



Best play in Dots and Boxes endgames

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Abstract

We give very simple algorithms for best play in the simplest kind of Dots and Boxes endgames: those that consist entirely of loops and long chains. In every such endgame we compute the margin of victory, assuming both players maximize the numbers of boxes they capture, and specify a move that leads to that result. We improve on results by Buzzard and Ciere on the same problem: our algorithms examine only the current position and do not need to consider the game tree at all.

Keywords Dots and Boxes · Nimstring · Control strategy

Mathematics Subject Classification 91A46

1 Introduction

Dots and Boxes is one of the few over-the-table games that has a rich mathematical theory and is also played at a high level by many people without any special interest in mathematics. There are several websites where one can play online against other people, such as littlegolem.net and yourturnmyturn.com.

We assume familiarity with the game, but give a quick statement of the rules. See Sect. 2 for some elements of good play, like *loops* and *long chains*, the *hard-hearted handout*, and the *long chain rule*. For further background we recommend Berlekamp (2000) or Berlekamp et al. (1982, Chapter 16).

The game begins with a grid of dots. When it is your turn, you draw an edge joining two (horizontally or vertically) adjacent dots that are not yet joined. If you complete at least one box this way, then you move again immediately. You continue in this way until you move without completing a box (when play passes to your opponent) or you complete the last box (when the game ends). Whoever completes more boxes wins.

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The board size is typically 5×5 , meaning 5 boxes by 5 boxes. Odd \times odd boards are best because they make ties impossible.

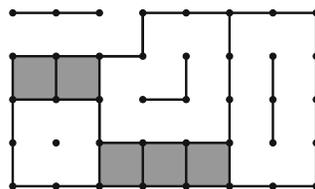
The first contribution of mathematics to Dots and Boxes is the idea of moving as though playing a simpler game called Nimstring. This is sometimes called Nimdots when applied to Dots and Boxes. The rules are the same except for who wins: in Nimstring, the loser is whoever completes the last box. This sounds like a radical change, but good Nimstring moves are often good Dots and Boxes moves. As the name suggests, Nimstring is amenable to the Sprague-Grundy theory for Nim-like games (Berlekamp et al. 1982, Ch. 16). This leads to a trick called the long chain rule. If neither player knows it, then the game proceeds mostly randomly. If only one knows it, then she will win. If both know it, then the game becomes challenging and interesting. The interest and challenge come from understanding the positions where there is a winning move, even when the Nimstring perspective suggests the game is lost.

An expert player who expects to lose the Nimstring game will try to steer the game toward such a position. Typically she does this by aiming for an endgame with many 3-chains and 4- and 6-loops. Therefore high-level play requires an understanding of such positions. This paper gives a complete analysis of endgames that consist entirely of loops and long chains. We restrict attention to such positions unless otherwise indicated. We were inspired by the study of these endgames by Buzzard and Ciere (2013). While their algorithms for determining the value of a game and an optimal move require only linear time, they involve trees of cases and subcases, and also a limited form of recursion. Our algorithms have very simple decision trees, and examine only the current position.

If it is your turn and you must choose a loop or long chain to open (play in), then we call you the *opener*. Once you choose this component C , it does not matter where in C you play. Your opponent will either take all those boxes, or reply with the hard-hearted handout (see below and Sect. 2). Where you play in C has no effect on these options.

Nimstring suggests that you have lost. We will see that you cannot win the endgame, in the sense that you cannot capture more of the remaining boxes than a skilled opponent. But if you captured more boxes than she did, before reaching the endgame, and you lose the endgame by only a little, then you may still be able to win. The strategy expressed in Theorem 1.1 is guaranteed to lose the endgame by as little as possible, and therefore gives your best chance of winning. Your opponent is called the *controller*, for reasons explained below.

A chain or loop of length N is called an N -chain or N -loop, and often indicated by simply writing N , attaching a subscript ℓ to indicate a loop. We use additive and exponential notation in the obvious way, for example $G = 3 + 4_\ell + 8_\ell^2$ means that G consists of a 3-chain, a 4-loop, two 8-loops and possibly some already-claimed boxes:



In actual play the claimed boxes (shaded here) would be marked with the players' initials. We ignore such boxes when discussing G because they do not affect play.

The *value* $v(G)$ of any endgame G means the margin by which the controller will beat the opener, assuming that they enter the endgame with a tie score and then play optimally. So $v(3 + 4_\ell + 8_\ell^2) = 1$ means that the opener can win if and only if she earned at least a 2-box advantage during whatever play led to this endgame. To lose this endgame by only one box, open the 4-loop first and then the 8-loops.

The *size* $\text{size}(G)$ of G means the number of (unclaimed) boxes in G .

Berlekamp introduced a variation on $v(G)$ called the *controlled value* (Berlekamp 2000, p. 84). He wrote $cv(G)$, but we will write $c(G)$. It is easy to compute in your head, and (non-obviously) carries a lot of information about $v(G)$. The definition is

$$c(G) = \text{size}(G) - 4(\# \text{ long chains}) - 8(\# \text{ loops}) + \text{tb}(G), \quad (1)$$

where $\text{tb}(G)$ is called the *terminal bonus* of G and defined by

$$\text{tb}(G) = \begin{cases} 0 & \text{if } G \text{ is empty} \\ 8 & \text{if } G = (\text{one or more loops}) \\ 6 & \text{if } G = (\text{one or more loops}) + (\text{one or more 3-chains}) \\ 4 & \text{otherwise.} \end{cases} \quad (2)$$

See Sect. 2 for why $c(G)$ is called the controlled value.

The *standard move* means to open a 3-chain if one is present, otherwise a shortest loop if a loop is present, and otherwise a shortest chain. We will prove the following result in Sect. 3.

Theorem 1.1 (Opener strategy) *Suppose G is a nonempty Dots and Boxes position that consists of loops and long chains. In each of the following cases, opening the shortest loop is optimal:*

- (i) $c(G) \geq 2$ and $G = 3 + (\text{one or more loops})$;
- (ii) $c(G) \in \{0, \pm 1\}$ and $G = 4_\ell + (\text{anything except } 3 + 3 + 3)$;
- (iii) $c(G) \leq -2$ and $G = 4_\ell + 3 + H$, where $4 \mid \text{size}(H)$ and H has no 3-chains.

In all other cases the standard move is optimal.

The “phase transition” when $c(G)$ changes from 1 to 2 was discovered by Berlekamp and Scott (2002) and Scott (2000). The transition from -2 to -1 appears to be new.

Example 1.2 Suppose G consists of five 3-chains, a 4-loop and an 8-loop. We have $c(G) = 27 - 4 \cdot 5 - 8 \cdot 2 + 6 = -3$. The theorem says that opening the components in the order 3, 3, 4_ℓ , 3, 3, 8_ℓ , 3 is optimal. We used a computer to check this, and also that the only other optimal line of play is 3, 4_ℓ , 3, 3, 3, 8_ℓ , 3. This shows that there is a certain subtlety to the order in which one must open the components, and suggests that Theorem 1.1 may be the simplest strategy possible.

If you are the controller, and the opener has just opened a loop or long chain, then the decision you face is whether to keep control or give it up. *Giving up control* means that you take all of the boxes in the just-opened component. Unless that ends the game you must move again, so you become the opener and your opponent the controller. As the new opener, you can use Theorem 1.1 to choose which component to open. *Keeping control* means that you take all but a few of the boxes in the opened component (4 for a loop or 2 for a long chain). This is called the hard-hearted handout: a handout because you are giving your opponent those boxes, and hard-hearted because after she takes them she must open the next component. In this case she remains the opener and you remain the controller.

Now we consider the situation where the opener has just opened component C of a position G . It is easy to see that the controller should keep control if $v(G - C)$ is larger than the number of boxes (2 or 4) given away in the hard-hearted handout, and give up control if $v(G - C)$ is smaller. She may choose either option in case of equality. We use subtractive notation in the obvious way: $G - C$ means the position got from G by removing C . So the following well-known result gives an optimal strategy for the controller [4, Section 5].

Theorem 1.3 (Controller strategy) *If the opener has just opened a component C of G , then the following gives an optimal move for the controller. Keep control if C is a loop and $v(G - C) > 4$, or if C is chain and $v(G - C) > 2$; otherwise give up control. \square*

To use this strategy the controller must be able to recognize when $v(G - C) > 4$ or 2. The following theorems enable this. They are corollaries of Theorem 4.1, which gives all values explicitly.

