

$$\begin{pmatrix} 3 & \bar{\lambda} & \bar{\lambda} \\ \lambda & 3 & \bar{\lambda} \\ \lambda & \lambda & 3 \end{pmatrix}.$$

Proof. Observe that $\hat{s}^\perp \subseteq L/\theta L$ has 6 nonzero isotropic elements and 14 of each norm 3 (a nonzero square in \mathbb{F}_7). Therefore M contains no roots, and since each norm 3 element of $L/\theta L$ is represented by exactly

one norm 3 element of L , M has exactly 14 norm 3 vectors. It is easy to check that the three displayed vectors lie in M . Since M is preserved by cyclic permutation, they yield all 14 norm 3 vectors.

It is easy to check that $\langle e_i | e_j \rangle = \bar{\lambda}$ if $i < j$, so the inner product matrix is as stated. Since L/M is 1-dimensional over $\mathcal{O}/\theta\mathcal{O} \cong \mathbb{F}_7$, we see that $\det M$ is 7 times $\det L$. Since the e_i have inner product matrix of determinant 7, they form a basis for M . \square

Since Mumford describes his Hermitian lattice in terms of a basis with this same inner product matrix, our M is a copy of it. (Once we suspected Mumford's lattice lay inside L , there was only one candidate for it, and searching for the e_i 's realizing his inner product matrix was easy. Using the known $2 \times F_{21}$ symmetry, we could without loss take e_2 and e_3 as stated. Then there were just three possibilities for e_1 .)

Modulo θ , the matrix is the all 3's matrix, hence has rank 1. The following result is needed for theorem 3.3, on the index of $\Gamma_L \cap \Gamma_M$ in Γ_L and Γ_M , but not elsewhere.

Lemma 2.6. *M is a sublattice of exactly 8 unimodular Hermitian lattices, one of which is L and the rest of which are isometric to \mathcal{O}^3 . In particular, the isometry group of M is the subgroup $2 \times F_{21}$ of $\text{Isom } L$ preserving M .*

Proof. Since $\det M = 7$, any unimodular Hermitian superlattice L' contains it of index 7. So L' corresponds to a 1-dimensional subspace S' of $M/\theta M$. That is, $L' = \langle M, \frac{1}{\theta}v \rangle$ where $v \in M$ represents any nonzero element of S' . In order for L' to be integral, S' must be isotropic. Since the rank of $\langle | \rangle$ on $M/\theta M$ is 1, there are exactly 8 possibilities for S' , hence 8 unimodular Hermitian superlattices L' . One of these is L ; write S for its corresponding line in $M/\theta M$. Note that S contains (the reductions mod θ of) no norm 7 vectors, because L has no norm 1 vectors. Each $S' \neq S$ contains (the reductions mod θ of) at most 6 norm 7 vectors, because L' contains a norm 1 vector for every norm 7 vector of M projecting into S' . (Each norm 1 vector spans a summand, so L' can have at most 6 norm 1 vectors, and if it has 6 then it is a copy of \mathcal{O}^3 .) Since M has 42 norm 7 vectors (lemma 2.4), the subspaces $S' \neq S$ have on average $42/7 = 6$ (images of) norm 7 vectors. It follows that each has exactly 6, and that each $L' \neq L$ is a copy of \mathcal{O}^3 . \square

3. THE ISOMETRY GROUPS Γ_L AND Γ_M OF L AND M OVER $\mathbb{Z}[\frac{1}{2}]$

In this section we regard the isometry groups of L and M as group schemes over \mathbb{Z} and study the groups Γ_L and Γ_M of points over $\mathbb{Z}[\frac{1}{2}]$. Γ_M is Mumford's Γ_1 . We show that $P\Gamma_L$ and $P\Gamma_M$ are densest possible

lattices in $\mathrm{PGL}_3(\mathbb{Q}_2)$, and find generators for them. We also show by an independent argument that $\Gamma_L \cap \Gamma_M$ has index 8 in Γ_L and Γ_M .

