MAXIMAL INTEGRAL ORTHOGONAL GROUPS

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Abstract. Consider a nondegenerate quadratic space V over the rational or p-adic numbers. Over the rational numbers, suppose V is indefinite of dimension at least 3. Among the orthogonal groups of lattices in V, we determine which are maximal under inclusion. For each maximal group, we find all the elementary lattices it preserves.

1. Introduction

Suppose V is a nondegenerate quadratic space over $\mathbb{Q}$ resp. $\mathbb{Q}_p$. Among the subgroups of its orthogonal group $\mathrm{O}(V)$ are the orthogonal groups of lattices in V. Such a subgroup will be called *maximal* if it is maximal under inclusion *among all such subgroups*. This is the only notion of maximality we will consider; in the p-adic case it is equivalent to being maximal among compact subgroups of $\mathrm{O}(V)$. In this paper we find all of $\mathrm{O}(V)$'s maximal subgroups, unconditionally in the local case, and under the assumption that V is indefinite of dimension ≥ 3 in the global case.

A *lattice* in V means a $\mathbb{Z}$- resp. $\mathbb{Z}_p$-submodule L for which the natural map from $L \otimes \mathbb{Q}$ resp. $L \otimes \mathbb{Q}_p$ to V is an isomorphism. The *dual lattice* L^* means the sublattice of V consisting of all vectors having integral (ie $\mathbb{Z}$- resp. $\mathbb{Z}_p$-valued) inner products with all elements of L. We call L *integral* if $L \subseteq L^*$. In this case, the finite group L^*/L is called the *discriminant group* of L and written $\Delta(L)$. We call L *elementary* if it is integral and $\Delta(L)$ is a direct sum of cyclic groups of prime order.

If L is any lattice in V, then it is standard that its orthogonal group $\mathrm{O}(L)$ preserves an elementary lattice in V. (Following Watson [7], first scale L to make it integral. Then, supposing $\Delta(L)$ has an element of order p^2 with p a prime, enlarge L by adjoining $pL^* \cap \frac{1}{p}L$. This gives an integral lattice with smaller discriminant group. Repeating this process gives a sequence of integral lattices, with ascending isometry

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groups, terminating with an elementary lattice.) Therefore we may restrict attention to elementary lattices when searching for the maximal subgroups of $\mathrm{O}(V)$.

Theorem 1.1 (Elementary $\mathbb{Z}_p$-lattices with maximal isometry groups)**.** *Suppose p is a prime, E is an elementary $\mathbb{Z}_p$-lattice, and $V = E \otimes \mathbb{Q}_p$.*

(1) *If p is odd, then $\mathrm{O}(E)$ is maximal, and preserves no elementary lattice in V other than E.*
(2) *If $p = 2$, then $\mathrm{O}(E)$ is maximal if and only if E is one of*

$$1^{\mathrm{odd}}_{\cdots}2^{\mathrm{odd}}_{\cdots} \qquad 1^{\cdots}_{\mathrm{II}}2^{\cdots}_{\mathrm{II}} \qquad \begin{matrix} 1^{\mathrm{odd}}_{\cdots}2^{\cdots}_{\mathrm{II}} \\ 1^{\cdots}_{\mathrm{II}}2^{\mathrm{odd}}_{\cdots} \end{matrix} \qquad \begin{matrix} 1^{\cdots}_{\pm 2}2^{\cdots}_{\mathrm{II}} \\ 1^{\cdots}_{\mathrm{II}}2^{\cdots}_{\pm 2} \end{matrix} \qquad \begin{matrix} (1^{+1}_{\pm 1} \text{ or } 1^{-1}_{\pm 3})2^{\cdots}_{0} \\ 1^{\cdots}_{0}(2^{+1}_{\pm 1} \text{ or } 2^{-1}_{\pm 3}) \end{matrix} \qquad \begin{matrix} 1^{\cdots}_{0} \\ 2^{\cdots}_{0}. \end{matrix} \tag{1.1}$$

We assume a command of 2-adic lattices in the language developed by Conway and Sloane; see section 2 for details and references. The notation "$\cdots$" means "any/all possible values that are consistent with the data given explicitly". The next theorem describes when several elementary 2-adic lattices share the same maximal orthogonal group. The lattices E_{even}, E_{ex} and E^{ex} appearing in the statement are the even sublattice of E and the "exceptional" sublattice and superlattice. When relevant, the last two are defined in lemmas 4.4 and 4.5.

Theorem 1.2 (Maximal orthogonal groups over $\mathbb{Z}_2$)**.** *Suppose V is a nondegenerate quadratic space over $\mathbb{Q}_2$. Then each maximal subgroup of $\mathrm{O}(V)$ coincides with $\mathrm{O}(E)$ for exactly one lattice E in V with isometry type among*

$$1^{\mathrm{odd}}_{\cdots}2^{\mathrm{odd}}_{\cdots} \qquad 1^{\cdots}_{\mathrm{II}}2^{\cdots}_{\mathrm{II}} \qquad 1^{\mathrm{odd}}_{\cdots}2^{\cdots}_{\mathrm{II}} \qquad 1^{\cdots}_{\mathrm{II}}2^{\mathrm{odd}}_{\cdots} \qquad 1^{\cdots}_{\pm 2}2^{\cdots}_{\mathrm{II}} \tag{1.2}$$

Furthermore, $\mathrm{O}(E)$ preserves no elementary lattice in V other than E, except as follows:

E from (1.2)	*elementary lattice with same orthogonal group*
$1^{\cdots}_{\mathrm{II}}2^{+2}_{\mathrm{II}}$	$E^{\mathrm{ex}} \cong 1^{\cdots}_{0}$
$1^{+2}_{\mathrm{II}}2^{\cdots}_{\mathrm{II}}$	$E_{\mathrm{ex}} \cong 2^{\cdots}_{0}$
$(1^{+3}_{\pm 1} \text{ or } 1^{-3}_{\pm 3})2^{\cdots}_{\mathrm{II}}$	$E_{\mathrm{ex}} \cong (1^{+1}_{\pm 1} \text{ or } 1^{-1}_{\pm 3})2^{\cdots}_{0}$
$1^{\cdots}_{\mathrm{II}}(2^{+3}_{\pm 1} \text{ or } 2^{-3}_{\pm 3})$	$E^{\mathrm{ex}} \cong 1^{\cdots}_{0}(2^{+1}_{\pm 1} \text{ or } 2^{-1}_{\pm 3})$
$1^{\pm d}_{t \in \{2,-2\}}2^{\cdots}_{\mathrm{II}}$	$E_{\mathrm{even}} \cong 1^{+(d-2)}_{\mathrm{II}}2^{\cdots}_{\pm t}$

In particular, only if $E \cong 1^{+2}_{\mathrm{II}}2^{+2}_{\mathrm{II}}$ does $\mathrm{O}(E)$ preserves more than two elementary lattices in V. In that case it preserves exactly three, the other two being isometric to 1^{+4}_{0} and 2^{+4}_{0}.

Corollary 1.3 (Maximal compact subgroups)**.** *Suppose p is a prime and V is a nondegenerate quadratic space over $\mathbb{Q}_p$. If p is odd, define*

$\mathcal{M}$ as the set of elementary lattices in V. Otherwise, define $\mathcal{M}$ as the set of lattices in V having one of the forms (1.2)*. Then $E \mapsto \mathrm{O}(E)$ is a bijection from $\mathcal{M}$ to the set of maximal compact subgroups of* $\mathrm{O}(V)$. □

Our final results shows that the global situation is a simple combination of the local situations. If E is an elementary $\mathbb{Z}$-lattice, such that one of the 2-adic operations $E_2 \mapsto (E_2)_{\text{even}}$, $(E_2)_{\text{ex}}$ or $(E_2)^{\text{ex}}$ appearing in theorem 1.2 is defined, then we regard that operation as also acting on E. It alters E_2 as before and leaves E_p unchanged for all odd p.

Theorem 1.4 (Elementary $\mathbb{Z}$-lattices with maximal isometry groups)**.** *Suppose L is an indefinite elementary $\mathbb{Z}$-lattice of dimension ≥ 3, and $V = L \otimes \mathbb{Q}$. Then $\mathrm{O}(L)$ is maximal in $\mathrm{O}(V)$ if and only if $\mathrm{O}(L_2)$ is maximal in $\mathrm{O}(V_2)$, ie if and only if L_2 appears in* (1.1)*.*

Furthermore, in this case

1. *$M \mapsto M_2$ is a bijection from the $\mathrm{O}(L)$-invariant elementary lattices in V to the $\mathrm{O}(L_2)$-invariant elementary lattices in V_2.*
2. *There is a unique $\mathrm{O}(L)$-invariant elementary lattice E whose completion at 2 appears in* (1.2)*.*
3. *The elementary lattices in V, that are preserved by $\mathrm{O}(L) = \mathrm{O}(E)$, are E and whichever of E_{even}, E_{ex} and E^{ex} arise by altering E_2 as in theorem 1.2*
4. *There are at most two $\mathrm{O}(L)$-invariant elementary lattices in V, except that there are exactly three if $L_2 \cong 1_{\mathrm{II}}^{+2}2_{\mathrm{II}}^{+2}$, 1_0^{+4} or 2_0^{+4}.*

Corollary 1.5 (Maximal orthogonal groups over $\mathbb{Z}$)**.** *Suppose V is an nondegenerate quadratic space over $\mathbb{Q}$, indefinite of dimension ≥ 3. Let $\mathcal{M}$ be the set of elementary lattices E in V for which E_2 appears in* (1.2)*. Then $E \mapsto \mathrm{O}(E)$ is a bijection from $\mathcal{M}$ to the set of maximal subgroups of* $\mathrm{O}(V)$. □

We summarize the paper; details and additional background appear in the indicated sections. As preparation for the local case at a fixed prime p, section 3 considers the action of $\mathrm{O}(V)$ on a simplicial complex $\mathcal{E}$ whose vertices are the elementary lattices in V. This complex is CAT(0), a metric space property that implies: the fixed-point set, of any compact group of isometries, is nonempty and convex. For example, affine buildings are well-known to be CAT(0). The complex $\mathcal{E}$ resembles the affine Bruhat-Tits building for $\mathrm{SO}(V)$, but is different if $p = 2$. It seems likely that its full subcomplex, on the vertices corresponding to the set $\mathcal{M}$ of lattices from corollary 1.3, coincides with the Bruhat-Tits building.