Theorem 1.4 (Values > 4) *We have $v(G) > 4$ if and only if $c(G) > 4$.*

Theorem 1.5 (Values > 2) *Suppose G is nonempty. Then $v(G) > 2$ if and only if: either $c(G) > 2$, or else G satisfies one of the two alternatives*

- (1) G has exactly one 3-chain and $\text{size}(G) \equiv 3 \pmod{4}$
- (2) G has no 3-chains and $\text{size}(G) \not\equiv 2 \pmod{4}$,

and one of the two alternatives

- (a) $c(G) + 4f(G) > 2$ and $c(G) \equiv \pm 3$ or $4 \pmod{8}$
- (b) $c(G) + 4f(G) < 2$ and $f(G)$ is even,

where $f(G)$ is the number of 4-loops in G .

In particular, if two or more 3-chains are present, then $v(G) > 2$ if and only if $c(G) > 2$. The same holds if just one is present and $\text{size}(G) \not\equiv 3 \pmod{4}$. In the remaining cases we prefer Theorem 4.2(3) to the complicated second pair of alternatives. It gives $v(G)$ as the result of a simple process applied to a simple starting number, and is much easier to remember.

Section 5 gives some consequences for mid-game strategy. For example, the player who is opener when the endgame begins will probably lose if she has only a 1-box advantage, but very likely win with a 2-box advantage. This assumes an odd \times odd board and that enough 3-chains and 4- and 6-loops are present to make $c(G) < 2$.

Players often sacrifice a box or two during the midgame to create the “right” number of long chains. (The goal is to be controller in the endgame, because in Nimstring the controller always wins. See the long chain rule in Sect. 2.) Our results show that sacrificing one box is safe but sacrificing two is suicide, unless the resulting endgame is quite special.

As a sanity check, we used a computer to verify Theorems 1.1, 1.4, 1.5, 4.1, 4.2, 5.1 and 5.2. for all positions that consist of loops and long chains and have size at most 11^2 . There are around 1.5 million positions, after taking into account loop- and chain-amalgamation [4, Section 7]. We computed their values recursively, using (3)–(4), and then compared the results with the statements of our theorems.

In Nimstring, positions consisting of long loops and long chains are examples of “loony” endgames, which roughly means that no matter how the first player moves, the second may choose to keep control or give it up. See Berlekamp (2000), Berlekamp and Scott (2002) and Berlekamp et al. (1982) for a precise definition and for analyses of many loony Nimstring endgames, such as vines, crosses, dippers, earmuffs and bracelets. Sometimes these come up in actual Dots and Boxes play, so it would be nice to generalize our algorithms.

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2 Background

In the first part of this section we review some standard elements of good play. The main references are Berlekamp et al. (1982, Ch. 16) and Berlekamp (2000). In the second part we discuss some more-technical material that we will need.

Because a player moves again immediately if she completes at least one box, her turn may consist of more than one move. It often happens that the segment completing a box also forms the third edge of another box. In this case the extra move earned by completing the first box enables her to complete the second box, which in turn might allow her to complete a third box, and so on. In this way she might capture an entire chain of boxes in a single turn.

A sample position appears in Fig. 1. During earlier play the author completed three boxes and placed his initial inside them to mark them as his. This makes it look like he is winning, but it is his turn and no good move is available. Any move in the loop on the right lets his opponent C (for “controller”) capture all 10 boxes there. Similarly, any move in the chain on the left lets her capture all 12 of its boxes. When a child, the author would have moved in the loop, expecting her to capture those 10 boxes and then open the 12-chain for him to capture. He would win, 15–10.

Unfortunately for the author, C knows the *hard-hearted handout*: when he offers her the 10 boxes, she takes all but 4, for example as shown in the top line of play. Dotted lines indicate her moves. Note the 2×1 and 1×2 rectangles that she could have claimed but chose not to. Your poor author wins those 4 boxes but then must move again, opening the 12-chain for her to capture entire. Therefore C wins, 18–7. If

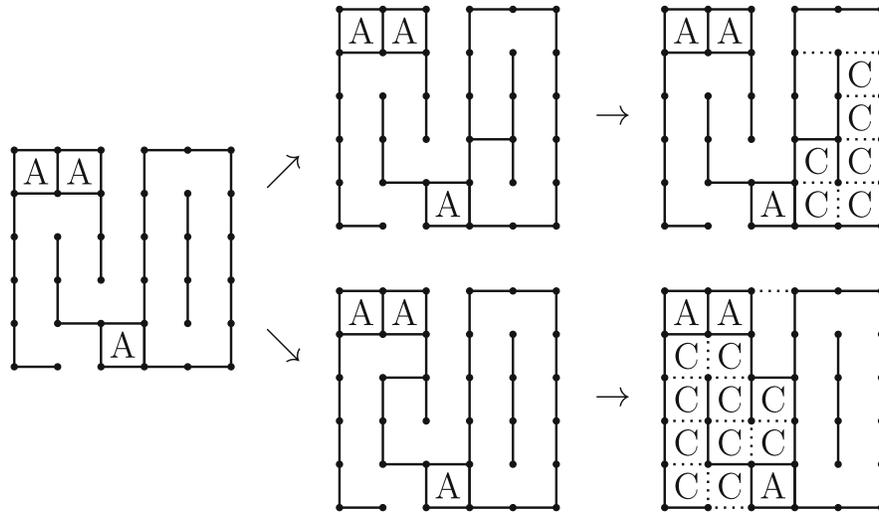


Fig. 1 A Dots and Boxes position consisting of a 12-chain and a 10-loop, and two lines of play demonstrating the hard-hearted handout

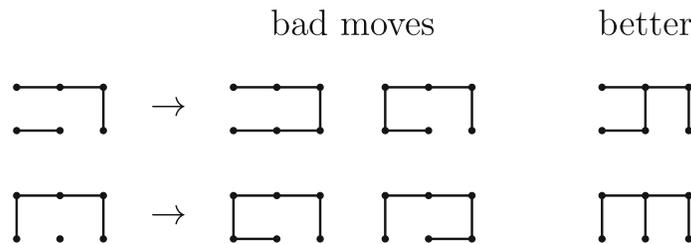


Fig. 2 Opening a 2-chain in one of the “bad” ways allows the opponent to choose between capturing both boxes and then moving again, or replying with the hard-hearted handout. Bisecting the 2-chain as shown on the right removes the opponent’s second option

the author opens the chain instead of the loop, then C takes all but two of its boxes, as shown in the bottom line of play. This is the chain version of the hard-hearted handout. In this case the author would lose 20-5. (A few people play with a rule that forbids the hard-hearted handout: if one can complete a box, then one must. But forced greedy strategies make games dull.)

A chain of length 2 does not present the same problem, because if you split it into two single boxes, then your opponent has no choice but to open the next component. Two examples appear in Fig. 2. The “long” in “long chain” restricts attention to chains of length 3 or more. In this paper we are restricting attention to positions consisting of loops and long chains, so chains of length 2 will not appear. So henceforth we will write “chain” for long chain, sometimes saying “long chain” just for emphasis.

If C always used the hard-hearted handout, except in the last component, then she would capture all but 2 from each chain and all but 4 from each loop, except that she would capture all of the last component. If the loops and chains are not too small, then this leads to very lopsided scores, with the loser being whoever opens the first loop or long chain. This is the connection between Dots and Boxes and nimstring, and leads to the *long chain rule* (Berlekamp et al. 1982, Ch. 16). This rule predicts that the first resp. second player will win if there are an even resp. odd number of long chains in

the endgame (assuming an odd \times odd board). So in advanced play, the early game and midgame are all about trying to create chains, or obstruct their creation, so that the desired number (mod 2) are formed.

As in the introduction, whichever player has no choice but to open some chain or loop is the *opener*, and whoever replies to such a move is the *controller*. In Fig. 1 the author was the opener and his opponent the controller. The players sometimes exchange roles during a game, because in some positions the controller should give up control by taking all of an opened chain or loop. By doing this she becomes the opener and her opponent becomes the controller.

For example, if the position consists of $N > 1$ chains of length 3, one of which was just opened, then a controller who keeps control except at the end will give her opponent 2 boxes from each chain but the last. She scores $(N - 1) + 3$ to her opponent's $2(N - 1)$. This is suboptimal if $N > 3$; indeed it leads to her losing if $N > 4$. It is better to take all 3 boxes of the opened chain. Being forced to move again, she is now the opener: she opens a chain and her opponent (now the controller) faces a similar decision. In one line of optimal play, the players take turns giving up control, except that whoever responds in the second-to-last chain keeps control, giving up 2 boxes there but getting all 3 in the last chain. If N is even, then the original controller will be the one to do this, and wins $\frac{3}{2}N + 1$ to $\frac{3}{2}N - 1$. When N is odd, she still wins $\frac{3}{2}N + \frac{1}{2}$ to $\frac{3}{2}N - \frac{1}{2}$.

This article is about deciding what to open when you must open something, and deciding whether to keep control when you have it.