Regarding $\mathrm{Isom} L$ and $\mathrm{Isom} M$ as group schemes amounts to the following. If A is any commutative ring then the A -points of $\mathrm{Isom} L$ are the $(\mathcal{O} \otimes_{\mathbb{Z}} A)$ -linear transformations of $L \otimes_{\mathbb{Z}} A$ that preserve the unique $(\mathcal{O} \otimes_{\mathbb{Z}} A)$ -sesquilinear extension of $\langle | \rangle$. And similarly for $\mathrm{Isom} M$. The groups of \mathbb{Z} -points of these group schemes are the finite groups $\mathrm{Isom} L \cong L_3(2) \times 2$ and $\mathrm{Isom} M \cong F_{21} \times 2$ from the previous section. We define Γ_L and Γ_M to be their groups of $\mathbb{Z}[\frac{1}{2}]$ -points. By the theory of arithmetic groups [19, p. 1], their central quotients $P\Gamma_L$ and $P\Gamma_M$ are lattices in $\mathrm{PGL}_3(\mathbb{Q}_2)$. Actually all we need from the general theory is discreteness, which is the easy part; cocompactness is part of theorems 3.1 and 3.2.

We write $L[\frac{1}{2}]$ for the Hermitian $\mathcal{O}[\frac{1}{2}]$ -lattice $L \otimes_{\mathcal{O}} \mathcal{O}[\frac{1}{2}]$ and similarly for $M[\frac{1}{2}]$. It is easy to find extra isometries of $L[\frac{1}{2}]$ and $M[\frac{1}{2}]$. For L we observe that the roots $(2, 0, 0)$, $(0, \bar{\lambda}, \bar{\lambda})$ and $(0, \bar{\lambda}, -\bar{\lambda})$ are mutually orthogonal. Therefore the isometry sending the roots $(2, 0, 0)$, $(0, 2, 0)$, $(0, 0, 2)$ of the standard frame to them lies in Γ_L . We call this transformation β_L , namely

$$\beta_L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\lambda & 1/\lambda \\ 0 & 1/\lambda & -1/\lambda \end{pmatrix}$$

Similarly, the cyclic permutation $e_1 \rightarrow e_2 \rightarrow e_3 \rightarrow e_1$ does not quite preserve inner products in M , because $\langle e_2 | e_1 \rangle$ and $\langle e_3 | e_1 \rangle$ are λ not $\bar{\lambda}$. However, this can be fixed up by multiplying by $\lambda/\bar{\lambda}$. That is, Γ_M contains the isometry $\beta_M : e_1 \rightarrow e_2 \rightarrow e_3 \rightarrow (\lambda/\bar{\lambda})e_1$. This is a slightly more convenient isometry than Mumford's ρ [22, p. 241], and appears in [16, p. 639]. Its cube is obviously the scalar $\lambda/\bar{\lambda}$.

The next two theorems refer to the normalization of Haar measure on $\mathrm{PGL}_3(\mathbb{Q}_2)$ introduced in section 1, namely the one for which $\mathrm{PGL}_3(\mathbb{Z}_2)$ has mass 1.

Theorem 3.1. *$P\Gamma_M$ is a lattice in $\mathrm{PGL}_3(\mathbb{Q}_2)$ of covolume $1/21$ and is generated by F_{21} and β_M . It acts transitively on the vertices, edges and 2-simplices of \mathcal{B} , with stabilizers F_{21} , $\mathbb{Z}/3$ and $\mathbb{Z}/3$ respectively.*

Proof. Consider the vertex v of \mathcal{B} corresponding to $M_2 := M \otimes_{\mathcal{O}} \mathbb{Z}_2$. We know that F_{21} acts on v 's neighbors with two orbits, corresponding to the points and lines of $\mathbb{P}(M/\lambda M) = \mathbb{P}(M_2/2M_2)$. On the other hand, β_M cyclically permutes the three (scalar classes of) \mathbb{Z}_2 -lattices

$$\langle e_1, e_2, e_3 \rangle, \quad \langle e_1, e_2, \frac{1}{\lambda} e_3 \rangle, \quad \langle e_1, \frac{1}{\lambda} e_2, \frac{1}{\lambda} e_3 \rangle.$$