Theorem 3.6 shows that the neighbors in $\mathcal{E}$ of a given elementary lattice E are indexed by certain subspaces of E/pE^* and $\Delta(E)$. This makes it possible to understand how $O(E)$ permutes them. Section 4 is devoted to working out which ones are $O(E)$-invariant. When there are none, it follows that $O(E)$ is maximal; see the argument after lemma 3.4. This always applies if p is odd, but the $p = 2$ case is complicated and requires much more work.

Section 4 also examines the neighbors which are invariant under the spinor norm 1 subgroup $\Theta(E)$ of $SO(E)$. This lets us use the strong approximation theorem in section 5 to study the global case. In most cases this is easy, but sometimes $\Theta(E)$ is "not big enough" and additional effort is required.

Our motivation for this work is the classification problem of integer lattices L, of Lorentzian signature, whose isometry groups are generated up to finite index by reflections. The resulting Coxeter groups act on hyperbolic space with finite-volume fundamental domains, and are interesting from many perspectives. The cases with $O(L)$ maximal are the most interesting. As described on p. 9 of Scharlau's proposed classification in the rank 4 case [6], we need to be able to recognize the maximal groups and distinguish them from each other. We wrote this paper so that we could rigorously prove his classification and extend it to higher rank.

2. Background

The norm r^2 of a vector r means the self inner product $r \cdot r$.

We will follow the convention that $\subseteq$ and $\subset$ indicate containment and strict containment respectively. And similarly for $\supseteq$ and $\supset$.

In this paper, all vector spaces are finite-dimensional with ground field $\mathbb{Q}$, $\mathbb{Q}_p$ or $\mathbb{F}_p$. A $\mathbb{Z}$- resp. $\mathbb{Z}_p$-lattice means a free $\mathbb{Z}$- resp. $\mathbb{Z}_p$-module equipped a symmetric bilinear form taking values in $\mathbb{Q}$ resp. $\mathbb{Q}_p$. Unfortunately, "lattice" has another common meaning that we will also need. In a clearly marked passage at the beginning of section 3, a lattice *in a given vector space* V over $\mathbb{Q}$ resp. $\mathbb{Q}_p$, means a $\mathbb{Z}$- resp. $\mathbb{Z}_p$-submodule that spans V. Usually V will be a quadratic space, in which case the definitions are equivalent.

If $d_1, \ldots, d_n$ are numbers, then $\langle d_1, \ldots, d_n \rangle$ means the lattice having inner product matrix with $d_1, \ldots, d_n$ down the main diagonal and 0's elsewhere. Otherwise, the brackets $\langle \cdots \rangle$ contain lattice vectors, resp. elements and/or subgroups of a group. Then the notation means the sublattice resp. subgroup that they generate.

We already defined the dual L^* of a nondegenerate lattice L, integrality, and (for integral lattices) the discriminant group $\Delta(L)$. The determinant $\det(L)$ of L means the determinant of the inner product matrix of any basis for L. This lies in $\mathbb{Q}$ resp. $\mathbb{Q}_p$, and is well-defined up to squares of units in $\mathbb{Z}$ resp. $\mathbb{Z}_p$. For p a prime and L a $\mathbb{Z}$- or $\mathbb{Z}_p$-lattice, we write $\det_p(L)$ for the power of p involved in $\det(L)$. When L is integral we also write $\Delta_p(L)$ for the subgroup of $\Delta(L)$ consisting of elements with p-power order; it is standard that $|\Delta_p(L)| = \det_p(L)$.

We also defined elementary lattices and what we mean by a maximal subgroup of $\mathrm{O}(V)$, and explained why every lattice stabilizer preserves an elementary lattice. A lattice over $\mathbb{Z}$ or $\mathbb{Z}_2$ is called *even* if all its elements have even norm, and *odd* otherwise.

A root of a lattice L means a primitive lattice vector, of nonzero norm, whose reflection $x \mapsto x - 2\frac{x \cdot r}{r^2} r$ preserves L. This reflection fixes pointwise its "mirror" $v^\perp$, and negates v. Given a primitive vector $r \in L$, it is standard that r is a root of L if and only if $r \cdot L \subseteq \frac{1}{2} r^2 \mathbb{Z}$. In lemma 5.2 we give a sufficient local condition for the existence of roots of given norms in $\mathbb{Z}$-lattices.

Suppose L is integral. Then $\Delta(L)$ has a nondegenerate symmetric bilinear pairing taking values in $\mathbb{Q}/\mathbb{Z}$ resp. $\mathbb{Q}_p/\mathbb{Z}_p$. Given two elements of $\Delta(L)$, one chooses vectors in L^* representing them, evaluates their inner product, and reduces mod $\mathbb{Z}$ resp. $\mathbb{Z}_p$. Similarly, vectors in L/pL have well-defined inner products mod p. Two special phenomena occur for $p = 2$. First, if L is integral, then the norm of an element of $L/2L$, a priori well-defined mod 2, is actually well-defined mod 4. Second, if L is even, then norms of elements of $\Delta(L)$, a priori well-defined mod 1, are actually well-defined mod 2.

We use a superscript $^\times$ to indicate the group of units of a ring, eg $\mathbb{Z}_p^\times$ and $\mathbb{Q}_p^\times$. The spinor norm is a homomorphism $\mathrm{SO}(V) \to \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ resp. $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$; see [4, Cor. 3 of Thm. 10.3.1]. If $g \in \mathrm{SO}(V)$ is expressed as the composition of the reflections in $r_1, \ldots, r_k \in V$, then its spinor norm is the square class of $r_1^2 \cdots r_k^2$. That this is a well-defined homomorphism is standard (albeit non-obvious). The kernel of the spinor norm homomorphism is written $\Theta(V)$. If L is a lattice in V, then $\mathrm{O}(L)$ and $\Theta(L)$ are defined as the subgroups of $\mathrm{O}(V)$ and $\Theta(V)$ that preserve L. If L is a $\mathbb{Z}$-lattice in V, then we write L_p for $L \otimes \mathbb{Z}_p$ and V_p for $V \otimes \mathbb{Q}_p$. We will need the following form of the strong approximation theorem [4, Cor. to Thm. 10.7.1]: if L is indefinite with dimension ≥ 3, then $\Theta(L)$ is dense in each $\Theta(L_p)$.

The Conway-Sloane calculus for p-adic lattices is explained in [5] and (in more detail for $p = 2$) in [2]. Briefly, for odd p, an isometry type of unimodular p-adic lattice L is represented by a symbol $1^{\pm d}$, where $d = \dim L$ and the sign is the Kronecker symbol $(\frac{\det L}{p})$. In particular, the superscript is not a signed integer but two separate pieces of information. Each isometry type of unimodular 2-adic lattice is represented by a symbol $1_{\mathrm{II}}^{\pm d}$ or $1_t^{\pm d}$, where II is a formal symbol and $t \in \mathbb{Z}/8$. The superscript has the same meaning as for odd p. The *type* of L is defined as I or II according to whether L is odd or even. A subscript II means L has type II. The two-dimensional even unimodular lattices are $1_{\mathrm{II}}^{+2} \cong \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and $1_{\mathrm{II}}^{-2} \cong \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. Otherwise, L has type I and is diagonalizable over $\mathbb{Z}_2$, and the subscript is the sum (mod 8) of the norms of any orthogonal basis. It is called the *oddity* of L. When we wish to indicate that L has type I, but do not need to name the subscript, we sometimes write it as I.

Now suppose L is got by scaling the inner product on a unimodular $\mathbb{Z}_p$-lattice by p^k. The type (I or II) and symbol of L are defined to be that of the unimodular lattice, except that the central "1" in the symbol is replaced by the scale factor p^k. For example, $2_{\mathrm{II}}^{+2} \cong \left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$ over $\mathbb{Z}_2$ and $49^{-2} \cong \left(\begin{smallmatrix} 49 & 0 \\ 0 & -49 \end{smallmatrix}\right)$ over $\mathbb{Z}_7$. A chain of such symbols, for example $1_3^{-7}2_2^{+2}$ or $(\frac{1}{7})^{+1}49^{-2}$, indicates the direct sum of those lattices. Every Jordan decomposition of a $\mathbb{Z}_p$-lattice yields such a chain, which we call the Conway-Sloane symbol (of that Jordan decomposition). For odd p, any two Jordan decompositions are equivalent by a lattice isometry, so one can speak of *the* Conway-Sloane symbol of the lattice.

The direct sum of two scaled unimodular p-adic lattices, with the same scale, is easy to express in terms of the notation: dimensions add, signs multiply, and subscripts add (when present). The addition of subscripts includes the formal rules $\mathrm{II} + \mathrm{II} = \mathrm{II}$ and $\mathrm{II} + t = t$. For example, $1_{\mathrm{II}}^{+2}2_2^{+2}4_{\mathrm{II}}^{-2} \oplus 1_5^{-3}2_5^{+3}4_{\mathrm{II}}^{+2} \cong 1_5^{-5}2_{-1}^{+5}4_{\mathrm{II}}^{-4}$.

Over $\mathbb{Z}_2$, a lattice may have inequivalent Jordan decompositions, hence several distinct symbols. Two symbols represent isometric lattices if and only if they are related by sequences of certain "moves". These are called *sign walking* (example: $1_5^{+3}2_{\mathrm{II}}^{+2} \cong 1_1^{-3}2_{\mathrm{II}}^{-2}$) and *oddity fusion* (example: $1_3^{-7}2_2^{+2} \cong 1_5^{-7}2_0^{+2}$). See the references for details, including an alternate notation (example: $[1^{-7}2^{+2}]_5$) in which all oddity is already fused.

When we include "$\cdots$", or other ambiguous notations in one of these symbols, we mean "any/all possible values that are consistent with data given explicitly". Multiple ambiguities in the same context are not assumed to have any particular relationship with each other, except that all ambiguous signs $\pm$ and $\mp$ are linked in the standard way. For example, in (1.1), $1_{\dots}^{\mathrm{odd}}2_{\dots}^{\mathrm{odd}}$ means "any elementary $\mathbb{Z}_2$-lattice whose Jordan constituents are odd-dimensional". The signs may be arbitrary, and the subscripts may be arbitrary subject to the constituents being legal (ie that such a lattice exists). Legality amounts to:

(i) type II constituents have even dimension,
(ii) every constituent's subscript and dimension have the same parity,
(iii) all scalings of $1^{+1}_{\pm 3}$, $1^{-1}_{\pm 1}$, 1^{+2}_{4}, 1^{-2}_{0} are illegal, and
(iv) every 0-dimensional constituent has sign + and type II.