The rest of this section is more technical. We have said that we will consider only Dots and Boxes positions G consisting of loops and long chains, where long means of length ≥ 3 . This is not literally true in two cases. Immediately after the opener has opened a component C , and whenever the controller keeps control, one or two opened components are present. But we will be able to phrase all our analysis in terms of $G - C$. So we adopt the convention that unless opened components are specifically allowed, Dots and Boxes positions consist of *unopened* loops and long chains. We also note that the length of a loop must be even and at least 4, by the geometry of the grid.

If G is a Dots and Boxes position, then we define

- size(G) number of boxes not yet claimed
- $\theta(G)$ number of 3-chains
- $f(G)$ number of 4-loops
- $s(G)$ number of 6-loops
- $v(G)$ value—see Sect. 1 and below
- $c(G)$ controlled value—see (1) and below
- tb(G) terminal bonus—see (2)

Recall that when discussing G we ignore any boxes captured earlier in the game, so size(G) is not the total number of boxes in the grid, but rather the number of unclaimed boxes. We call G even or odd according to whether size(G) is even or odd. Usually the symbol G will represent whatever position is of central interest, so we will use the abbreviations θ , f , s , v and c for $\theta(G)$, $f(G)$, $s(G)$, $v(G)$ and $c(G)$. We will still use functional notation for other positions, for example $\theta(G + 3) = \theta + 1$.

We recall the meaning of the value $v(G)$ from Sect. 1: the opener will lose by $v(G)$ boxes if both players maximize the number of boxes they take. The trivial example is that $v(G) = \text{size}(G)$ when G consists of a single component. After the opener opens it, the controller “gives up control” by claiming all the boxes and ending the game. The next example is $v(3 + 3) = 2$. The opener opens a chain, and the controller keeps control (replying with the hard-hearted handout). The opener accepts the 2 boxes in the handout and opens the other chain, all of which the controller claims. So $v(3 + 3) = 4 - 2 = 2$.

Once the opener has opened a component C , the controller has really only two choices. First, she may keep control and then play optimally, in which case she will finish the game with

$$\begin{aligned} &(\text{size}(C) - 4) + v(G - C) \text{ if } C \text{ is a chain} \\ &(\text{size}(C) - 8) + v(G - C) \text{ if } C \text{ is a loop} \end{aligned}$$

more boxes than her opponent. Second, she may give up control and then play optimally, in which case she will win by $\text{size}(C) - v(G - C)$. The minus sign appears because when the controller gives up control, she enters the position $G - C$ as the opener, so her opponent is the one who will score $v(G - C)$. We write $v(G; C)$ for the higher of these two margins of victory. One should think of this as the value of G , given that the opener has just opened C .

Obviously the controller will choose the higher-scoring option, leading to

$$v(G; C) = \begin{cases} (\text{size}(C) - 2) + |v(G - C) - 2| & \text{if } C \text{ is a chain} \\ (\text{size}(C) - 4) + |v(G - C) - 4| & \text{if } C \text{ is a loop.} \end{cases} \quad (3)$$

We will use this repeatedly without specific reference. Another way to express this is that if $v(G - C) = 2$ resp. 4 and C is a chain resp. loop, then keeping and giving up control are equally good options. If $v(G - C)$ is more than this, then the controller should keep control. If it is less, then she should give up control.

Now we switch perspectives. Faced with a nonempty position G , the opener will obviously prefer to open whichever component C minimizes $v(G; C)$. So

$$v(G) = \min_C v(G; C), \quad (4)$$

where C varies over the components of G . The previous paragraph shows that each $v(G; C)$ is nonnegative, which proves $v(G) \geq 0$. This justifies our assertion that the opener can never win the endgame. Also, combining (4) and the previous paragraph gives a way to evaluate $v(G)$ recursively. As an example we work out how to play the position $G = 3^n$, meaning that G consists of n many 3-chains. The opener has no real choice about which component to open, so $v(3^n) = v(3^n; 3) = 1 + |v(3^{n-1}) - 2|$ whenever $n > 0$. Induction gives $v(3^n) = 0, 3, 2, 1, 2, 1, 2, 1, \dots$ when $n = 0, 1, 2, 3, 4, 5, 6, 7, \dots$. So the controller must keep control if $n = 2$, and must give up control if either $n = 1$ or n is even and larger than 2. In all other cases the choice of

keeping or giving up control makes no difference. The simplest rule is to keep control if and only if $n = 2$.

Now we can explain the controlled value $c(G)$, defined in (1). Another useful formula for it is

$$c(G) = \sum_i (c_i - 4) + \sum_j (l_j - 8) + \text{tb}(G), \quad (5)$$

where c_1, c_2, \dots are the lengths of the chains, l_1, l_2, \dots are the lengths of the loops, and the terminal bonus was defined in (2). Berlekamp introduced the idea of a controller who follows the *control strategy* of always keeping control, except in a few cases when she obviously should not. There is disagreement in the literature about what exactly the control strategy is. Berlekamp's description in Berlekamp (2000, p. 84) is: keep control until all unopened components are chains of length < 4 or loops of length < 8 , "and then play optimally". He states that this guarantees a score of at least $c(G)$. Berlekamp and Scott (2002, p. 318) say: keep control until at least the last- or second-to-last turn. But this strategy fails to score $c(G)$ if $G = 4_\ell + 3 + 3$. An opener who opens the 4-loop will win by 2, even though $c(G) = 0$. Buzzard and Ciere [4, Section 9] prove that their version guarantees a score of at least $c(G)$. They say to keep control until the last move, unless a loop has just been opened and only 3-chains remain. In the latter case, give up control, and then play optimally (which is easy in a union of 3-chains).

Our version is: when the opener opens a component C , give up control if and only if $\text{tb}(G - C) < \text{tb}(G)$. We do not need to say what to open when not in control, because only 3-chains will remain.

Theorem 2.1 (Controlled value) *Suppose G is a Dots and Boxes position consisting of loops and long chains. A controller who follows the control strategy will score at least $c(G)$. Against a skilled opponent who knows the controller is using this strategy, the controller will score exactly $c(G)$.*

Proof We induct on the number of components, the case of empty G being trivial. We write K resp. O for the player who is initially the controller resp. opener. If O opens component C , and $\text{tb}(G - C) \geq \text{tb}(G)$, then K will keep control. By induction, she will score at least $c(G - C)$ in $G - C$. So the score in G will be at least

$$\begin{aligned} & \left(\begin{array}{ll} \text{size}(C) - 4 & \text{if } C \text{ is a chain} \\ \text{size}(C) - 8 & \text{if } C \text{ is a loop} \end{array} \right) + c(G - C) \\ & = c(G) + \text{tb}(G - C) - \text{tb}(G) \geq c(G). \end{aligned}$$

There are only two cases in which removing C can decrease the terminal bonus. In the first case, C is the last component, and K will score $\text{size}(C) = c(G)$. In the other case, C is a loop and $G - C$ consists of $k \geq 1$ many 3-chains. So K will give up control immediately, capturing all of C . Then O will keep control j many times, for some $0 \leq j \leq k$. Thereafter, and only if $j < k$, O gives up control. Thereafter, and only if $j < k - 1$, K keeps control except for the last component. By examining the candidates

for j one sees that K scores at least $\text{size}(C) - 3$ (if $k = 1$) or $\text{size}(C) - 2$ (if $k = 2$) or $\text{size}(C) - k + 2$ (if $k > 2$). In every case this is at least $c(G) = \text{size}(C) - k - 2$.

Now we suppose O knows that K is using the control strategy. We give a strategy for O that leads to a final score of $c(G)$. If G has a chain of length ≥ 4 , or no chains at all, or no loops at all, then O opens all loops and then all chains. This leads K to keep control until the last component. Otherwise, G has at least one loop, at least one 3-chain, and no longer chains. In this case O opens all but one of the chains, then all of the loops. When the last loop is opened, K will give up control, and O will capture the remaining 3-chain. \square

We close this section with a few easy and useful relations between $v(G)$, $c(G)$ and $\text{size}(G)$. There are also two more relations that we will prove later. First, adding the hypothesis that G is even to part (3) strengthens its conclusion to $\text{size}(G) \equiv c(G) \equiv v(G) \pmod{4}$. (See Theorem 4.1.) Second, our Theorem 3.2 contains the result of Berlekamp and Scott that $c(G) \geq 2 \implies c(G) = v(G)$.

Lemma 2.2 *Suppose a Dots and Boxes position G consists of loops and long chains, and C is a component of G . Then*

- (1) $v(G) \geq c(G)$;
- (2) $\text{size}(G) \equiv c(G) \equiv v(G) \equiv v(G; C) \pmod{2}$;
- (3) $\text{size}(G) \equiv c(G) \pmod{4}$ if G has no 3-chains.

Proof (1) The controller may guarantee a score of at least $c(G)$ by using the control strategy.