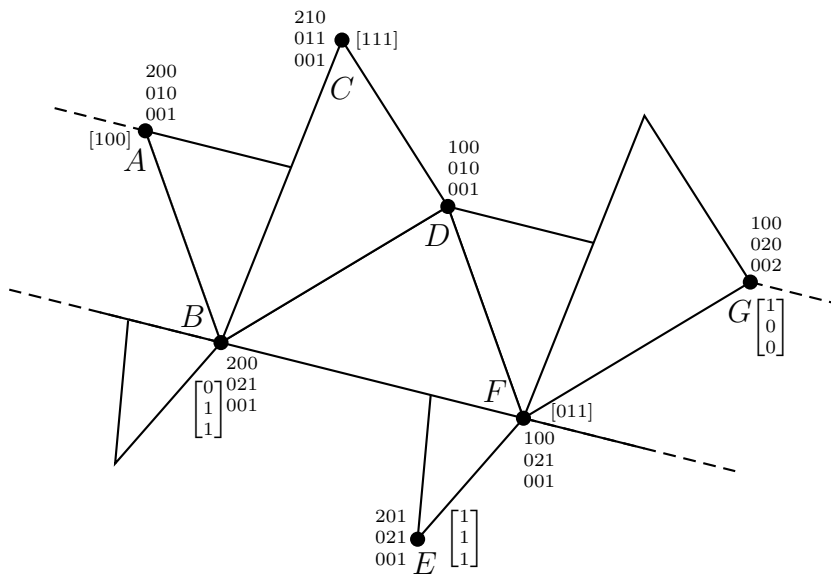


FIGURE 3.1. The subcomplex X of \mathcal{B} used in the proof of theorem 3.2; β_L acts by rotating about the centerline of the strip and sliding everything to the left.

This is a rotation of order 3 around the center of a 2-simplex containing v . It follows that v is $\langle F_{21}, \beta_M \rangle$ -equivalent to each of its neighbors. Since the same holds for the neighbors, induction proves transitivity on vertices. Transitivity on edges is also clear, and transitivity on 2-simplices follows from F_{21} 's transitive action on the edges in the link of v .

We have shown that $\langle F_{21}, \beta_M \rangle$ has covolume $\leq 1/21$. Since $P\Gamma_M$ contains it and has covolume $\geq 1/21$ by lemma 1.8, these groups coincide and have covolume exactly $1/21$. That the vertex stabilizers are no larger than F_{21} also follows from that lemma, and the structures of the other stabilizers follow. \square

Theorem 3.2. $P\Gamma_L$ is a lattice in $\mathrm{PGL}_3(\mathbb{Q}_2)$ of covolume $1/21$ and is generated by $L_3(2)$ and β_L . It acts with two orbits on vertices of \mathcal{B} , with stabilizers $L_3(2)$ and S_4 .

Proof. The main step is to show that $\langle L_3(2), \beta_L \rangle$ acts with ≤ 2 orbits on vertices. Consider the subcomplex X of \mathcal{B} pictured in figure 3.1. (It is the fixed-point set of the dihedral group $D_8 \subseteq L_3(2)$ generated by the negations of evenly many coordinates, together with the simultaneous negation of the first coordinate and exchange of the second and third.) We have named seven vertices A, \dots, G and given bases for

the \mathbb{Z}_2 -lattices in \mathbb{Q}_2^3 they represent (the columns of the 3×3 arrays). Obviously D corresponds to \mathbb{Z}_2^3 , and one can check that C represents $L \otimes_{\mathbb{O}} \mathbb{Z}_2$. For A, B, C, E, F, G we have also indicated their positions in the link of D by giving the corresponding subspace of $\mathbb{Z}_2^3/2\mathbb{Z}_2^3 \cong \mathbb{F}_2^3$. A column vector represents its span and a row vector represents its kernel (thinking of it as a linear function).

One can check that β_L acts on X by rotating everything around the centerline of the main strip and shifting to the left by half a notch. In particular, the vertices along the edges of the strip are all $\langle \beta_L \rangle$ -equivalent, as are the tips of the triangular flaps. We claim that every vertex of \mathcal{B} is $\langle L_3(2), \beta_L \rangle$ -equivalent to C or D . It suffices by induction to verify this for every vertex adjacent to C or D . Any neighbor of C is $L_3(2)$ -equivalent to B or D , and we just saw that $\beta_L(D) = B$. The stabilizer of D in $L_3(2)$ is the subgroup S_4 preserving the standard frame in L . It acts on \mathbb{F}_2^3 by the S_3 of coordinate permutations. The 14 neighbors of D correspond to the nonzero row and column vectors over \mathbb{F}_2 . Under the S_4 symmetry, every neighbor is equivalent to A, B, C, E, F or G . Since $\beta_L(E) = C$ and A, B, F and G are $\langle \beta_L \rangle$ -equivalent to D , we have proven our claim.