For a more complex example, the last line in the table in theorem 1.2 applies to all 2-adic lattices $E \cong$ "$1^{\pm d}_{t\in\{2,-2\}} 2^{\cdots}_{\mathrm{II}}$". That is, to all $E \cong 1^{\delta d}_t 2^{\varepsilon e}_{\mathrm{II}}$, where $t \in \{2,-2\} \subseteq \mathbb{Z}/8$ (by hypothesis), d is a positive even integer (even because t is, nonzero because its constituent has type I), e is a nonnegative even integer (since its constituent has type II), and the signs δ, ε may be chosen arbitrarily (except that $\varepsilon = +$ when $e = 0$). The illegality of constituents 1^{+2}_4 and 1^{-2}_0 is irrelevant because $t \neq 0, 4$. The table says "$E_{\mathrm{even}} \cong 1^{+(d-2)}_{\mathrm{II}} 2^{\cdots}_{\pm t}$". See (4.3) for the computation of even sublattices; here we consider only the meaning of the notation. We will deduce that it means $E_{\mathrm{even}} \cong 1^{+(d-2)}_{\mathrm{II}} 2^{\delta\varepsilon(e+2)}_{\delta t}$. The subscript δt reflects that δ is our name for the sign written "$\pm$" in the table. The second constituent's sign is $\delta\varepsilon$, because the product of all signs is an invariant of quadratic spaces. The second constituent has dimension $e + 2$, because the total dimension is also invariant. This finishes the "decryption" of the notation. If $d > 2$, then sign walking allows the alternate description $E_{\mathrm{even}} \cong 1^{-(d-2)}_{\mathrm{II}} 2^{(-\delta\varepsilon)(e+2)}_{\delta t+4}$. But when $d = 2$, this would include the illegal constituent "1^{-0}_{II}".

Next we recall an operation on p-adic lattices called "rescaled duality". For our purposes, it turns out that understanding an elementary lattice is equivalent to understanding its rescaled dual. Attaching a superscript in brackets to a lattice (resp. quadratic space), eg $L^{[p]}$, indicates the same underlying module (resp. vector space) with all inner products scaled by that factor. If L is a nondegenerate p-adic lattice, then its "rescaled dual" L' means the lattice in $(L \otimes \mathbb{Q}_p)^{[p]}$ defined by $L' = L^{*[p]}$. If a lattice L has index p^k in another lattice M, then L' contains M', also with index p^k. This follows immediately from the corresponding property of dual lattices.

Jordan decompositions of a p-adic lattice L will always be expressed in the form $L = \oplus L_i^{[p^i]}$ where the L_i are unimodular and i varies over a finite subset of $\mathbb{Z}$. Given such a Jordan decomposition, we have

$$L^* = \oplus\, p^{-i} L_i^{[p^i]} \quad \text{and therefore} \quad L' = \oplus\, p^{-i} L_i^{[p^{1+i}]} \cong \oplus\, L_i^{[p^{1-i}]}$$

(Note that $p^{-i} L_i \cong L_i^{[p^{-2i}]}$.) It follows that $L'' \cong L$. Furthermore, if L is elementary, ie $L = L_0 \oplus L_1^{[p]}$, then $L' \cong L_1 \oplus L_0^{[p]}$. That is, priming "swaps" the unimodular and non-unimodular Jordan constituents of an elementary lattice. For example, $(1^{+5} p^{-3})' \cong 1^{-3} p^{+5}$ when p is odd, and $(1_{\mathrm{II}}^{-6}2_3^{+3})' \cong 1_3^{+3}2_{\mathrm{II}}^{-6}$, and $(1_3^{-7})' \cong 2_3^{-7}$ for $p=2$. We record this for later use:

Lemma 2.1 (Rescaled duality). *Suppose V is a nondegenerate quadratic space over $\mathbb{Q}_p$. Then the rescaled duality operation $L \mapsto L'$ sends the elementary lattices in V bijectively to those in $V^{[p]}$, reverses inclusions, and preserves indices of inclusion.* □

Rescaled duality has some additional interesting properties that are not essential for us; see the end of section 3.

3. The simplicial complex $\mathcal{E}$ of elementary lattices

Fix a prime p and a vector space V over $\mathbb{Q}_p$ of dimension n. In this paragraph and the next, a lattice means a finitely generated $\mathbb{Z}_p$-submodule of V which spans V. The homothety classes of lattices in V are the vertices of a certain $(n-1)$-dimensional simplicial complex $\mathcal{B}$, called the affine Bruhat-Tits building for $\mathrm{SL}(V)$. For each lattice L, we write v_L for the vertex of $\mathcal{B}$ it represents. Two vertices are joined by an edge just if they are represented by lattices L, M satisfying $L \supset M \supset pL$. (This is a symmetric relation.) More generally, a set of vertices span a simplex just if they are represented by lattices $L_0, \ldots, L_k$ for which $L_0 \supset L_1 \supset \cdots \supset L_k \supset pL_0$. See [1, Sec. 6.9] for background.

We metrize $\mathcal{B}$ in the standard way, identifying each $(n-1)$-simplex with a fixed simplex of Coxeter type $\widetilde{A}_{n-1}$ in Euclidean space $\mathbb{R}^{n-1}$. The recipe is that if lattices $L_0 \supset \cdots \supset L_{n-1} \supset pL_0$ represent the vertices $v_0, \ldots, v_{n-1}$ of an $(n-1)$-simplex σ, then the facets opposite v_i, v_{i+1} make dihedral angle $\pi/3$, where we read subscripts mod n. All other dihedral angles are $\pi/2$. This makes $\mathcal{B}$ into a complete CAT(0) metric space [1, Thm. 11.16(2)] [3, Thm. II.10A.4], on which $\mathrm{GL}(V)$ acts by isometries. ([3] is a standard reference for CAT(0) spaces.)

Now suppose V is equipped with a nondegenerate inner product. Then the lattices of the previous two paragraphs are lattices in this paper's usual sense. We write δ for the duality map $\delta(L) = L^*$ on the set of lattices in V, and also for the induced automorphism $v_L \mapsto v_{L^*}$ of $\mathcal{B}$. Because $\mathcal{B}$ is CAT(0), geodesics (isometric embeddings of intervals) are unique given their endpoints. It follows that the fixed-point set $\mathcal{E}$ is nonempty, convex (a geodesic lies in $\mathcal{E}$ if its endpoints do) and CAT(0). The orthogonal group $\mathrm{O}(V)$ acts on $\mathcal{E}$, because orthogonal transformations respect duality.

Except in degenerate cases, $\mathcal{E}$ is not a subcomplex of $\mathcal{B}$. But it still has a natural simplicial complex structure. The simplices are the fixed-point sets of δ inside the δ-invariant simplices of $\mathcal{B}$. The following easy lemma shows that these fixed-point sets are simplices:

Lemma 3.1. *Suppose a group acts on a simplex. Then its fixed point set is also a simplex.*

Proof. Writing σ for the simplex and $v_0, \ldots, v_{n-1} \in \mathbb{R}^{n-1}$ for its vertices, every point of σ has a unique expression as a convex combination of the v_i, ie as $\sum a_i v_i$ where the a_i are nonnegative and sum to 1. The point is invariant if and only if $a_i = a_j$ whenever v_i and v_j lie in the same orbit. Therefore the fixed-point set is the convex hull of the barycenters of the orbits of the v_i. No point in this convex hull has more than one expression as a convex combination of the barycenters, or else that point would have more than one expression as a convex combination of the v_i. Therefore the barycenters are the vertices of a nondegenerate $(k-1)$-simplex in $\mathbb{R}^{n-1}$, where k is the number of barycenters. □

Lemma 3.2 (Vertices of $\mathcal{E}$)**.**

(1) *A vertex v of $\mathcal{B}$ is δ-invariant just if it is represented by a lattice E satisfying $E = E^*$ or pE^*.*
(2) *A vertex v of $\mathcal{B}$ is adjacent (or equal) to $\delta(v)$ just if $\{v, \delta(v)\} = \{v_E, v_{E^*}\}$ for some elementary lattice E.*
(3) *The vertices of $\mathcal{E}$ are in bijection with the elementary lattices in V, with such a lattice E corresponding to*

$$w_E = \begin{cases} v_E & \text{if } v_E = v_{E^*} \\ \text{the midpoint of the edge } \overline{v_E v_{E^*}} & \text{otherwise.} \end{cases}$$

Proof. (1) follows from (2), because an elementary lattice E and its dual represent the same vertex of $\mathcal{B}$ if and only if $E = E^*$ or $E = pE^*$.

(2) Suppose E is elementary, with Jordan decomposition $E = E_0 \oplus E_1^{[p]}$. Then $E^* = E_0 \oplus \frac{1}{p} E_1^{[p]}$ lies between E and $\frac{1}{p}E$, so v_{E^*} is adjacent (or equal) to v_E. Now suppose a vertex v of $\mathcal{B}$ is adjacent (or equal) to $\delta(v)$. Represent v by a lattice L with Jordan decomposition $\oplus_{i \geq 0} L_i^{[p^i]}$. Let m resp. M be the smallest resp. largest i for which $L_i \neq 0$. Then $L^* = \oplus_{i=m}^{M} p^{-i} L_i^{[p^i]}$, and the smallest multiple of L^* that lies in L is $p^M L^* = \oplus_{i=m}^{M} p^{M-i} L_i^{[p^i]}$. Since v and $\delta(v)$ are adjacent (or equal), this must contain pL, which forces $m = M - 1$ or M. That is, L has at most 2 constituents, and if there are 2 then their scales are consecutive powers of p. By replacing L by a homothetic lattice, we may suppose $L = L_0 \oplus L_1^{[p]}$ or $L_{-1}^{[1/p]} \oplus L_0$. (Note: $p^i L \cong L^{[p^{2i}]}$.) We define E as L or L^* respectively. Then E is an elementary lattice with $\{v_E, v_{E^*}\} = \{v, \delta(v)\}$.