(2) Both $c(G)$ and $v(G)$ have the same parity as $\text{size}(G)$ because they are margins of victory under certain lines of play. For the last congruence, it follows from Eq. (3) (not part (3) of the lemma) that $v(G; C)$ has the same or different parity as $v(G - C)$ according to whether $\text{size}(C)$ is even or odd. Also, $\text{size}(G)$ has the same or different parity as $\text{size}(G - C)$ under the same conditions. Since $\text{size}(G - C)$ and $v(G - C)$ have the same parity, it follows that $v(G; C) \equiv \text{size}(G) \pmod{2}$.

(3) When no 3-chains are present the terminal bonus is divisible by 4. So our claim follows from the definition of $c(G)$ in (1). \square

3 Opener strategy

Our goal in this section is to prove Theorem 1.1, under the standing hypothesis that G is a Dots and Boxes position consisting of unopened loops and long chains. Although our proof is logically independent of the work of Buzzard and Ciere (2013), and organized very differently, we would not have been able to formulate Theorem 1.1 or the lemmas below without reference to it. We regret that we do not use their lovely chain- and loop-amalgamation arguments.

Lemma 3.1 (Very small positions) *Suppose G has one component, or consists of a 3-chain and a loop. Then $v(G) = c(G)$. In the second case, opening the loop is optimal.*

Proof The first case is obvious. For the second case one examines the four possible lines of play. (Opening the 3-chain is not optimal.) \square

The next theorem summarizes the key points of the lemma after it. After establishing it we will use the implication $(c \geq 2) \implies (v = c)$ many times without specific reference. The lemma and its proof are essentially the same as lemma 15 from Buzzard and Ciere (2013), that one can “fly the plane without crashing” in the sense of Berlekamp and Scott (2002).

Theorem 3.2 (Large controlled values) *Suppose $c(G) \geq 2$. Then $v(G) = c(G)$ and the following gives an optimal move:*

- (1) *Open a 3-chain (if G has one and at least one other chain).*
- (2) *Otherwise, open a shortest loop (if G has a loop).*
- (3) *Otherwise, open a shortest chain.*

Lemma 3.3 (Large controlled values—details) *Suppose $c(G) \geq 2$.*

- (1) *If C is any component satisfying*

$$\text{tb}(G - C) = \text{tb}(G) \text{ and } c(G - C) \geq \begin{cases} 2 & \text{if } C \text{ is a chain} \\ 4 & \text{if } C \text{ is a loop,} \end{cases}$$

then opening C is optimal.

- (2) *If G has a 3-chain, 4-loop or 6-loop, whose removal does not alter the terminal bonus, then opening it is optimal.*
- (3) *Suppose G has no 3-chains, or exactly one 3-chain and no other chains. If G has a loop, then opening a shortest loop is optimal.*
- (4) *If G has no loops, then opening a shortest chain is optimal.*
- (5) *The value $v(G)$ is equal to the controlled value $c(G)$.*

Proof We induct on the number of components. If G has one component, then (1)–(2) are vacuous and (3)–(5) are trivial. So suppose G has at least two components.

- (1) Write l for the length of C , and assume first that C is a loop. We have

$$v(G; C) = (l - 4) + |v(G - C) - 4|.$$

Since $c(G - C) \geq 4$ by the hypothesis of (1), $v(G - C) = c(G - C) \geq 4$ follows by induction. So we have

$$v(G; C) = (l - 4) + c(G - C) - 4 = (l - 8) + c(G - C).$$

Since G and $G - C$ have the same terminal bonus, $c(G - C) = c(G) - (l - 8)$. Plugging this in gives $v(G; C) = c(G)$. Together with Lemma 2.2 this gives

$$c(G) \leq v(G) \leq v(G; C) = c(G).$$

Therefore $v(G; C) = v(G)$, which proves optimality, and $v(G) = c(G)$. If C is a chain, then the argument is the same with all 4s replaced by 2s and all 8s by 4s.

(2) First suppose G has a 3-chain with $\text{tb}(G - 3) = \text{tb}(G)$. Then $c(G - 3) = 1 + c(G) \geq 3$. So (1) applies to the 3-chain. And similarly for a 4-loop or 6-loop.

(3) By hypothesis, G consists of either at least one loop and a 3-chain, or at least one loop and possibly some chains of length ≥ 4 . Let C be a shortest loop. In the special case $G = C + 3$, opening C is optimal by Lemma 3.1. So suppose otherwise: that G has a second loop, or a chain of length ≥ 4 . In either case the removal of C leaves the terminal bonus invariant. If $C = 4_\ell$ or 6_ℓ , then we are done by (2). There are two remaining cases. First, G consists of a 3-chain and two or more loops of length ≥ 8 . Second, G consists of at least one loop of length ≥ 8 and possibly some chains of length ≥ 4 . In each case we write out formula (5) for $c(G - C)$ and use the absence of 4- and 6-loops. The results in the two cases are

$$c(G - C) = (3 - 4) + (\text{nonnegative terms}) + 6 \geq 5$$

$$\text{and } c(G - C) = (\text{nonnegative terms}) + (4 \text{ or } 8) \geq 4,$$

respectively. So (1) shows that opening C is optimal.

(4) Because G consists of at least 2 chains, removing any one of them leaves the terminal bonus unchanged. Let C be shortest possible. If it is a 3-chain, then we appeal to (2). Otherwise we mimic the argument at the end of the proof of (3) and get

$$c(G - C) = (\text{nonnegative terms}) + 4 \geq 4.$$

Since the right side is at least 2, it follows from (1) that opening C is optimal.

(5) One of (2)–(4) applies to G . Their proofs show that either G consists of a 3-chain and a loop, or else G has a component C satisfying (1). In the first case we appeal to Lemma 3.1 for the equality $v = c$. In the second case we observed $v = c$ during the proof of (1). \square

Lemma 3.4 (Example of 6_ℓ -optimality) *Suppose $G \neq \emptyset$ has no 3-chains or 4-loops, and $c(G) \leq 4$. Then*

$$v(G) = 3, 2, 3, 4 \text{ if } \text{size}(G) \equiv 1, 2, 3, 4 \pmod{4}, \text{ respectively.}$$

Furthermore, if $c(G) < 2$, then G has at least two 6-loops and opening one of them is optimal.

Proof We write w for the function appearing on the right, namely

$$w(H) = 3, 2, 3, 4 \text{ if } \text{size}(H) \equiv 1, 2, 3, 4 \pmod{4}, \text{ respectively,}$$

for every position H . We must prove $v(G) = w(G)$. First we establish the cases $c = 2, 3, 4$. By the absence of 3-chains, Lemma 2.2(3) gives $\text{size}(G) \equiv c \pmod{4}$. So the definition of w gives $w(G) = c$. And Theorem 3.2 gives $v = c$, completing the proof when $c \geq 2$.

So we may suppose $c < 2$. We write this out using formula (5), as

$$2 > c = -\theta - 4f - 2s + (\text{nonnegative terms}) + \text{tb}(G).$$

By $\text{tb}(G) \geq 4$ and $\theta = f = 0$ we get $s \geq 2$. Since there are at least two loops, the removal of any loop leaves the terminal bonus unchanged. In particular, $c(G - 6_\ell) = c + 2$. Also, G has a third component because $c(G) < 2$ and $c(6_\ell + 6_\ell) = 4$. Having made these preparations, we now induct on the number of components.

First we claim $v(G - 6_\ell) = w(G - 6_\ell)$. If $c(G - 6_\ell) \geq 2$, then $c(G - 6_\ell) = 2$ or 3 , which are cases already proven. This includes the base case that G has 3 components, because then $G - 6_\ell = 6_\ell + C'$ with $C' \neq 3, 4_\ell$, which forces $c(G - 6_\ell) \geq 2$. On the other hand, if $c(G - 6_\ell) < 2$, then induction gives $v(G - 6_\ell) = w(G - 6_\ell)$.

Next we claim $v(G; 6_\ell) = w(G)$. The previous paragraph gives us

$$v(G; 6_\ell) = 2 + |w(G - 6_\ell) - 4| = \begin{cases} 3 & \text{if } \text{size}(G - 6_\ell) \text{ is odd} \\ 2 & \text{if } \text{size}(G - 6_\ell) \equiv 4 \pmod{4} \\ 4 & \text{if } \text{size}(G - 6_\ell) \equiv 2 \pmod{4}. \end{cases} \quad (6)$$

Because the sizes of G and $G - 6_\ell$ differ by $2 \pmod{4}$, the right side of (6) is $w(G)$, proving the claim. Once we prove that opening a 6-loop is optimal it will follow that $v = v(G; 6_\ell) = w(G)$ and the induction will be complete.