Since $\langle L_3(2), \beta_L \rangle$ has covolume $\leq 1/168 + 1/24 = 1/21$ and $P\Gamma_L$ has covolume $\geq 1/21$ (lemma 1.8), these two groups coincide. Lemma 1.8 also implies that the stabilizers of C and D are no larger than the visible $L_3(2)$ and S_4 . \square

A key relation between Γ_L and Γ_M is the following. It is independent of the other theorems in this section. In fact, together with either of theorems 3.1 and 3.2 it implies the other, except for the explicit generating sets.

Theorem 3.3. $\Gamma_L \cap \Gamma_M$ has index 8 in each of Γ_L and Γ_M .

Proof. This boils down to two claims: $L[\frac{1}{2}]$ contains exactly 8 copies of $M[\frac{1}{2}]$, on which Γ_L acts transitively, and $M[\frac{1}{2}]$ lies in exactly 8 copies of $L[\frac{1}{2}]$, on which Γ_M acts transitively.

For the first claim, any sublattice of $L[\frac{1}{2}]$ isometric to $M[\frac{1}{2}]$ must have index 7 since $\det M = 7$ and L is unimodular. The index 7 sublattices correspond to the hyperplanes in $L[\frac{1}{2}]/\theta L[\frac{1}{2}] = L/\theta L$, which we studied in section 2. The orthogonal complements of the plus and minus points of $\mathbb{P}(L/\theta L)$ give \mathcal{O} -sublattices M' with $M'[\frac{1}{2}]$ not isomorphic to $M[\frac{1}{2}]$. (The reduction of $\langle | \rangle$ to $M'/\theta M'$ has rank 2 not 1.) Since $L_3(2) \subseteq \Gamma_L$ acts transitively on the 8 isotropic points of $\mathbb{P}(L/\theta L)$, the claim is proven.

The second claim is similar. By lemma 2.5, the unimodular lattices containing $M[\frac{1}{2}]$ are $L[\frac{1}{2}]$ and 7 copies of $\mathcal{O}[\frac{1}{2}]^3$. Happily, from the definition of L in section 2 it is obvious that $L[\frac{1}{2}] = \mathcal{O}[\frac{1}{2}]^3$. To prove transitivity, suppose $L[\frac{1}{2}]'$ is one of 7, and choose any isometry carrying it to $L[\frac{1}{2}]$. This sends $M[\frac{1}{2}]$ to one of 8 sublattices of $L[\frac{1}{2}]$. Following this by an isometry of $L[\frac{1}{2}]$ sending this image of $M[\frac{1}{2}]$ back to $M[\frac{1}{2}]$, we see that $L[\frac{1}{2}]'$ is Γ_M -equivalent to $L[\frac{1}{2}]$. \square

4. UNIQUENESS

In this section we show that $P\Gamma_L$ and $P\Gamma_M$ are the only two densest lattices in $\mathrm{PGL}_3(\mathbb{Q}_2)$. We will use the fact that $\mathrm{PGL}_3(\mathbb{Q}_2)$ contains only one conjugacy classes of subgroups isomorphic to F_{21} resp. $L_3(2)$. One way to see this is that the finite group has only two faithful characters of degree 3, which are exchanged by an outer automorphism.

Theorem 4.1. *Suppose $P\Gamma$ is a lattice in $\mathrm{PGL}_3(\mathbb{Q}_2)$ with smallest possible covolume. Then its covolume is $1/21$ and it is conjugate to $P\Gamma_L$ or $P\Gamma_M$.*

Proof. The covolume claim is proven in lemma 1.8, which also shows that $P\Gamma$ either acts transitively on vertices of \mathcal{B} with stabilizer isomorphic to F_{21} , or has two orbits, with stabilizers isomorphic to $L_3(2)$ and S_4 .