(3) We apply lemma 3.1 to the subgroup $\mathbb{Z}/2$ of $\mathrm{Aut}(\mathcal{B})$ generated by δ. This shows that the vertices of $\mathcal{E}$ are (i) the δ-invariant vertices

of $\mathcal{B}$, and (ii) the midpoints of segments $\overline{v\,\delta(v)}$, where v is a vertex of $\mathcal{B}$ that is not equal to $\delta(v)$ but is adjacent to it. Now we may quote (1)–(2). □

Lemma 3.3. *For every $g \in \mathrm{O}(V)$, g acts trivially on every simplex of $\mathcal{E}$ that it preserves.*

Proof. Suppose the given simplex has vertices $w_{E_1}, \ldots, w_{E_k}$, where E_1, $\ldots, E_k$ are elementary lattices. By the definition of $\mathcal{E}$, this simplex is the fixed point set of δ in some δ-invariant simplex of $\mathcal{B}$. Every simplex of $\mathcal{B}$ containing w_{E_i} contains v_{E_i} and $v_{E_i^*}$, so collectively the v_{E_i} and $v_{E_i^*}$ form the vertices of the smallest simplex σ of $\mathcal{B}$ that contains all the w_{E_i}. Obviously δ acts on σ, with fixed point set equal to the given simplex of $\mathcal{E}$. Because g permutes the w_{E_i}, it permutes the vertices of σ. So it is enough to prove that g acts trivially on σ. This is essentially a standard fact about how $\mathrm{GL}(V)$ permutes the $\mathrm{SL}(V)$-orbits of vertices in $\mathcal{B}$. (The key property is that every orthogonal transformation has determinant in $\mathbb{Z}_p^\times$.) But we give a direct proof.

Choose any vertex of σ, say v_{L_0} with L_0 a lattice in V; we must show that $g(L_0)$ is homothetic to L_0. Choose additional lattices $L_1, \ldots, L_l$, that represent the other vertices of σ and satisfy $L_0 \supset L_1 \supset \cdots \supset L_l \supset pL_0$. Writing p^{c_i} for $[L_0 : L_i]$, we have $\det(L_i) = p^{2c_i}\det(L_0)$, and for $i \neq 0$ we also have $0 < c_i < n = \dim V$. The determinants of the lattices homothetic to L_i are $p^{2c_i+2nj}\det(L_0)$, where j varies over $\mathbb{Z}$. Observe

$$\det(g(L_0)) = \det(g)^2 \det(L_0) = (\pm 1)^2 \det(L_0) = \det(L_0).$$

Because $g(L_0)$ is homothetic to some L_i, we have $0 = 2c_i + 2nj$ for some i and j, which forces $i = j = 0$. □

Lemma 3.4. *If G is a subgroup of $\mathrm{O}(V)$ with compact closure, for example the isometry group of a lattice in V, then the fixed-point set of G in $\mathcal{E}$ is nonempty, and is the convex hull of the G-invariant vertices of $\mathcal{E}$.*

Proof. Any compact group, acting on any complete CAT(0) metric space, has nonempty convex fixed-point set. (See [1, Thm 11.23] for the existence of a fixed point; convexity follows from the uniqueness of geodesics.) Now suppose G fixes a point of $\mathcal{E}$, and write σ for the unique smallest simplex of $\mathcal{E}$ that contains it. Lemma 3.3 shows that each vertex of σ is G-invariant, proving the last claim. □

We can now explain our strategy for proving theorems 1.1 and 1.2. We are searching for the lattice stabilizers in $\mathrm{O}(V)$ that are maximal among lattice stabilizers. By lemma 3.4, this is the same as looking for the vertex-stabilizers in $\mathrm{O}(V)$, under its action on $\mathcal{E}$, that are maximal among vertex-stabilizers. It is elementary that a vertex-stabilizer is maximal in this sense if, and only if, it is the full stabilizer of each vertex that it fixes. So our problem is reduced to the following: for each elementary lattice E in V, determine whether $\mathrm{O}(E)$ is the full $\mathrm{O}(V)$-stabilizer of every $\mathrm{O}(E)$-invariant vertex in $\mathcal{E}$.

By lemma 3.4, any two $\mathrm{O}(E)$-invariant vertices of $\mathcal{E}$ are joined by a chain of such vertices. Therefore we may find all $\mathrm{O}(E)$-invariant elementary lattices by first examining the $\mathrm{O}(E)$-invariant neighbors of w_E, then their $\mathrm{O}(E)$-invariant neighbors, and so on. If at any point in this exploration process we find that $\mathrm{O}(E)$ fixes some vertex w_F, but is strictly smaller than $\mathrm{O}(F)$, then we recognize $\mathrm{O}(E)$ as non-maximal, and stop. Otherwise, we will eventually find all $\mathrm{O}(E)$-invariant vertices. And in the process of finding them, we will have checked that $\mathrm{O}(E)$ is the full stabilizer of each, proving $\mathrm{O}(E)$ maximal.

We carry out this procedure in section 4, and happily, it terminates almost immediately. On one hand, non-maximality of $\mathrm{O}(E)$ can always be detected after one step. On the other, if $\mathrm{O}(E)$ is maximal then its fixed point set turns out to be small: a vertex, an edge (only over $\mathbb{Z}_2$), or a triangle (in a single case, over $\mathbb{Z}_2$). The calculations require a description of the neighbors of w_E, in which the $\mathrm{O}(E)$-action is manifest. Theorem 3.6 gives such a description.

Lemma 3.5 (Simplices of $\mathcal{E}$)**.** *A set of elementary lattices in V represent the vertices of a simplex of $\mathcal{E}$ if and only if they are totally ordered under inclusion.*

We will sometimes say that elementary lattices E and F are adjacent, as shorthand for w_E and w_F being adjacent. By the lemma, this is equivalent to one of E and F strictly containing the other.

Proof. Write $E_0, \ldots, E_k$ for the lattices. If $E_0 \supset E_1 \supset \cdots \supset E_k$, then

$$E_k^* \supset \cdots \supset E_0^* \supseteq E_0 \supset \cdots \supset E_k \supseteq pE_k^*,$$

the first $\supseteq$ using the integrality of E_0 and the second using that E_k is elementary. So the v_{E_i} and $v_{E_i^*}$ are the vertices of a δ-invariant simplex of $\mathcal{B}$. By definition, the corresponding simplex of $\mathcal{E}$ has vertices $w_{E_0}, \ldots, w_{E_k}$.

Next we prove the key ingredient for the converse: if E and F are elementary lattices with v_E and v_F adjacent, then one of them contains the other strictly. By adjacency, $E \supset p^i F \supset pE$ for some $i \in \mathbb{Z}$. If $i = 0$ then $E \supset F$, while if $i = 1$ then $F \supset E$. Otherwise, we obtain a contradiction by taking determinants and recalling $n = \dim V$:

$$\det_p(E) < p^{2ni} \det_p(F) < p^{2n} \det_p(E). \tag{3.1}$$

That E and F are elementary gives $\det_p(E), \det_p(F) \in \{1, p, \ldots, p^n\}$. If $i < 0$ then the first inequality in (3.1) is a contradiction, while if $i > 1$ then the second one is.

Now suppose $w_{E_0}, \ldots, w_{E_k}$ are the vertices of a simplex of $\mathcal{E}$. By definition, this simplex is the set of δ-fixed points in some δ-invariant simplex σ of $\mathcal{B}$. Like every simplex of $\mathcal{B}$ containing the w_{E_i}, σ contains the v_{E_i} and $v_{E_i^*}$. In particular, the v_{E_i} are mutually adjacent in $\mathcal{B}$. So the previous paragraph shows that the E_i are totally ordered under inclusion. □

Our next step is to describe the elementary lattices adjacent to a given elementary lattice E. We will express the answer in terms of canonical bilinear forms on $\Delta(E)$ and E/pE^*. Because E is elementary, the natural $(\mathbb{Q}_p/\mathbb{Z}_p)$-valued nondegenerate pairing on $\Delta(E) = E^*/E$ takes values in $\frac{1}{p}\mathbb{Z}_p/\mathbb{Z}_p$. There is also a natural $(\mathbb{Z}_p/p\mathbb{Z}_p)$-valued bilinear pairing on E/pE^*, got by reducing mod p the inner products of lattice vectors. This is nondegenerate because E is elementary. If W is either of these inner product spaces, and S a subspace, then $S^\perp$ means the subspace of W whose elements pair trivially with all elements of S. By nondegeneracy, $S^{\perp\perp} = S$. We call S totally isotropic if $S \subseteq S^\perp$. (Caution: when discussing isotropic vectors and totally isotropic subspaces in the case $p = 2$, we consider only the bilinear pairings, not any related quadratic forms. For example, if E is even then *every* 1-dimensional subspace of $E/2E^*$ is totally isotropic.)

Theorem 3.6 (Links in $\mathcal{E}$)**.** *Suppose E is an elementary lattice. Then the link of w_E in $\mathcal{E}$ is the join of the flag complexes of the nonzero totally isotropic subspaces of $\Delta(E)$ and E/pE^*.*

In more detail, the vertices adjacent to w_E are represented bijectively by the following elementary lattices:

(1) *the preimages in E^* of the nonzero totally isotropic subspaces of $\Delta(E)$, and*
(2) *the preimages in E of the orthogonal complements of the nonzero totally isotropic subspaces of E/pE^*.*

Furthermore, given some nonzero totally isotropic subspaces of $\Delta(E)$ and E/pE^*, the corresponding elementary lattices represent the vertices, of a simplex in the link of w_E, if and only if the given subspaces of $\Delta(E)$ resp. E/pE^* are totally ordered under inclusion.

Remarks 3.7. (i) A flag in a vector space means a set of subspaces that are totally ordered under inclusion. The flag complex of any set Z of subspaces means the simplicial complex with vertex set Z, where a set of these subspaces spans a simplex if and only if they form a flag.

(ii) Given simplicial complexes with disjoint vertex sets X and Y, their join is a simplicial complex with vertex set $X \cup Y$. A set σ of vertices spans a simplex in the join if and only if $X \cap \sigma$ and $Y \cap \sigma$ span simplices in the given complexes.

Proof. By lemma 3.5, if an elementary lattice F represents a neighbor of E in $\mathcal{E}$, then either $F \supset E$ or $E \supset F$. In the first case, F's integrality gives $E^* \supseteq F \supset E$. So F is the preimage in E^* of a nonzero subspace of $E^*/E = \Delta(E)$. In the second case we have $E \supset F \supseteq pF^* \supset pE^*$, using that F is elementary and $E \supset F$. So F is the preimage in E of a subspace of E/pE^*. We have reduced the enumeration of neighbors of E to the problem: given a lattice F satisfying $E^* \supset F \supset E$ resp. $E \supset F \supset pE^*$, express the condition that F is elementary in terms of its image $\bar{F}$ in $\Delta(E)$ resp. E/pE^*.

First suppose $E^* \supset F \supset E$. If F is elementary, then it is integral, so the $(\frac{1}{p}\mathbb{Z}_p)/\mathbb{Z}_p$-valued inner product on $\bar{F} \subset \Delta(E)$ vanishes identically. Conversely, if $\bar{F}$ is totally isotropic, then the same argument shows that F is integral. From this follows $E^* \supset F^* \supseteq F \supset E$, so $\Delta(F)$ is a subquotient of $\Delta(E)$. Since $\Delta(E)$ is elementary abelian, $\Delta(F)$ is also, ie F is elementary.