Suppose that some component $C \neq 6_\ell$ has $v(G; C) < v(G; 6_\ell)$. We have

$$2 \leq v(G; C) \equiv_{\text{mod } 2} v(G; 6_\ell) \leq 4. \quad (7)$$

The first inequality comes from (3) because C is a chain of length ≥ 4 or a loop of length ≥ 8 . The congruence is by Lemma 2.2(2), and we saw $v(G; 6_\ell) \leq 4$ in the previous paragraph. This forces

$$v(G; C) = 2 \quad \text{and} \quad v(G; 6_\ell) = 4. \quad (8)$$

Together with (3), the first shows that C is a 4-chain with $v(G - 4) = 2$. Together with (6), the second shows that $\text{size}(G) \equiv 4 \pmod{4}$.

This is impossible because $v(G - 4) \equiv 4 \pmod{4}$: by induction if $c(G - 4) < 2$, or by

$$v(G - 4) = c(G - 4) \equiv \text{size}(G - 4) \equiv 4 \pmod{4}$$

if $c(G - 4) \geq 2$. (The first congruence is Lemma 2.2.) □

We will re-use the arguments establishing (7) and (8) several times, without giving the details each time.

Lemma 3.5 (First example of 3-optimality) *Suppose $c(G) < 2$ and that G has no 4-loops, but does have a 3-chain. Then opening a 3-chain is optimal, and*

$$v(G) = \begin{cases} 2 & \text{if } G \text{ is even} \\ 3 & \text{if } \theta = 1 \text{ and } \text{size}(G) \equiv 3 \pmod{4} \\ 1 & \text{otherwise.} \end{cases} \quad (9)$$

Proof As we did in the proof of Lemma 3.4, we write w for the function on the right side of (9).

We will prove $v(G) = w(G)$ by induction on θ . In both the base case and the inductive step the strategy is to show $v(G; 3) = w(G)$. Given this, the optimality of a 3-chain follows from

$$2 \leq v(G; C) \equiv_{\text{mod } 2} v(G; 3) = w(G) \leq 3$$

for every component C other than a 3-chain.

Now for the induction. By the hypothesis $c < 2$, G does not consist of a single 3-chain, so there is another component. This implies that the terminal bonus rises by at most 2 if a 3-chain is removed. In particular, $c(G - 3) \leq c + 3 \leq 4$. In the case $\theta = 1$, $v(G - 3)$ is given by Lemma 3.4. This yields

$$v(G; 3) = 1 + |v(G - 3) - 2| = \begin{cases} 2 & \text{if size}(G - 3) \text{ is odd} \\ 3 & \text{if size}(G - 3) \equiv 4 \pmod 4 \\ 1 & \text{if size}(G - 3) \equiv 2 \pmod 4. \end{cases}$$

This is visibly equal to $w(G)$. Now suppose $\theta > 1$. We must show that $v(G; 3)$ is equal to whichever of 1 and 2 has the same parity as G . It is enough to show $v(G; 3) \in \{1, 2\}$. This will follow once we prove $v(G - 3) \in \{1, 2, 3\}$. We observe $c(G - 3) = c + 1 \leq 2$. In the case $c(G - 3) = 2$ we have $v(G - 3) = c(G - 3) = 2$. In the case $c(G - 3) < 2$ we have $v(G - 3) \in \{1, 2, 3\}$ by induction. This finishes the proof. \square

Lemma 3.6 (3-chains + 4-loops) *Suppose G consists of 3-chains and 4-loops. Then*

$$v(G) = \begin{cases} (\text{whichever of } 0, 4) \equiv \text{size}(G) \pmod 8 & \text{if } \theta = 0 \\ 3 & \text{if } G = 3 + 4_\ell^{\text{even}} \\ (\text{whichever of } 1, 2) \equiv \text{size}(G) \pmod 2 & \text{otherwise.} \end{cases}$$

Furthermore,

- (1) *Opening a 4-loop is optimal if and only if θ is even or 1.*
- (2) *Opening a 3-chain is optimal if and only if $\theta \geq 2$ or f is even.*

Proof The following table displays the value of every position consisting of 3-chains and 4-loops. The top left entry corresponds to the empty game, with value 0. The values of positions consisting of 3-chains appear in the left column. We worked them out inductively in Sect. 2, as an example of using (3). A similar calculation gives the values of positions consisting of 4-loops, which appear in the top row. The remaining entries follow from (4): each is the smaller of

$$v(G; 4_\ell) = |(\text{the entry to the left}) - 4|$$

and $v(G; 3) = 1 + |(\text{the entry above}) - 2|.$

It follows by induction that the table gives $v(G)$, which justifies all our claims.

Best play in Dots and Boxes endgames

		f	0	1	2	3	4	5	6
θ	0		0	4	0	4	0	4	0
	1		3	1	3	1	3	1	3
	2		2	2	2	2	2	2	2
	3		1	1	1	1	1	1	1

Starting with $\theta = 2$, the rows alternate between all 2s and all 1s. □

Lemma 3.7 (First example of 4_ℓ -optimality) *Suppose G has a 4-loop and $c(G) \geq -2$. Then*

- (1) $v(G) = |c(G)|$ unless $G = 3 + 3 + 4_\ell$, in which case $v(G) = 2$;
- (2) opening a 4-loop is optimal unless $G = 3 + 3 + 3 + 4_\ell$.

Proof First we treat the case that removing a 4-loop changes the terminal bonus. This can only happen when G consists of a 4-loop and zero or more 3-chains. The condition $c \geq -2$ shows that the number of 3-chains is 0, 1, 2, 3 or 4. In these cases we have $c = 4, 1, 0, -1$ or -2 , and Lemma 3.6 gives $v = 4, 1, 2, 1$ or 2 respectively. As claimed, $v = |c|$ unless $G = 3 + 3 + 4_\ell$. For the optimality of opening a 4-loop in $3 + 4_\ell, 3^2 + 4_\ell$ and $3^4 + 4_\ell$, we refer to Lemma 3.6.

Now suppose removing a 4-loop leaves the terminal bonus invariant. So $c(G - 4_\ell) = c + 4$. We first prove $v(G; 4_\ell) = |c|$. By $c \geq -2$ we have $c(G - 4_\ell) \geq 2$, so $v(G - 4_\ell) = c(G - 4_\ell)$. If $c = -2$ resp. -1 , then $c(G - 4_\ell) = 2$ resp. 3 , so $v(G - 4_\ell) = 2$ resp. 3 , so $v(G; 4_\ell) = 2$ resp. 1 , which equals $|c|$. And if $c \geq 0$, then

$$v(G; 4_\ell) = |c(G - 4_\ell) - 4| = (c + 4) - 4 = c = |c|.$$

This completes the proof that $v(G; 4_\ell) = |c|$.

If $c \geq 2$, then the optimality of opening a 4-loop is Lemma 3.3(2). Otherwise, $v(G; 4_\ell) = |c| = 0, 1$ or 2 by the previous paragraph. If C is a component other than a 4-loop, then

$$1 \leq v(G; C) \equiv_{\text{mod } 2} v(G; 4_\ell) \leq 2.$$

So $v(G; C)$ cannot be less than $v(G; 4_\ell)$. This proves the optimality of a 4-loop, hence $v(G) = v(G; 4_\ell) = |c|$. □

Lemma 3.8 (Second example of 4_ℓ -optimality) *Suppose $c < 2$ and that $G \neq \emptyset$ has no 3-chains. If G has a 4-loop, then opening it is optimal. Regardless of whether G has a 4-loop, if $c + 4f \geq 2$, then*

$$v = 0, 1, 2, 3, 4 \text{ in the cases } c \equiv 0, \pm 1, \pm 2, \pm 3, 4 \pmod 8 \tag{10}$$

while if $c + 4f < 2$, then

$$v = \begin{cases} 2 & \text{if } c \equiv 2 \pmod{4} \\ \text{otherwise:} & \\ \begin{cases} \text{(whichever of } 0,1) \equiv c \pmod{2} & \text{if } f \text{ is odd} \\ \text{(whichever of } 3,4) \equiv c \pmod{2} & \text{if } f \text{ is even.} \end{cases} & \end{cases} \quad (11)$$

Remark 3.9 In (11) we could replace every occurrence of c by $\text{size}(G)$, because the absence of 3-chains implies $\text{size}(G) \equiv c(G) \pmod{4}$.

Proof We begin with three special cases. First, suppose G has no 4-loops. Then $c + 4f = c < 2$, so we are asserting that $v(G)$ is given by (11). This is justified by Lemma 3.4.

Second, suppose that G is a union of 4-loops. Then $c + 4f = -4f + 8 + 4f \geq 2$, so we are asserting that $v(G)$ is given by (10). This is justified by Lemma 3.6.

Third, suppose G has a 4-loop and that $c = -2, -1, 0$, or 1 . Then $c + 4f \geq 2$, so we are asserting that $v(G)$ is given by (10), namely $v(G) = 2, 1, 0$, or 1 respectively. This is justified by Lemma 3.7.