We begin by assuming the latter case and proving $P\Gamma$ conjugate to $P\Gamma_L$. As in the proof of theorem 3.2, let C be the vertex in \mathcal{B} corresponding to L and consider its neighbors B and D and the transformation $\beta_L \in \Gamma_L$ sending D to B . Because $\mathrm{PGL}_3(\mathbb{Q}_2)$ contains a unique conjugacy class of $L_3(2)$'s, we may suppose that $P\Gamma$ contains the $P\Gamma_L$ -stabilizer $L_3(2)$ of C . By lemma 1.7, the $P\Gamma$ -stabilizers of B and D are subgroups S_4 of this $L_3(2)$. By the assumed transitivity of $P\Gamma$ on vertices with stabilizer S_4 , it contains some $\beta \in \mathrm{PGL}_3(\mathbb{Q}_2)$ sending D to B . So $\beta \circ \beta_L^{-1}$ normalizes the S_4 fixing B . Now, this S_4 is self-normalizing in $\mathrm{PGL}_3(\mathbb{Q}_2)$ since S_4 has no outer automorphisms and the centralizer is trivial (by irreducibility). So β and β_L differ by an element of this S_4 . In particular, $P\Gamma$ contains β_L . By theorem 3.2 we have $P\Gamma_L = \langle L_3(2), \beta_L \rangle$, so we have shown that $P\Gamma$ contains $P\Gamma_L$. By the maximality of the latter group, we have equality.

The case of $P\Gamma$ transitive on vertices with stabilizer F_{21} is similar in spirit, but messier because the inclusions $S_4 \subseteq L_3(2)$ are replaced by $\mathbb{Z}/3 \subseteq F_{21}$, and $\mathbb{Z}/3$ is very far from self-normalizing in $\mathrm{PGL}_3(\mathbb{Q}_2)$. Before embarking on the details, we observe that each triangle in \mathcal{B} has $P\Gamma$ -stabilizer $\mathbb{Z}/3$, acting on it by cyclically permuting its vertices.

This is a restatement of the simple transitivity of $P\Gamma$ on pairs (vertex of \mathcal{B} , triangle containing it), which can be checked by considering the action of F_{21} on the link of its fixed vertex.

Write v for the vertex of \mathcal{B} corresponding to $M_2 := M \otimes \mathbb{Z}_2$, preserved by the group $F_{21} \times 2$ of \mathbb{Z} -points of Γ_M . We will need Mumford's explicit generators

$$\sigma = \begin{pmatrix} 1 & 0 & \lambda \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 + \lambda \\ 0 & 1 & \lambda \end{pmatrix}$$

of orders 3 and 7 respectively. These matrices come from Mumford's description of M as a Hermitian \mathcal{O} -lattice in the $\mathbb{Q}(\lambda)$ -vector space $\mathbb{Q}(\zeta_7)$, with basis $1, \zeta_7, \zeta_7^2$. Namely, τ is multiplication by ζ_7 and σ is the Galois automorphism $\zeta_7 \rightarrow \zeta_7^2$. As a 3-element of $L_3(2)$, σ preserves a unique point and a unique line of $\mathbb{P}(M_2/2M_2)$. We write p and l for the corresponding neighbors of v . By working out the action of τ on $\mathbb{P}(M_2/2M_2)$, one can check that $\tau^3(l)$ is a neighbor of p .

Since $\mathrm{PGL}_3(\mathbb{Q}_2)$ contains a unique conjugacy class of F_{21} 's, we may suppose $P\Gamma$ contains the $P\Gamma_M$ -stabilizer of v . Let $\Gamma_0 \subseteq \mathrm{GL}_3(\mathbb{Q}_2)$ be generated by σ , τ and an $\alpha \in \mathrm{GL}_3(\mathbb{Q}_2)$ lying over the element of $P\Gamma$ that rotates the triangle with vertices $v, p, \tau^3(l)$ as shown:

$$(4.1) \quad \begin{array}{c} \bullet \tau^3(l) \\ \diagup \quad \diagdown \\ \alpha \\ \diagdown \quad \diagup \\ \bullet v \quad \bullet p \\ \diagleft \quad \diagright \\ \bullet l \end{array}$$

By our remarks above on the $P\Gamma$ -stabilizer of a 2-simplex, α is uniquely defined (up to scalars) and its cube is a scalar. We will replace α by its product with a scalar whenever convenient. Note that $\gamma := \alpha\tau^3$ sends the edge \overline{lv} to \overline{vp} . Since these two edges have the same $P\Gamma$ -stabilizer $\langle \sigma \rangle$, γ normalizes $\langle \sigma \rangle$. We will consider the case that γ centralizes σ and then the case that it inverts σ .