Next suppose $E \supset F \supset pE^*$. Because E/pE^* is nondegenerate, the following are equivalent: (a) $\bar{F}$ is the orthogonal complement of a totally isotropic subspace of E/pE^*, and (b) $\bar{F}^\perp \subseteq \bar{F}$. So our goal is to prove that $\bar{F}^\perp \subseteq \bar{F}$ if and only if F is elementary. Dualizing $E \supset F \supset pE^*$ gives $E \supset pF^* \supset pE^*$, so we may speak of the subspace $\overline{pF^*}$ of E/pE^* corresponding to pF^*. Unwrapping definitions shows $\overline{pF^*} = \bar{F}^\perp$. This reduces us to proving that $pF^* \subseteq F$ if and only if F is elementary, which is a restatement of the definition.

This finishes the identification of the neighbors of E. The theorem's final statement follows immediately from lemma 3.5. (Note: the passage in (2), from nonzero totally isotropic subspaces of E/pE^* to elementary lattices strictly contained in E, reverses inclusions. But this does not affect well-orderedness under inclusion.) □

Remark 3.8. Implicit in the proof is a symmetry between the two cases of theorem 3.6, arising from the equality $E/pE^* = \Delta(E')$. So the neighbors of w_E in $\mathcal{E}$ are indexed by the nonzero totally isotropic subspaces of $\Delta(E)$ and $\Delta(E')$. The symmetry is obvious in the presence of a Jordan decomposition; suppose $E = E_0 \oplus E_1^{[p]}$, so $E' \cong E_0^{[p]} \oplus E_1$. Then E_0/pE_0 maps isomorphically to E/pE^* and to $\Delta(E')$.

To formulate this more intrinsically, observe that $E^{[p]*} = \frac{1}{p} \cdot E^{*[p]}$ for any lattice E, ie $p \cdot E^{[p]*} = E'$. From this follows

$$pE'^* = p \cdot (E')^* = p \cdot (p \cdot E^{[p]*})^* = p \cdot (\tfrac{1}{p} \cdot E^{[p]**}) = E^{[p]}$$

When E is elementary, so that E' is also, this gives

$$E'/pE'^* = E^{*[p]}/E^{[p]} = E^*/E = \Delta(E)$$

and symmetrically $E/pE^* = \Delta(E')$. The bilinear pairings on E/pE^* and $\Delta(E')$ coincide under this identification. (To compare them, multiply the pairing on $\Delta(E')$ by p, so that it takes values in $\mathbb{Z}_p/p\mathbb{Z}_p$ rather than $\frac{1}{p}\mathbb{Z}_p/\mathbb{Z}_p$.)

4. The local case

We continue to fix a prime p and write V for a nondegenerate quadratic space over $\mathbb{Q}_p$. In this section we prove theorems 1.1 and 1.2 by carrying out the program explained after lemma 3.4. That is, for each elementary lattice $E \subseteq V$, we will work out the $\mathrm{O}(E)$-invariant neighbors of w_E in $\mathcal{E}$, and use this iteratively to either show that $\mathrm{O}(E)$ is nonmaximal, or to show that it is maximal and find all $\mathrm{O}(E)$-invariant vertices of $\mathcal{E}$. For our local results, ie theorems 1.1 and 1.2, all that matters below is the invariance of various subspaces of $\Delta(E)$ or E/pE^* under $\mathrm{O}(E)$. The more detailed information contained in lemmas 4.1–4.5, concerning the spinor norm 1 subgroup $\Theta(E)$ of $\mathrm{SO}(E)$, and certain reflections in $\mathrm{O}(E)$, is only needed for the global case in the next section.

Lemma 4.1 (Unimodular orthogonal groups over $\mathbb{Z}_{\mathrm{odd}}$). *Suppose p is an odd prime and U is a unimodular $\mathbb{Z}_p$-lattice.*

1. *If $\dim U > 2$, then U/pU is irreducible under $\Theta(U)$.*
2. *If $U \cong \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and $r \in U$ with $p \nmid r^2$, then the reflection in r exchanges the two isotropic lines in U/pU.*

We remark that the lemma addresses exactly the cases that U/pU contains isotropic vectors.

Proof. (1) First suppose U is the lattice of symmetric 2×2 matrices over $\mathbb{Z}_p$, the quadratic form being the determinant. As usual $\mathrm{SL}_2\mathbb{Q}_p$ acts isometrically on $V = U \otimes \mathbb{Q}_p$, with each $g \in \mathrm{SL}_2\mathbb{Q}_p$ acting by $x \mapsto g^T x g$. The map $\mathrm{SL}_2(\mathbb{Q}_p) \to \mathrm{SO}(V)$ is well-known to be the spin double cover, whose image is $\Theta(V)$ (see [4, example 4, p. 193] for the essence of the argument). The restriction to $\mathrm{SL}_2(\mathbb{Z}_p)$ maps into $\Theta(U)$, and acts on U/pU by $\mathrm{SL}_2(\mathbb{F}_p)$'s usual action on the space of symmetric bilinear forms on $\mathbb{F}_p^2$. For $p > 2$, this is well-known to be irreducible. Up to scale, there is a unique unimodular 3-dimensional $\mathbb{Z}_p$-lattice, so we have finished the proof for $\dim V = 3$. Then an easy induction finishes the proof.

(2) Writing $\bar{r}$ for the image of r in U/pU, the reflection in r acts on U/pU as the reflection in $\bar{r}$, which makes sense because $\bar{r}$ is not isotropic. This reflection leaves invariant the span of $\bar{r}$, and also $\bar{r}^\perp$, but no other lines. These two lines are anisotropic, so the reflection must exchange the two isotropic lines. □

The last part of the next theorem is the odd-p case of theorem 1.1.

Theorem 4.2. *Suppose E is an elementary $\mathbb{Z}_p$-lattice with p odd.*

1. *There is no $\Theta(E)$-invariant elementary proper sublattice of E, except perhaps if E's unimodular constituent is isometric to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. In that case, for every root r with $p \nmid r^2$, the reflection in r exchanges the two elementary proper sublattices of E.*
2. *There is no $\Theta(E)$-invariant elementary proper superlattice of E, except perhaps if E's non-unimodular constituent is isometric to $\left(\begin{smallmatrix} 0 & p \\ p & 0 \end{smallmatrix}\right)$. In that case, for every root r with $p \,|\, r^2$, the reflection in r exchanges the two elementary proper superlattices of E.*
3. *$\mathrm{O}(E)$ is maximal, and E is the only $\mathrm{O}(E)$-invariant elementary lattice in $E \otimes \mathbb{Q}_p$.*

Proof. (1) By theorem 3.6, the elementary lattices contained in E correspond to the totally isotropic subspaces of the nondegenerate inner product space E/pE^*. This space equals E_0/pE_0 for any choice of Jordan decomposition $E = E_0 \oplus E_1^{[p]}$. We recall that the oddness of p implies that any two Jordan decompositions are $\mathrm{O}(E)$-equivalent. First consider the exceptional case, in which the unimodular constituent of every Jordan decomposition is isometric to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and r is a root of E with $p \nmid r^2$. Because $\frac{1}{2} \in \mathbb{Z}_p^\times$, every root generates a summand of E. So $\langle r \rangle$ lies in the unimodular constituent E_0 of some Jordan decomposition. We are assuming $E_0 \cong \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, so $E/pE^* = E_0/pE_0$ has exactly two isotropic lines. And lemma 4.1(2) shows that the reflection in r exchanges them, finishing the proof in the special case.

The generic case is simpler; choose any Jordan decomposition $E_0 \oplus E_1^{[p]}$. Since $E_0 \not\cong \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, E/pE^* possesses no isotropic vectors unless $\dim E_0 \geq 3$. By lemma 4.1(1), $\Theta(E_0) \subseteq \Theta(E)$ acts irreducibly on $E_0/pE_0 = E/pE^*$. So there are no nonzero $\Theta(E)$-invariant totally isotropic subspaces.

(2) Essentially the same, with E/pE^* replaced by $\Delta(E)$, which equals $\frac{1}{p}E_1^{[p]}/E_1^{[p]} \cong E_1/pE_1$ for any Jordan decomposition $E = E_0 \oplus E_1^{[p]}$. In the exceptional case, one may choose the Jordan decomposition so that $E_1^{[p]}$ contains r, and then apply the same argument with E_1 in place of E_0.

(3) Theorem 3.6 shows that the neighbors of w_E in $\mathcal{E}$ correspond to the elementary lattices that E properly contains, or is properly contained in. So (1)–(2) imply that no neighbor of w_E is $\mathrm{O}(E)$-invariant. It follows from lemma 3.4 that w_E is the only fixed point of $\mathrm{O}(E)$. Since the vertices of $\mathcal{E}$ correspond to the elementary lattices in $E \otimes \mathbb{Q}_p$, E is the only $\mathrm{O}(E)$-invariant elementary lattice. $\mathrm{O}(E)$ is maximal because it is the full $\mathrm{O}(V)$-stabilizer of every vertex of $\mathcal{E}$ that it fixes. □

In the rest of this section we take $p = 2$. The strategy is the same, but everything is more delicate. We recall that a nondegenerate symmetric bilinear form, on an $\mathbb{F}_2$ vector space W, distinguishes the *characteristic vector* c_W of W. To construct it, observe that the norm function $W \to \mathbb{F}_2$, namely $x \mapsto x \cdot x$, is linear (this uses $2 = 0$). We define c_W as the unique element of W for which this linear function coincides with $x \mapsto x \cdot c_W$.

As in theorem 3.6, "isotropic" and "totally isotropic" refer only whatever symmetric bilinear form is present, not to any quadratic form which might also be present.