For the general case we use induction on the number of components. So suppose that every position with fewer components than G , that satisfies the hypotheses of the lemma, also satisfies its conclusions. Regarding the cases already treated as base cases, we may suppose that $c(G) < -2$ and that G has a 4-loop and a component other than a 4-loop, hence a longer loop or a chain of length ≥ 4 . The presence of this extra component shows that removing a 4-loop from G does not change the terminal bonus. In particular, $c(G - 4_\ell) = c + 4 < -2 + 4 = 2$. Therefore induction applies to $G - 4_\ell$. In particular, $v(G - 4_\ell)$ is given by whichever of (10) and (11) applies to $G - 4_\ell$. To figure out which one applies we observe

$$c(G - 4_\ell) + 4f(G - 4_\ell) = c(G) + 4f(G). \quad (12)$$

This shows that whichever of (10) and (11) claims to apply to G does indeed apply to $G - 4_\ell$.

As we did in the proof of Lemma 3.4, we write $w(\cdot)$ for the function on positions given by (10) and (11). Our next step is to prove that $v(G; 4_\ell)$ is equal to what the lemma claims is $v(G)$. That is, we will prove $v(G; 4_\ell) = w(G)$. Regardless of which of (10) and (11) applies to $G - 4_\ell$, we have $v(G - 4_\ell) = w(G - 4_\ell) \in \{0, 1, 2, 3, 4\}$. It follows that $v(G; 4_\ell) = 4 - v(G - 4_\ell)$. It is also easy to see that $w(G) = 4 - w(G - 4_\ell)$: for (10) this uses the fact that $c(G)$ and $c(G - 4_\ell)$ differ by $4 \pmod{8}$, while for (11) this uses the fact that $f(G)$ and $f(G - 4_\ell)$ have different parities. It follows that $v(G; 4_\ell) = w(G)$.

Now we prove the optimality of a 4-loop. Suppose for a contradiction that C is a component of G with $v(G; C) < v(G; 4_\ell)$. By

$$2 \leq v(G; C) \equiv_{\text{mod } 2} v(G; 4_\ell) \leq 4$$

we must have $v(G; C) = 2$ and $w(G) = v(G; 4_\ell) = 4$. The first of these forces C to be a 4-chain with $v(G - 4) = 2$, or a 6-loop with $v(G - 6_\ell) = 4$. Removing a 6_ℓ leaves the terminal bonus invariant, while removing a 4-chain increases it by 0 or 4. In either case we have

$$c(G - C) \leq c(G) + 4 < -2 + 4 = 2.$$

By induction, the lemma describes $v(G - C)$.

If C is a 4-chain, then $v(G - 4) = 2$ forces $c(G - 4) \equiv 2 \pmod{4}$, regardless of whether (10) or (11) applies to $G - 4$. But then the absence of 3-chains shows

$$c(G) \equiv \text{size}(G) \equiv \text{size}(G - 4) \equiv c(G - 4) \equiv 2 \pmod{4}.$$

This gives $w(G) = 2$, regardless of whether (10) or (11) applies to G . This contradicts our earlier conclusion that $w(G) = 4$. If C is a 6-loop, then $v(G - 6_\ell) = 4$ forces $c(G - 6_\ell) \equiv 0 \pmod{4}$, regardless of whether (10) or (11) applies to $G - 6_\ell$. This forces $c(G) \equiv 2 \pmod{4}$, leading to the same contradiction.

We have proven the optimality of opening a 4-loop. So $v(G) = v(G; 4_\ell) = w(G)$ follows and the proof is complete. \square

The next two lemmas, taken together, are an analogue of sections 11–13 of Buzzard and Ciere (2013). But the formulations and arguments are quite different.

Lemma 3.10 (Third example of 4_ℓ -optimality) *Suppose G has exactly one 3-chain, and $\text{size}(G) \equiv 3 \pmod{4}$ and $c < 2$. If G has a 4-loop, then opening it is optimal. Regardless of whether G has a 4-loop, if $c + 4f \geq 2$, then*

$$v(G) = 1 \text{ or } 3 \text{ according to whether } c(G) \equiv \pm 1 \text{ or } \pm 3 \pmod{8} \quad (13)$$

while if $c + 4f < 2$, then

$$v(G) = 1 \text{ or } 3 \text{ according to whether } f \text{ is odd or even.} \quad (14)$$

Proof This is similar in structure to the previous proof. Note that c is odd because G is. We begin with three special cases. First, if G has no 4-loops, then $c + 4f < 2$, so the lemma claims $v = 3$. This is justified by Lemma 3.5.

Second, if G has a 4-loop and $c(G) = \pm 1$, then $c + 4f \geq 2$, so the lemma claims that $v = 1$ and opening a 4-loop is optimal. This is justified by Lemma 3.7. (The exception $3 + 3 + 3 + 4_\ell$ there is irrelevant because $\theta = 1$.)

Third, suppose that G consists of a 3-chain and $f > 0$ many 4-loops. Then $c + 4f = -1 - 4f + 6 + 4f = 5$, so the lemma asserts that $v(G) = 1$ or 3 according to whether $c \equiv \pm 1$ or $\pm 3 \pmod{8}$. The cases $c \equiv -1$ or $3 \pmod{8}$ do not occur, and the cases $c \equiv 1$ or $-3 \pmod{8}$ are equivalent to f being odd or even, respectively. So we may quote Lemma 3.6.

For the general case we induct on the number of components. So suppose every position with fewer components than G , that satisfies the hypotheses of the lemma,

also satisfies its conclusions. Regarding the cases already treated as base cases, we may suppose that $c < -2$, and that G contains a 4-loop and also a component that is neither a 3-chain nor a 4-loop. In particular, removing a 4-loop does not alter the terminal bonus. So $c(G - 4_\ell) = 4 + c < 2$. By induction, the lemma describes $v(G - 4_\ell)$. In particular, $v(G - 4_\ell) = 1$ or 3 . These lead to $v(G; 4_\ell) = 3$ or 1 respectively.

Next we claim $v(G; 3) = 3$. We have $c(G - 3) \leq c + 4 < 2$, so Lemma 3.8 computes $v(G - 3)$. Which case of that lemma applies depends on the value of $c(G - 3) + 4f(G - 3)$. But both cases give

$$v(G - 3) \equiv \text{size}(G - 3) \equiv \text{size}(G) - 3 \equiv 0 \pmod{4}.$$

So $v(G - 3) \in \{0, 4\}$, which implies $v(G; 3) = 1 + |(0 \text{ or } 4) - 2| = 3$. This proves the optimality of opening a 4-loop, because any component $C \neq 3, 4_\ell$ has $v(G; C)$ odd and at least 2.

All that remains is to justify the stated value of $v(G)$. We know $v(G) = v(G; 4_\ell) = 4 - v(G - 4_\ell)$. On the other hand, the equality (12) from the previous proof also holds here, so that whichever of (13) and (14) purports to describe $v(G)$ does indeed describe $v(G - 4_\ell)$. Examining these two formulas shows that $4 - v(G - 4_\ell)$ is the claimed value of $v(G)$. For (13) this uses the fact that $c(G)$ and $c(G - 4_\ell)$ differ by 4, while for (14) this uses the fact that $f(G)$ and $f(G - 4_\ell)$ have different parities. \square

Lemma 3.11 (Second example of 3-optimality) *Suppose $c < -1$ and that either*

- (1) $\theta \geq 2$, or
- (2) $\theta = 1$ and $\text{size}(G) \not\equiv 3 \pmod{4}$.

Then opening a 3-chain is optimal, and $v = 1$ or 2 according to whether G is odd or even.

Proof We begin by treating three special cases. First, if G has no 4-loops, then we quote Lemma 3.5. Second, if G consists of 4-loops and 3-chains, then we quote Lemma 3.6.

Third, suppose that G consists of a 3-chain and at least one loop. Then G is odd, so our hypothesis on $\text{size}(G)$ forces $\text{size}(G) \equiv 1 \pmod{4}$. Also $c(G - 3) = 3 + c(G) < 2$, so Lemma 3.8 describes $v(G - 3)$. In order to apply it we need to know $c(G - 3)$ modulo either 8 or 4, depending on which of (10) or (11) applies to $G - 3$. Because $\theta(G - 3) = 0$, we have $c(G - 3) \equiv \text{size}(G - 3) \equiv 2 \pmod{4}$. Now Lemma 3.8 gives $v(G - 3) = 2$, no matter which of (10) and (11) applies to $G - 3$. From $v(G - 3) = 2$ we get $v(G; 3) = 1$. Since $v(G)$ is odd and bounded above by $v(G; 3)$, the 3-chain is optimal and $v(G) = 1$.