The centralizer of σ in $\mathrm{GL}_3(\mathbb{Q}_2)$ is the product of the scalars and

$$\{\pi_1 + a\pi_2 + b\sigma_2 \mid a, b \in \mathbb{Q}_2, \text{ not both } 0\} \subseteq \mathrm{GL}_3(\mathbb{Q}_2)$$

where $\pi_1 = \frac{1}{3}(1 + \sigma + \sigma^2)$ is the projection to σ 's fixed space, $\pi_2 = I_3 - \pi_1$ is the projection to the span of σ 's other eigenspaces, and $\sigma_2 = \sigma \circ \pi_2$. This is because the image of π_2 is irreducible as a $\mathbb{Q}_2[\langle \sigma \rangle]$ -module, making it into a 1-dimensional vector space over $\mathbb{Q}_2(\zeta_3)$. Expressing α in terms of γ , we get $\alpha = (\pi_1 + a\pi_2 + b\sigma_2)\tau^{-3}$, and the equation $\alpha^3 = (\text{scalar})$ imposes conditions on a and b , namely the vanishing of 8 polynomials in $\mathbb{Q}(\lambda)[a, b]$. These are unwieldy enough that we used the PARI/GP software [25] to handle the algebra. One uses Gaussian

elimination on the a^3 , a^2b , a^2 , ab^2 and ab coefficients, obtaining a relation $a = (\text{polynomial in } b)$. After eliminating a in favor of b , one of the relations becomes $b^3 - b = 0$, so $b \in \{0, \pm 1\}$. Of these, $b = 0$ does not satisfy the other relations, while $b = \pm 1$ do. But these give the trivial solutions $\gamma = \sigma^{\pm 1}$, and the corresponding $\alpha = \sigma^{\pm 1}\tau^{-3}$ fix v rather than acting as in (4.1). So we have eliminated the case that γ centralizes σ .

To treat the case of γ inverting σ , we note that $\alpha_M := \sigma\tau^5\beta_M^{-1}\tau^{-5}\sigma^{-1}$ is an element of Γ_M acting as in (4.1), and that the corresponding $\gamma_M := \alpha_M\tau^3$ inverts σ . Therefore γ has the form $\gamma_M \circ (\pi_1 + a\pi_2 + b\sigma_2)$. As before, the equation $\alpha^3 = (\text{scalar})$ imposes conditions on a and b . Gaussian elimination yields a relation $f(b)a + g(b) = 0$ with f and g polynomials and f of degree 1. After checking that $f(b) \neq 0$ (it turns out that $f(b) = 0$ implies $g(b) \neq 0$), one solves for a in terms of b . Then eliminating a gives a family of polynomials in b , all of which must vanish. It happens that all are divisible by b , so $b = 0$ is a solution. This leads to $a = 1$, hence $\alpha = \alpha_M$ and $\langle \sigma, \tau, \beta_M \rangle \subseteq \Gamma_0$. By theorem 3.1, the projectivization of the left side is $P\Gamma_M$. Since $P\Gamma_0 \subseteq P\Gamma \subseteq P\Gamma_M \subseteq P\Gamma_0$, all three groups coincide.

If $b \neq 0$ then we divide the polynomials by as many powers of b as possible and take their gcd in $\mathbb{Q}(\lambda)[b]$, which turns out to have degree 1. Solving for b , one obtains α . It turns out that this α is a scalar times a matrix with entries in \mathcal{O} and odd determinant. That is, it represents an element of $\text{PGL}(M_2)$. Therefore it fixes v rather than sending it to p , so this solution for b is spurious. This completes the proof. \square

REFERENCES

- [1] Barlow, R. N., Zero-cycles on Mumford's surface. *Math. Proc. Camb. Phil. Soc.* **126** (1999), no. 3, 505-510.
- [2] Belolipetsky, Mikhail, On volumes of arithmetic quotients of $\text{SO}(1, n)$, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **3** (2004), no. 4, 749-770.
- [3] Belolipetsky, Mikhail, Addendum to: "On volumes of arithmetic quotients of $\text{SO}(1, n)$ ", *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **6** (2007), no. 2, 263-268.
- [4] Belolipetsky, Mikhail and Emery, Vincent, On volumes of arithmetic quotients of $\text{PO}(n, 1)^\circ$, n odd, *arXiv:1001.4670v1*.
- [5] Conway, J. H., Curtis, R. T., Norton, S. P., Parker, R. A., Wilson, R. A., *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups*. Oxford University Press, Eynsham, 1985.
- [6] Bruhat, Francois and Tits, Jacques, Groupes réductifs sur un corps local, I. Données radicielles valuées, *Publ. Math. IHES* 41 (1972) 5-251.
- [7] Capdeboscq (Korchagina), Inna and Thomas Anne, Lattices in complete rank 2 Kac-Moody groups, *arXiv:0907.1350*
- [8] Cornelissen, Gunther; Kato, Fumiharu and Kontogeorgis, Aristides, Discontinuous groups in positive characteristic and automorphisms of Mumford curves, *Math. Ann.* **320** (2001), no. 1, 55-85.