Lemma 4.3 (Unimodular orthogonal groups over $\mathbb{Z}_2$)**.** *Suppose U is a unimodular $\mathbb{Z}_2$-lattice and c is the characteristic vector of $U/2U$.*

1. *If $U \not\cong 1_{\mathrm{II}}^{+2}$ is even, then $\Theta(U)$ acts irreducibly on $U/2U$.*
2. *Suppose U is odd, and odd-dimensional. Then $U/2U = c^\perp \oplus \langle c \rangle$. If U is neither $1_{\pm 1}^{+3}$ nor $1_{\pm 3}^{-3}$, then $\Theta(U)$ acts irreducibly on $c^\perp$.*
3. *Suppose U is odd, and even-dimensional. Then $c \in c^\perp$. If U is neither 1_0^{+4} nor 1_4^{-4}, then $\langle c \rangle$ is the only nonzero $\Theta(U)$-invariant totally isotropic subspace of $U/2U$.*
4. *Suppose U is one of the exceptions 1_{II}^{+2}, $1_{\pm 1}^{+3}$, $1_{\pm 3}^{-3}$, 1_0^{+4} or 1_4^{-4} from (1)–(3). Then U possesses vectors with norm 2 mod 4, and the images in $U/2U$ of all such vectors span an $\mathrm{O}(U)$-invariant totally isotropic subspace S. Furthermore, writing $R \in \mathrm{O}(U)$ for the reflection in any one such vector,*
   (a) *if $U \cong 1_{\mathrm{II}}^{+2}$, $1_{\pm 1}^{+3}$ or $1_{\pm 3}^{-3}$, then $\dim S = 1$ and S is the only nonzero R-invariant totally isotropic subspace of $U/2U$.*
   (b) *if $U \cong 1_0^{+4}$ or 1_4^{-4}, then $\dim S = 2$ and S contains c. Furthermore, $\langle c \rangle$ and S are the only nonzero totally isotropic subspaces of $U/2U$ that are invariant under R and $\Theta(U)$.*

Proof. (1) We are assuming U is even unimodular. We work out three cases explicitly. First, $U = 1_{\mathrm{II}}^{-2}$ is represented by the A_2 root lattice. The obvious $\mathbb{Z}/3$, inside the Weyl group $W(A_2) \cong S_3$, acts irreducibly on $U/2U$ and lies in $\Theta(U)$.

Second, $U = 1_{\mathrm{II}}^{+4}$ is represented by the root lattice $A_2 \oplus A_2$. We will exhibit an element of $\Theta(U)$ that exchanges the summands. Choose orthogonal roots a, b in the first summand, and orthogonal roots a', b' in the second, such that $a^2 = a'^2 = 2$ and $b^2 = b'^2 = 6$. Reflection in $a - a'$, composed with reflection in $b - b'$, composed with negation of one of the summands, exchanges the summands and has spinor norm 1. Together with $(\mathbb{Z}/3)^2 \subseteq W(A_2)^2$, it generates a subgroup of $\Theta(U)$ that acts irreducibly on $U/2U$.

Third, $U = 1_{\mathrm{II}}^{-4}$ is represented by the A_4 root lattice, and $\Theta(U)$ contains the commutator subgroup of the Weyl group $W(A_4) \cong S_5$. This is the alternating group A_5, which acts on $U/2U$ with orbit sizes 1, 5 and 10, hence irreducibly. The higher-dimensional cases follow by decomposing U as a sum of copies of 1_{II}^{+2} and 1_{II}^{-2} and using induction.

In the proofs of (2)–(4) we will use the following direct sum decompositions:

$$1_t^{\pm n} \cong \begin{cases} 1_t^{+1} \oplus 1_{\mathrm{II}}^{\pm(n-1)} & \text{if } t \in \{1, -1\} \\ 1_t^{-1} \oplus 1_{\mathrm{II}}^{\mp(n-1)} & \text{if } t \in \{3, -3\} \\ 1_t^{\pm 2} \oplus 1_{\mathrm{II}}^{+(n-2)} \cong 1_t^{\mp 2} \oplus 1_{\mathrm{II}}^{-(n-2)} & \text{if } t \in \{2, -2\} \\ 1_0^{+2} \oplus 1_{\mathrm{II}}^{\pm(n-2)} & \text{if } t = 0 \\ 1_4^{-2} \oplus 1_{\mathrm{II}}^{\mp(n-2)} & \text{if } t = 4 \end{cases} \tag{4.1}$$

In the middle case, the second decomposition exists if and only if $n > 2$.

(2) We are assuming $U \cong 1_{\mathrm{I}}^{\mathrm{odd}}$. One of the first 2 cases of (4.1) applies. The first summand, modulo 2, equals $\langle c \rangle \subseteq U/2U$, and the second summand, modulo 2, equals $c^\perp$. Now suppose U is neither $1_{\pm 1}^{+3}$ nor $1_{\pm 3}^{-3}$. Then the second summand is even but not 1_{II}^{+2}, so the claimed irreducibility follows from (1).

(3) We are assuming $U \cong 1_{\mathrm{I}}^{\mathrm{even}}$. First suppose $\dim U = 2$, so $U \cong \langle a, b \rangle$ with a, b odd. Then the characteristic vector $c = (1,1) \in \mathbb{F}_2^2$ spans the only nonzero totally isotropic subspace of $U/2U$. Now suppose $\dim U > 2$. One of the last 3 cases of (4.1) applies. The first summand, mod 2, contains c. Write B for the second summand, so $c^\perp = \langle c \rangle \oplus B/2B$. The rest of (3) assumes $U \ncong 1_0^{+4}$ or 1_4^{-4}. Then B is (or can be chosen to be) different from 1_{II}^{+2}. By (1), $\Theta(B)$ acts irreducibly on $B/2B \cong c^\perp/\langle c \rangle$. Now, seeking a contradiction, suppose $S \neq \langle c \rangle$ is a nonzero $\Theta(U)$-invariant totally isotropic subspace of $U/2U$. Total

isotropy implies $S \subseteq c^\perp$. By the irreducibility of $\Theta(B)$ on $B/2B$, S must project onto $B/2B$. Because c^2 is even, and $B/2B$ contains a pair of vectors with odd inner product, S does too. This contradicts total isotropy.

(4) The listed exceptional cases are

$$U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus \big(\text{nothing}, \langle \pm 1\rangle, \langle \pm 3\rangle, \langle 1,-1\rangle \text{ or } \langle 1,3\rangle\big) \tag{4.2}$$

A norm 2 root is visible in the first summand. The lemma writes $R \in \mathrm{O}(U)$ for the reflection in any one lattice vector with norm 2 mod 4, which we will call r. Because $4 \nmid r^2$, r is primitive in U, and therefore its image $\bar{r} \in U/2U$ is nonzero. R acts on $U/2U$ by the orthogonal transvection $x \mapsto x + (x \cdot \bar{r})\bar{r}$.

(4a) First suppose $U \cong \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then every element of U with norm 2 mod 4 projects to $(1,1) \in U/2U$. So $(1,1)$ spans S (which is therefore totally isotropic) and equals $\bar{r}$. Its transvection exchanges the other two nonzero elements of $U/2U$. This finishes the proof if $U \cong \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Now suppose $U \cong \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus \langle \pm 1 \text{ or } \pm 3\rangle$. We have $c = (0,0,1) \in \mathbb{F}_2^3$. Every totally isotropic subspace of $U/2U$ lies in $c^\perp$, which is the mod 2 reduction of the summand $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ of U. Every element of U with norm 2 mod 4 projects to $(1,1,0) \in U/2U$. So $(1,1,0)$ spans S (which is therefore totally isotropic) and equals $\bar{r}$. The two nonzero elements of $c^\perp$, other than $\bar{r}$, are exchanged by R. So S is the only nonzero R-invariant totally isotropic subspace of $U/2U$.

(4b) We are assuming $U \cong \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus \langle 1, -1 \text{ or } 3\rangle$. We define $u, v \in U$ by $u = (1,1,0,0)$ and $v = (1,1,1,1)$. Their images $\bar{u}, \bar{v}$ in $U/2U$ are the only mod 2 classes represented by vectors with norm 2 mod 4. This gives several things. First, S is their span, which is totally isotropic because $u \cdot v$ is even. Second, one checks $c = \bar{u} + \bar{v} \in S$. Finally, $\bar{r} = \bar{u}$ or $\bar{v}$.

We must show that if T is a nonzero $\langle R, \Theta(U)\rangle$-invariant totally isotropic subspace of $U/2U$, then $T = \langle c\rangle$ or S. First, T consists of isotropic vectors and hence lies in $c^\perp$. Next, T lies in S. To see this, suppose to the contrary that T contains some

$$x \in c^\perp - S = \big\{(1,0,0,0), (0,1,0,0), (1,0,1,1), (0,1,1,1)\big\}.$$

Then $x \cdot \bar{r} \neq 0$ (one checks this for both possibilities for $\bar{r}$). It follows that T also contains $R(x) = x + \bar{r}$, which has nonzero pairing with x, contrary to total isotropy.

It remains to show that $\langle c\rangle$ is the only 1-dimensional $\langle R, \Theta(U)\rangle$-invariant subspace of S. It is enough to show that $\bar{u}$ and $\bar{v}$ are $\Theta(U)$-equivalent. First we show they are $\mathrm{O}(U)$-equivalent. Observe that the span of $(1,0,0,0)$ and u resp. v is a copy of 1_{II}^{+2}; we will call it E_u

resp. E_v. It is classical that in an integral 2-adic lattice, any isometry between even unimodular sublattices extends to an isometry of the ambient lattice. (See [2, Lemma 7.2], or the corollary on p. 122 of [4].) So some $g \in \mathrm{O}(U)$ sends E_u to E_v. Obviously g sends vectors with norm $\equiv 2 \bmod 4$ to vectors with norm $\equiv 2 \bmod 4$. Reducing mod 2 shows that g sends $\bar{u}$ to $\bar{v}$.

Next we replace g by its compositions with various reflections of U, to arrange that $g \in \Theta(U)$. We will not lose the property $g(\bar{u}) = \bar{v}$, because every reflection we will use fixes $\bar{u}$ and $\bar{v}$. (In particular, reflections in odd norm roots act trivially on $U/2U$.) First, if $g \notin \mathrm{SO}(U)$, then we postcompose g with the reflection in $(0,0,1,0)$. So we may take $g \in \mathrm{SO}(U)$. Next, if the spinor norm of g is even, then we postcompose g with the reflections in u and $(0,0,1,0)$. So we may suppose g has odd spinor norm. Finally, $\langle 1,-1\rangle$ and $\langle 1,3\rangle$ possess vectors of every odd norm. After postcomposing g with the reflections in some such vectors, we may suppose $g \in \Theta(U)$. □

Lemma 4.4 (Invariant elementary sublattices)**.** *Suppose E is an elementary 2-adic lattice.*

(1) *Suppose $E \cong 1_{\mathrm{I}}^{\mathrm{even}} 2_{\cdots}^{\cdots}$, but $E \not\cong (1_0^{+4} \,\mathrm{or}\, 1_4^{-4})2_{\mathrm{II}}^{\cdots}$. Then its even sublattice E_{even} is the only $\Theta(E)$-invariant elementary proper sublattice of E.*

(2) *Suppose $E \cong \big(1_{\mathrm{II}}^{+2},\, 1_{\pm 1}^{+3},\, 1_{\pm 3}^{-3},\, 1_0^{+4}$ or $1_4^{-4}\big)2_{\mathrm{II}}^{\cdots}$. Then E contains vectors with norm $\equiv 2$ mod 4; let S be the subspace of $E/2E^*$ spanned by their images and let $R \in \mathrm{O}(E)$ be the reflection in any one of them. Define the* **exceptional sublattice** *E_{ex} as the preimage in E of $S^\perp \subseteq E/2E^*$.*

   (a) *If $E \cong \big(1_{\mathrm{II}}^{+2},\, 1_{\pm 1}^{+3}$, or $1_{\pm 3}^{-3}\big)2_{\mathrm{II}}^{\cdots}$, then E_{ex} is the only R-invariant elementary proper sublattice of E.*

   (b) *If $E \cong (1_0^{+4} \,\mathrm{or}\, 1_4^{-4})2_{\mathrm{II}}^{\cdots}$, then E_{ex} and E_{even} are the only $\langle R, \Theta(E)\rangle$-invariant elementary proper sublattices of E.*

(3) *Otherwise, E has no $\Theta(E)$-invariant elementary proper sublattice.*

The subscript $_{\mathrm{even}}$, indicating the even sublattice, is not related to the superscript $^{\mathrm{even}}$, which we often use to indicate that a Jordan constituent has even dimension.