We have reduced to the case that G has at least two chains, that it has a 4-loop, and that it has a component that is neither a 3-chain nor a 4-loop. It follows that the terminal bonus is invariant under the removal of either a 3-chain or a 4-loop. In particular, $c(G - 3) = 1 + c < 0$ and $c(G - 4_\ell) = 4 + c < 3$.

As a fourth special case, suppose $c = -2$. Then Lemma 3.7 applies to both G and $G - 3$, giving $v(G) = 2$ and $v(G - 3) = 1$. The first of these is our claimed value for $v(G)$, and the second shows $v(G; 3) = 2$. Since this equals $v(G)$, opening a 3-chain is optimal. Henceforth we suppose $c(G) < -2$.

Suppose G is a counterexample with fewest possible components; we will derive a contradiction. The main step is to show $v(G; 3) \in \{1, 2\}$. Suppose first that $\theta = 1$. Lemma 3.8 describes $v(G - 3)$. Because $c(G - 3) \equiv \text{size}(G - 3) \equiv 0 \pmod{4}$, it says $v(G - 3) = 1, 2$ or 3 . Then $v(G; 3) = 2, 1$ or 2 respectively. Now suppose $\theta = 2$ and $\text{size}(G) \equiv 2 \pmod{4}$. Then Lemma 3.10 gives $v(G - 3) \in \{1, 3\}$, hence $v(G; 3) = 2$. Finally, suppose that either $\theta = 2$ and $\text{size}(G) \equiv 2 \pmod{4}$, or that $\theta > 2$. Then the minimality of G shows that the current lemma describes $v(G - 3)$. In particular, $v(G - 3) \in \{1, 2\}$, so $v(G; 3) \in \{2, 1\}$ also. This finishes the proof of $v(G; 3) \in \{1, 2\}$.

Since G is a counterexample, it has a component C with $v(G; C) < v(G; 3)$. This is obviously impossible if $v(G; 3) = 1$, so we must have $v(G; 3) = 2$ and $v(G; C) = 0$. The latter forces C to be a 4-loop with $v(G - 4_\ell) = 4$. If $c(G - 4_\ell) < -1$, then induction would give the contradiction $v(G - 4_\ell) \in \{1, 2\}$. Therefore $c(G - 4_\ell) \geq -1$. On the other hand, $c(G - 4_\ell) = c + 4 < 2$, so we must have $c(G - 4_\ell) = -1, 0$ or 1 . There must be no 4-loops in $G - 4_\ell$, or else Lemma 3.7 would give the contradiction $v(G - 4_\ell) = |c(G - 4_\ell)| = (0 \text{ or } 1) \neq 4$. But then Lemma 3.5 gives the contradiction $v(G - 4_\ell) \in \{1, 2, 3\}$. \square

Proof of Theorem 1.1 This amounts to combining the results above. First suppose $c \geq 2$. If $G = 3 +$ (one or more loops), then Theorem 3.2 shows that opening the shortest loop is optimal, as claimed. Otherwise, the same theorem shows that opening a 3-chain is optimal, justifying our claim that the standard move is optimal.

Henceforth we may assume $c < 2$. We begin by showing that if G has no 3-chains or 4-loops then the standard move is optimal. Lemma 3.4 shows that G has a 6-loop, and that opening it (which is the standard move) is optimal. Next we generalize this by showing that if G has a 3-chain, but still no 4-loops, then the standard move (opening a 3-chain) is optimal. This is part of Lemma 3.5. Next we treat the special case $G = 4_\ell + 3 + 3 + 3$. We must prove that the standard move (opening a 3-chain) is optimal. This is part of Lemma 3.6.

We have reduced to the case that $c < 2$ and that $G \neq 4_\ell + 3 + 3 + 3$ has a 4-loop. If $c \in \{0, \pm 1\}$, then Lemma 3.7 shows that opening a 4-loop is optimal.

Finally, suppose $c \leq -2$. If G has no 3-chains, then we must prove that the standard move (opening a 4-loop) is optimal. This is part of Lemma 3.8. The only remaining cases are $G = 3 + 4_\ell + H$. If H has no 3-chains and $4 \mid \text{size}(H)$, then we must show that opening a 4-loop is optimal, which is part of Lemma 3.10. On the other hand, if H has a 3-chain, or $4 \nmid \text{size}(H)$, then we must show that the standard move (opening the 3-chain) is optimal. This is part of Lemma 3.11. \square

4 Game values and controller strategy

The controller's strategy in Theorem 1.3 depends on being able to recognize when a position G has value > 2 . We will give two ways to compute the value. First we give an explicit value in terms of $c(G)$, $f(G)$ and the overall size and shape of G . This is complicated. Then we give our preferred method, whose essential case computes $v(G)$ by starting with $c(G_0)$ for a certain smaller position G_0 , and then applying a simple process. The following result immediately implies Theorems 1.4 and 1.5.

Theorem 4.1 (Values—explicit) *Suppose G is a nonempty Dots and Boxes position consisting of loops and long chains. Then its value $v = v(G)$ is given by the first of the following cases that applies:*

- (1) *If $c \geq 2$, then $v = c$.*
- (2) *If $c = 0$ and $G = 4_\ell + (\text{anything except } 3 + 3)$, then $v = 0$.*
- (3) *If $\theta = 0$, or if $\theta = 1$ and $\text{size}(G) \equiv 3 \pmod{4}$, then*
 - (a) *if $c + 4f \geq 2$, then $v = 0, 1, 2, 3$ or 4 according to whether $c \equiv 0, \pm 1, \pm 2, \pm 3$ or $4 \pmod{8}$;*
 - (b) *if $c + 4f < 2$, then*
 - (i) *if G is odd, then $v = 1$ resp. 3 if f is odd resp. even;*
 - (ii) *if $\text{size}(G) \equiv 2 \pmod{4}$, then $v = 2$;*
 - (iii) *if $\text{size}(G) \equiv 0 \pmod{4}$, then $v = 0$ resp. 4 if f is odd resp. even.*
- (4) *In all other cases, $v = 1$ resp. 2 if G is odd resp. even.*

Proof (1) is part of Theorem 3.2. (2) is part of Lemma 3.7. (3) follows from Lemma 3.8 if $\theta = 0$, or Lemma 3.10 if $\theta = 1$.

(4) It is enough to prove $v \in \{1, 2\}$. Because case (1) does not apply to G , we know that $c < 2$. Because case (3) does not apply to G , we know that either $\theta \geq 2$, or else $\theta = 1$ and $\text{size}(G) \equiv 3 \pmod{4}$. In light of these restrictions, if G has no 4-loop, then Lemma 3.5 gives $v \in \{1, 2\}$.

So we suppose $f > 0$. If $c = \pm 1$, then $v = 1$ by Lemma 3.7. If $c = 0$, then we must have $G = 4_\ell + 3 + 3$, or else case (2) would have applied. This special case has value 2 by Lemma 3.6. Finally, if $c < -1$, then Lemma 3.11 gives $v \in \{1, 2\}$. \square

There is a way to replace the complicated case (3) by a simpler iterative procedure. In practice it is easier and less error-prone than working through the tree of subcases. It is easier to use than explain, so the reader might want to look ahead at Example 4.3.

We will define a union G_0 of components of G . The strategy for computing $v(G)$ is to show $v(G_0) = c(G_0)$ and that $v(G)$ is got from $v(G_0)$ by a simple process. Suppose $\theta \leq 1$. We define the *core* G_0 of G as the union of the following components of G . Its loops are all the loops of length ≥ 8 . Its chains are either all the chains of length ≥ 4 (if there are any) or the 3-chain (if G has one and no other chains). The core has the useful property that if H is the union of G_0 and possibly some more components of G , then $H_0 = G_0$. Write $\theta' \in \{0, 1\}$ for the number of 3-chains in $G - G_0$.

We define maps $\mathbb{Z} \rightarrow \mathbb{Z}$:

$$\Sigma(x) = |x - 4| + 2 \quad \Theta(x) = x - 1 \quad \Phi(x) = |x - 4|.$$

Only their values on $\mathbb{Z}_{\geq 0}$ will be important. We think of Σ as decreasing any $x \geq 2$ in steps of size 2 until x enters $[0, 4]$, where Σ acts as reflection across 3. And Φ decreases x in steps of size 4 until x enters $[0, 4]$, where Φ acts as reflection across 2. The operation Φ is from Lemma 21 and Corollary 22 of Buzzard and Ciere (2013).

Theorem 4.2 (Values—procedural) *Suppose G is a nonempty Dots and Boxes position consisting of loops and long chains. Then*

- (1) *if $c \geq 2$, then $v = c$;*

- (2) if $c = 0$, G has a 4-loop, and $G \neq 4_\ell + 3 + 3$, then $v = 0$;
- (3) if $\theta = 0$, or if $\theta = 1$ and $\text{size}(G) \equiv 3 \pmod 4$, then

$$v = \Phi^f \Theta^{\theta'} \Sigma^s(c(G_0)).$$

In all other cases, $v = 1$ or 2 according to the parity of G .