- [9] Gabai, David; Meyerhoff, Robert and Milley, Peter, Mom technology and hyperbolic 3-manifolds, *In the tradition of Ahlfors-Bers. V, Contemp. Math.* 510, pp. 84–107, Amer. Math. Soc., Providence, RI, 2010.
- [10] Gabai, David; Meyerhoff, Robert and Milley, Peter, Mom technology and volumes of hyperbolic 3-manifolds, *Comment. Math. Helv.* **86** (2011), no. 1, 145–188.
- [11] Golsefidy, Alireza, Lattices of minimum covolume in Chevalley groups over local fields of positive characteristic, *Duke Mathematical Journal* **146** (2009), no. 2, 227–251.
- [12] Hild, Thierry, The cusped hyperbolic orbifolds of minimal volume in dimensions less than ten, *J. Algebra* **313** (2007), no. 1, 208–222.
- [13] Hild, Thierry and Kellerhals, Ruth, The FCC lattice and the cusped hyperbolic 4-orbifold of minimal volume, *J. Lond. Math. Soc. (2)* **75** (2007), no. 3, 677–689.
- [14] Ishida, Masa-Nori, An elliptic surface covered by Mumford’s fake projective plane, *Tohoku Math. J. (2)* **40** (1988) no. 3, 367–396.
- [15] Kato, Fumiharu, Non-Archimedean orbifolds covered by Mumford curves, *J. Algebraic Geom.* **14** (2005) no. 1, 1–34.
- [16] Kato, Fumiharu, On the Shimura variety having Mumford’s fake projective plane as a connected component, *Math. Z.* **259** (2008) no. 3, 631–641.
- [17] Lubotzky, Alexander, Lattices of minimal covolume in SL_2 : a non-Archimedean analogue of Siegel’s theorem $\mu \geq \pi/21$, *J. Amer. Math. Soc.* **3** (1990), no. 4, 961–975.
- [18] Lubotzky, A. and Weigel, Th., Lattices of minimal covolume in SL_2 over local fields, *Proc. London Math. Soc. (3)* **78** (1999), no. 2, 283–333.
- [19] Margulis, G. A., *Discrete subgroups of semisimple Lie groups*, Springer-Verlag, Berlin, 1991.
- [20] Meyerhoff, Robert, The cusped hyperbolic 3-orbifold of minimum volume, *Bull. Amer. Math. Soc. (N.S.)* **13** (1985), no. 2, 154–156.
- [21] Meyerhoff, Robert, Sphere-packing and volume in hyperbolic 3-space, *Comment. Math. Helv.* **61** (1986), no. 2, 271–278.
- [22] Mumford, D. An algebraic surface with K ample, $(K^2) = 9$, $p_g = q = 0$. *Amer. J. Math.* **101** (1979) no. 1, 233–244.
- [23] Mustafin, G. A., Non-Archimedean uniformization (Russian), *Uspehi Mat. Nauk* **33** (1978) no. 1(199), 225–226.
- [24] Kurihara, Akira, Construction of p -adic unit balls and the Hirzebruch proportionality, *Amer. J. Math.* **102** (1980) no. 3, 565–648.
- [25] PARI/GP, version 2.4.3, Bordeaux, 2008, <http://pari.math.u-bordeaux.fr>
- [26] Prasad, Gopal and Yeung, Sai-Kee, Fake projective planes, *Invent Math.* **168** (2007) no. 2, 321–370. Addendum, *Invent. Math.* **182** (2010) no. 1, 213–227.
- [27] Siegel, C.L., Some remarks on discontinuous groups, *Ann. of Math. (2)* **46** (1945), 708–718.
- [28] Siegel, C. L., Symplectic Geometry, *Am. Jour. Math.* **65** (1943), 1–86.
- [29] Stover, Matthew, Volumes of Picard modular surfaces, to appear in *Proc. A.M.S.*

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