Proof. Theorem 3.6(2) shows that every elementary lattice, that is strictly contained in E, is the preimage of the orthogonal complement of a nonzero totally isotropic subspace of $E/2E^*$. The current proof consists of working out which such subspaces of $E/2E^*$ are invariant under $\Theta(E)$, or under R or $\langle \Theta(E), R\rangle$ in part (2). We will consider a Jordan decomposition $E = E_0 \oplus E_1^{[2]}$, and the key point is that $E/2E^* = E_0/2E_0$. This allows us to appeal to lemma 4.3 for information about how $\Theta(E_0) \subseteq \Theta(E)$ acts on $E_0/2E_0$. Some case splitting propagates from there, but the underlying reasoning is mostly uniform.

Case 1. Suppose E is even, but not of the form $1_{\mathrm{II}}^{+2} 2_{\mathrm{II}}^{\cdots}$. In particular, if $E_0 \cong 1_{\mathrm{II}}^{+2}$, then the second Jordan constituent has type I. In this case, sign walking shows $E \cong 1_{\mathrm{II}}^{-2} 2_{\mathrm{I}}^{\cdots}$. Replacing our original Jordan decomposition with this one, we have reduced to the case E_0 is even and not 1_{II}^{+2}. Appealing to lemma 4.3(1) shows that $\Theta(E_0)$ acts irreducibly on $E_0/2E_0 = E/2E^*$. So there is no nonzero $\Theta(E)$-invariant totally isotropic subspace of $E/2E^*$. Therefore E has no $\Theta(E)$-invariant elementary proper sublattices, as claimed by (3).

Case 2. Suppose $E \cong 1_{\cdots}^{\mathrm{odd}} 2_{\cdots}^{\cdots}$, but $E \not\cong (1_{\pm 1}^{+3} \text{ or } 1_{\pm 3}^{-3}) 2_{\mathrm{II}}^{\cdots}$. In particular, if $E_0 \cong 1_{\pm 1}^{+3}$ or $1_{\pm 3}^{-3}$, then the second Jordan constituent has type I, say $E = 1_t^{\varepsilon 3} 2_u^{\varepsilon' d}$. In this case, sign walking shows $E \cong 1_t^{-\varepsilon 3} 2_{u+4}^{-\varepsilon' d}$, and we replace our original Jordan decomposition with this one. We have reduced to the case that $E_0 \cong 1_{\mathrm{I}}^{\mathrm{odd}}$ but $E_0 \not\cong 1_{\pm 1}^{+3}, 1_{\pm 3}^{-3}$. Appealing to lemma 4.3(2) shows that $E_0/2E_0 = \langle c \rangle \oplus c^\perp$ and that $\Theta(E_0)$ acts irreducibly on $c^\perp$. Here c is the characteristic vector of $E_0/2E_0$. Since c is not isotropic, and $c^\perp$ is not totally isotropic, there are no nonzero $\Theta(E)$-invariant totally isotropic subspaces of $E/2E^*$. So E has no $\Theta(E)$-invariant elementary proper sublattices, as claimed in (3).

Case 3. Suppose $E_0 \cong 1_{\mathrm{I}}^{\mathrm{even}}$, but $E \not\cong (1_0^{+4} \text{ or } 1_4^{-4}) 2_{\mathrm{II}}^{\cdots}$. In particular, if $E_0 \cong 1_0^{+4}$ or 1_4^{-4}, then the second Jordan constituent has type I, say $E \cong 1_t^{\varepsilon 4} 2_u^{\varepsilon' d}$. In this case we use oddity fusion to increase t by some even number δ, and simultaneously decrease u by the same amount. If $d = 2$, then the legality of $2_{u-\delta}^{\varepsilon' d}$ constrains δ somewhat, but it is always possible to take $\delta \in \{2, -2\}$. After replacing our original Jordan decomposition with this one, we have $E_0 \not\cong 1_0^{+4}$ or 1_4^{-4}. Now, arguing as in the previous cases, using part (3) of lemma 4.3, shows that the only nonzero $\Theta(E_0)$-invariant totally isotropic subspace of $E_0/2E_0$ is $\langle c \rangle$. It follows that $\langle c \rangle$ is the only nonzero $\Theta(E)$-invariant totally isotropic subspace of $E/2E^*$. Therefore the only $\Theta(E)$-invariant elementary proper sublattice of E is the preimage in E of $c^\perp \subseteq E/2E^*$. This is E_{even}, as claimed in (1).

Case 4. The cases not yet treated are exactly the ones about which (2) makes assertions, namely

$$E \cong (1_{\mathrm{II}}^{+2},\ 1_{\pm 1}^{+3},\ 1_{\pm 3}^{-3},\ 1_0^{+4}, \text{ or } 1_4^{-4}) 2_{\mathrm{II}}^{\cdots}.$$

Because the second constituent consists of vectors with norms divisible by 4, the subspace $S \subseteq E/2E^*$ of the current lemma coincides with the subspace $S \subseteq E_0/2E_0$ of lemma 4.3(4), under the identification

$E/2E^* = E_0/2E_0$. Furthermore, writing $r \in E$ for a vector with norm 2 mod 4 whose reflection is R, the projection of r to E_0 also has norm 2 mod 4, and its reflection acts on $E/2E^*$ in the same way. So, for purposes of examining R-invariant subspaces of $E/2E^*$, we may suppose without loss that $r \in E_0$. Therefore lemma 4.3(4)'s R may be taken to be the current R. Quoting that lemma gives: the only nonzero $\langle \Theta(E_0), R\rangle$-invariant totally isotropic subspaces of $E_0/2E_0$ are

$$\begin{aligned} S \quad & \text{if } E \cong (1_{\mathrm{II}}^{+2},\ 1_{\pm1}^{+3},\ 1_{\pm3}^{-3})2_{\mathrm{II}}^{\cdots}, \quad \text{resp.} \\ S \text{ or } \langle c\rangle \quad & \text{if } E \cong (1_0^{+4} \text{ or } 1_4^{-4})2_{\mathrm{II}}^{\cdots}. \end{aligned}$$

Therefore the same holds with $\Theta(E_0)$ replaced by $\Theta(E)$ and $E_0/2E_0$ replaced by $E/2E^*$. It follows that the only $\langle \Theta(E), R\rangle$-invariant elementary proper sublattices of E are the preimages in E of $S^\perp$, resp. $S^\perp$ and $c^\perp$. These are E_{ex}, resp. E_{ex} and E_{even}. □

To use lemma 4.4, we will need to know E_{even} and E_{ex} explicitly. One can construct a copy of $1_t^{\pm2}$ by writing it as $\langle a, b\rangle$, with $a+b \equiv t$ mod 8 and the Kronecker symbol $(\frac{ab}{2})$ being the given sign. Then, choosing a basis for the even sublattice yields the $n = 2$ case of the following. The formula for the case of even $n > 2$ follows from this and (4.1).

$$(4.3) \qquad \text{for even } n, \qquad (1_t^{\pm n})_{\mathrm{even}} \cong \begin{cases} 1_{\mathrm{II}}^{+(n-2)}2_{\pm t}^{\pm 2} & \text{if } t \in \{2, -2\} \\ 1_{\mathrm{II}}^{\pm(n-2)}2_{\mathrm{II}}^{+2} & \text{if } t = 0 \\ 1_{\mathrm{II}}^{\mp(n-2)}2_{\mathrm{II}}^{-2} & \text{if } t = 4 \end{cases}$$

For a general elementary lattice E of the form $\cong 1_{\mathrm{I}}^{\mathrm{even}}2_{\cdots}^{\cdots}$, we get E_{even} by applying (4.3) to the unimodular constituent and combining the resulting $2_{\cdots}^{\pm2}$ with the second constituent $2_{\cdots}^{\cdots}$ of E.

If $E \cong \big(1_{\mathrm{II}}^{+2}, 1_{\pm1}^{+3}, 1_{\pm3}^{-3}, 1_0^{+4} \text{ or } 1_4^{-4}\big)2_{\mathrm{II}}^{\cdots}$, then we defined the exceptional sublattice E_{ex} in lemma 4.4(2). To compute it, write the unimodular constituent as

$$(4.4) \qquad E_0 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus \big(\text{nothing}, \langle \pm1\rangle, \langle \pm3\rangle, \langle 1, -1\rangle \text{ or } \langle 1, 3\rangle\big)$$

In the proof of lemma 4.3(4), we worked out the subspace S of $E/2E^*$. In the last two cases of (4.4), it is the span of the images of the lattice vectors $(1,1,0,0), (1,1,1,1) \in E_0$. By definition, a member of E lies in E_{ex} just if it has even inner product with these vectors, or equivalently with $(1,1,0,0), (0,0,1,1) \in E_0$. The sublattice of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ having even inner product with $(1,1)$ is $\langle 2, -2\rangle$, and the sublattice of $\langle 1, -1 \text{ or } 3\rangle$ having even inner product with $(1,1)$ is $\langle 1, -1 \text{ or } 3\rangle_{\mathrm{even}}$. So

$$(4.5) \quad (1_0^{+4})_{\mathrm{ex}} \cong \langle 2, -2\rangle \oplus 2_{\mathrm{II}}^{+2} \cong 2_0^{+4} \qquad (1_4^{-4})_{\mathrm{ex}} \cong \langle 2, -2\rangle \oplus 2_{\mathrm{II}}^{-2} \cong 2_0^{-4}$$

In the 2- resp. 3-dimensional cases of (4.4), S is the span of the image of $(1,1)$ resp. $(1,1,0) \in E_0$, leading to

$$(4.6) \qquad (1^{+2}_{\mathrm{II}})_{\mathrm{ex}} \cong 2^{+2}_0 \qquad (1^{+3}_{\pm 1})_{\mathrm{ex}} \cong 1^{+1}_{\pm 1} 2^{+2}_0 \qquad (1^{-3}_{\pm 3})_{\mathrm{ex}} \cong 1^{-1}_{\pm 3} 2^{+2}_0$$

As for E_{even}, the non-unimodular constituent of (4.5) or (4.6) should be merged with the second constituent $2^{\because}_{\mathrm{II}}$ of E when describing E_{ex}.