Example 4.3 Say $G = (8_\ell + 18) + 6_\ell^9 + 3 + 4_\ell^{101}$, where the parentheses indicate G_0 . Because $\theta = 1$ and $\text{size}(G) \equiv 3 \pmod 4$, part (3) of the theorem applies. We begin with $v(G_0) = c(G_0) = 18$. Adjoining the first seven 6-loops decreases this by 2 each time, leaving 4. Adjoining the remaining two 6-loops bounces this to 2 and back to 4 (reflecting across 3). Adjoining the 3-chain reduces this to 3. Adjoining the 4-loops bounces this between 3 and 1 an odd number of times, leaving $v(G) = 1$.

Proof of Theorem 4.2 Unlike Theorem 4.1, no case (1)–(3) has priority over any other. If two apply, then we are asserting that both are correct. Cases (1) and (2) are justified by Theorem 3.2 and Lemma 3.7 respectively. The “in all other cases” argument is the same as for Theorem 4.1.

For case (3) we first prove $v(G_0) = c(G_0)$. If $G_0 = \emptyset$, then both sides are 0, so suppose $G_0 \neq \emptyset$. Every component of G_0 is a loop of length ≥ 8 or a chain of length ≥ 4 , or a 3-chain. There is at most one 3-chain. Every term in the formula (5) for $c(G_0)$ is nonnegative, except that one term can be -1 . So $c(G_0) \geq \text{tb}(G_0) - 1 \geq 3$, which implies $v(G_0) = c(G_0)$.

Next we observe that the initial moves in an optimal strategy for the opener is to open all the 4-loops, then the 3-chain (if $G - G_0$ has one), then the 6-loops. This follows from Theorem 1.1. Therefore $v(G)$ can be got by working backwards from $v(G_0) = c(G_0)$. We defined Σ so that $v(H) = \Sigma(v(H - 6_\ell))$ for any position H in which opening a 6-loop is optimal. It follows that if $G - G_0$ has no 3-chain or 4-loops, then $v(G) = \Sigma^s(v(G_0))$. If $G - G_0$ has a 3-chain but no 4-loops, then instead we use $v(G) = 1 + |v(G - 3) - 2| = \Theta(v(G - 3))$. The second equality here comes from the fact that $G_0 \subseteq G - 3$ is nonempty (else the 3-chain would lie in G_0) without 3-chains or 4-loops, hence has value ≥ 2 by Lemma 3.4. We have proven $v = \Theta^{\theta'} \Sigma^s(c(G_0))$ if G has no 4-loops. For general G the same argument shows

$$v(G) = \Phi^f(G - (\text{all 4-loops})) = \Phi^f \Theta^{\theta'} \Sigma^s(c(G_0)). \quad \square$$

5 Consequences for midgame play

This section discusses how to use our results in actual play. After reaching a position consisting of loops and long chains, one can play optimally by following the strategies in the introduction. But before play gets that far, there are opportunities to influence the shape of the endgame.

We write from the perspective of a player (“you”) who expects to be the opener once the game settles down to a union of loops and long chains. This means that you expect to lose the Nimstring game. Assuming you are right in this expectation, your only chance of victory is to gain a large enough advantage in captured boxes before

the endgame. For a line of play under consideration, we write G for the resulting endgame and A for your advantage in boxes at the time play reaches that position. Players prefer odd \times odd boards because ties are impossible, so we restrict to this case. As a consequence, A and $\text{size}(G)$ have different parities. We also assume that you will create enough 3-chains and 4- and 6-loops so that $c(G) < 2$ and hence $v(G) \in \{0, \dots, 4\}$.

A simple rule of thumb is: if $A \leq 1$, then you will lose, while if $A \geq 2$, then you will win. This is not really true, but it is true “generically” in the sense that Theorem 4.1 shows that almost all positions with $c(G) < 2$ have value 1 or 2. If $A = 1$, then the only way you can win is to arrange for $v(G) = 0$. If $A = 2$ resp. 3, then the only way you can lose is for $v(G) = 3$ resp. 4. According to Theorem 4.1, each of these imposes very strong constraints on G . We now consider in more detail the possibilities for pairs (A, G) . Your goal, for given A , is to steer the game toward an endgame G satisfying $A - v(G) > 0$. We refer to Theorem 4.1 throughout the analysis. Any (A, G) with $A \leq 0$ is lost, while any (A, G) with $A \geq 4$ is a win.

First consider (A, G) with $A = 1$. You have probably lost, but if you can arrange $v(G) = 0$, then you will win. Your opponent will try to block the creation of a 4-loop. If she succeeds, then you will lose ($f = 0$ implies $v \neq 0$, hence $v \geq 2$ because G is even). So we assume that you can arrange for a 4-loop to be present. You will win if you can also arrange for $c = 0$, with the single exception $G = 4_\ell + 3 + 3$.

Our advice past this point involves the complicated case (3) of Theorem 4.1. See the next paragraph for a simplification. If you cannot arrange for $c = 0$, then you must aim for $\theta = 0$, because this is the only way to reach case (3a) or (3(b)iii). Even if you can achieve this, you must also aim for $4 \mid \text{size}(G)$. This is because $v = 0$ requires this in case (3(b)iii), and requires the stronger condition $c \equiv 0 \pmod 8$ in case (3a). This condition on c implies $4 \mid \text{size}(G)$, because the absence of 3-chains gives $\text{size}(G) \equiv c(G) \pmod 4$ by Lemma 2.2(3). If you can achieve both $\theta = 0$ and $4 \mid \text{size}(G)$, then you will win if you can also arrange for the number s of 6-loops to be small and $c \equiv 0 \pmod 8$, or if you can arrange for s to be large and f to be odd. Here and below we write “ s small” and “ s large” as stand-ins for the more precise statements $c + 4f \geq 2$ and $c + 4f < 2$ respectively. (Because $\theta = 0$, making $c + 4f$ small requires 6-loops.)

Actual play is unlikely to lead to many 6-loops, and one can ignore the “ s large/small” conditions without much loss. Ignoring them lets us simplify the advice in the previous paragraph: if you can arrange for $f > 0$, but cannot arrange for $c = 0$, then aim for $\theta = 0$ and $c \equiv 0 \pmod 8$. Of course, this simplified advice may mislead you into a losing position, or blind you to an opportunity to win. For small G it is easy to memorize the exceptions. The following theorem, verified by computer, makes it fairly easy to remember the small winnable G .

Theorem 5.1 *Suppose $0 < \text{size}(G) < 28$ and $c < 2$. Then $v = 0$ if and only if: either $G = 4_\ell + 6_\ell^2 + 4^1 \text{ or } 2$, or*

$$\theta = 0, f > 0, c \equiv 0 \pmod 8, \text{ and } G \neq 4_\ell^2 + 6_\ell^2 + 4.$$

Now consider (A, G) with $A = 2$, so G is odd. You will probably win, but if your opponent can arrange for $v(G) = 3$, then you will lose. If you can create two 3-chains, then you will win. Even a single 3-chain will do if $\text{size}(G) \equiv 3 \pmod{4}$. If you cannot achieve this, then you must arrange for both (a) and (b) of Theorem 1.5 to fail. That is, you must arrange for s to be small and $c(G) \equiv \pm 1 \pmod{8}$, or for s to be large and f to be odd. We sought an analogue of theorem 5.1, but could not find one with just a few exceptions.

Finally consider (A, G) with $A = 3$. This is similar to the $A = 2$ case but easier since you will win unless $v(G) = 4$. If you arrange for G to have a 3-chain, or $\text{size}(G) \equiv 2 \pmod{4}$, then you will win. (This uses the evenness of G .) Suppose you cannot achieve either of these, so $\theta = 0$ and $\text{size}(G) \equiv 0 \pmod{4}$. Then you must aim for s to be small and $c(G) \equiv 0 \pmod{8}$, or for s to be large and f to be odd. Here is an analogue of theorem 5.1, also verified by computer.

Theorem 5.2 *Suppose $\text{size}(G) < 28$ and $c < 2$. Then $v = 4$ if and only if: either $\theta = 0$ and $c \equiv 4 \pmod{8}$ and $G \neq 4_\ell + 6_\ell^2 + 4^{1 \text{ or } 2}$, or else*

$$G \in \{6_\ell^2 + 4^{1 \text{ or } 2 \text{ or } 3}, 6_\ell^2 + 4 + 4_\ell^2, 6_\ell^2 + 4 + 8_\ell, 6_\ell^3 + 6, 6_\ell^4\}.$$

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