Next we prove the superlattice analogue of lemma 4.4. We will use superscripts for the relevant invariant superlattices E^{char} and E^{ex} of E, just as we used subscripts for the sublattices E_{even} and E_{ex} in lemma 4.4. The *exceptional superlattice* E^{ex} is defined in lemma 4.5(2), and the *characteristic superlattice* E^{char} is defined as the span of E and any representative in E^* of the characteristic vector of $\Delta(E)$.

Lemma 4.5 (Invariant elementary superlattices)**.** *Suppose F is an elementary 2-adic lattice.*

(1) *Suppose $F \cong 1^{\because}_{\because} 2^{\mathrm{even}}_{\mathrm{I}}$, but $F \not\cong 1^{\because}_{\mathrm{II}}(2^{+4}_0 \text{ or } 2^{-4}_4)$. Then F^{char} is the only $\Theta(F)$-invariant elementary proper superlattice of F.*

(2) *Suppose $F \cong 1^{\because}_{\mathrm{II}}\big(2^{+2}_{\mathrm{II}},\ 2^{+3}_{\pm 1},\ 2^{-3}_{\pm 3},\ 2^{+4}_0$ or $2^{-4}_4\big)$. Then the norms of elements of $\Delta(F)$ are well-defined mod 2. Let $T \subseteq \Delta(F)$ be the span of those with norm 1 mod 2, and define the* **exceptional superlattice** *F^{ex} as the preimage in F^* of T. Also, F has roots of norm 4 mod 8; let R be the reflection in any one of them.*
   - (a) *If $F \cong 1^{\because}_{\mathrm{II}}\big(2^{+2}_{\mathrm{II}},\ 2^{+3}_{\pm 1}$, or $2^{-3}_{\pm 3}\big)$, then F^{ex} is the only R-invariant elementary lattice strictly containing F.*
   - (b) *If $F \cong 1^{\because}_{\mathrm{II}}(2^{+4}_0 \text{ or } 2^{-4}_4)$, then F^{char} and F^{ex} are the only $\langle R, \Theta(F)\rangle$-invariant elementary lattices that strictly contain F.*

(3) *Otherwise, no $\Theta(F)$-invariant elementary lattice strictly contains F.*

Before proving this we work out F^{char} and F^{ex}. For a, b odd, one obtains $\langle 2a, 2b\rangle^{\mathrm{char}}$ by adjoining the vector $(\frac{1}{2}, \frac{1}{2})$. The results are the $n = 2$ case of (4.7) below. For $F \cong 2^{\mathrm{even}}_{\mathrm{I}}$, decompose F as in (4.1), with all inner products doubled, and then apply the 2-dimensional case to the summand $2^{\pm 2}_{\mathrm{I}}$. This yields

$$(4.7) \qquad \text{for even } n, \qquad (2^{\pm n}_t)^{\mathrm{char}} \cong \begin{cases} 1^{\pm 2}_{\pm t} 2^{+(n-2)}_{\mathrm{II}} & \text{if } t \in \{2, -2\} \\ 1^{+2}_{\mathrm{II}} 2^{\pm (n-2)}_{\mathrm{II}} & \text{if } t = 0 \\ 1^{-2}_{\mathrm{II}} 2^{\mp (n-2)}_{\mathrm{II}} & \text{if } t = 4, \end{cases}$$

which (unsurprisingly) differs from (4.3) by rescaled duality. For $F \cong 1^{\because}_{\because} 2^{\mathrm{even}}_{\mathrm{I}}$, one applies this formula to the second constituent, and merges the unimodular part of the result with F's unimodular constituent. (Remark: F^{char} is defined for any elementary F. But the remaining cases are uninteresting: F^{char} coincides with F when $F \cong 1^{\cdots}_{\cdots}2^{\cdots}_{\text{II}}$, and is nonintegral when $F \cong 1^{\cdots}_{\cdots}2^{\text{odd}}_{\text{I}}$.)

Next suppose $F \cong 1^{\cdots}_{\text{II}}\big(2^{+2}_{\text{II}},\ 2^{+3}_{\pm1},\ 2^{-3}_{\pm3},\ 2^{+4}_{0}$ or $2^{-4}_{4}\big)$, as in part (2) of the lemma. Write F's second constituent as in (4.4) with all inner products doubled. The only element of $\Delta(F)$ with norm 1 mod 2 is represented by $(\frac{1}{2},\frac{1}{2})$ in the summand $\big(\begin{smallmatrix}0&2\\2&0\end{smallmatrix}\big)$, except in the last two cases, when the only other such element is represented by $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$. These span a totally isotropic subspace of $\Delta(F)$. (We recall the remark before theorem 3.6, that "total isotropy" refers to the vanishing of inner products mod 1, not the vanishing of norms mod 2.) So adjoining these vectors to F yields an $\mathrm{O}(F)$-invariant elementary superlattice of F. Working out the details yields

$$(4.8)\qquad \begin{gathered}(2^{+2}_{\text{II}})^{\text{ex}} \cong 1^{+2}_{0} \qquad (2^{+3}_{\pm1})^{\text{ex}} \cong 1^{+2}_{0}2^{+1}_{\pm1} \qquad (2^{-3}_{\pm3})^{\text{ex}} \cong 1^{+2}_{0}2^{-1}_{\pm3}\\ (2^{+4}_{0})^{\text{ex}} \cong 1^{+4}_{0} \qquad (2^{-4}_{4})^{\text{ex}} \cong 1^{-4}_{0}\end{gathered}$$

which is (unsurprisingly) the rescaled dual of (4.5)–(4.6). As before, the unimodular constituent of this result should be merged with the unimodular constituent of F.

Proof. Take $E = F^{[1/2]*}$, whose rescaled dual E' is F. By lemma 2.1, rescaled duality sends elementary lattices to elementary lattices, and reverses inclusions. So the elementary superlattices of F are the rescaled duals of the elementary sublattices of E. Using $E \cong E'' = F'$, we see that E satisfies the hypotheses of (1), (2a), (2b) or (3) of lemma 4.4 if and only if F satisfies the corresponding hypothesis of the current lemma. The rest of the proof transforms that lemma's results about E into our claims about F.

(1) We are assuming $F \cong 1^{\cdots}_{\cdots}2^{\text{even}}_{\text{I}}$ but $F \ncong 1^{\cdots}_{\text{II}}\big(2^{+4}_{0}$ or $2^{-4}_{4}\big)$. Scaling the inner product on V does not affect the spinor norm homomorphism $\mathrm{SO}(V) \to \mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2$. From this and $\mathrm{O}(E) = \mathrm{O}(F)$ follows $\Theta(E) = \Theta(F)$. The elementary proper sublattices of E, that are invariant under $\Theta(E)$, and listed in lemma 4.4(1). There is exactly one, namely E_{even}, so there exists a unique $\Theta(F)$-invariant elementary proper superlattice of F. Since F^{char} is such a lattice, we are done. We also get the unsurprising equality $F^{\text{char}} = (E_{\text{even}})'$.

(2) We are assuming $F \cong 1^{\cdots}_{\text{II}}\big(2^{+2}_{\text{II}}, 2^{+3}_{\pm1}, 2^{-3}_{\pm3}, 2^{+4}_{0}$ or $2^{-4}_{4}\big)$. The evenness of F implies the well-definedness of norms in $\Delta(F)$ mod 2, so the definition of T makes sense. As in the statement of the lemma, we write R for the reflection in any root t of F with norm 4 mod 8. (Such a root is visible in the second constituent of F.) Observe $t \cdot F \subseteq \frac{1}{2}t^2\mathbb{Z}_2 \subseteq 2\mathbb{Z}_2$,

and recall that $F^{[1/2]}$ is the $\mathbb{Z}$-module underlying F, with all inner products halved. Regarding t as an element of $F^{[1/2]}$, these imply (a) $t \cdot F^{[1/2]} \subseteq \mathbb{Z}_2$, ie $t \in F^{[1/2]*} = E$, and (b) $t^2 \equiv 2 \bmod 4$. Therefore the reflection called R in lemma 4.4(2) may be taken to be the current reflection R. With this preparation, and $\Theta(E) = \Theta(F)$ from the previous paragraph, lemma 4.4(2) lists the elementary proper sublattices of E that are invariant under R or $\langle R, \Theta(E)\rangle$. Their rescaled duals are the elementary proper superlattices of F with the same invariance properties. In particular, in subcase (2a) resp. (2b) the number of such superlattices is 1 resp. 2. We have exhibited this many such lattices, so our enumeration is complete. We also get the unsurprising equality $(E_{\text{ex}})' = F^{\text{ex}}$.

(3) We have already proven $\Theta(E) = \Theta(F)$. Lemma 4.4(3) shows the absence of $\Theta(E)$-invariant elementary proper sublattices of E. So there are no $\Theta(F)$-invariant elementary lattices that properly contain F. $\square$

The next theorem concludes the local analysis.

Theorem 4.6 (Table of elementary 2-adic lattices)**.** *For each elementary 2-adic lattice E, table 4.1 says whether $\mathrm{O}(E)$ is maximal under inclusion among all lattice stabilizers in $\mathrm{O}(V)$, where $V = E \otimes \mathbb{Q}_2$. When $\mathrm{O}(E)$ is maximal, the table lists all the other $\mathrm{O}(E)$-invariant elementary lattices in V, and a recipe for recovering E from each of them. When $\mathrm{O}(E)$ is not maximal, the table lists one or more $\mathrm{O}(E)$-invariant lattices in V, whose orthogonal groups are maximal and hence strictly contain $\mathrm{O}(E)$.*

Remarks 4.7 (Reading the table). (1) Every elementary 2-adic lattice E appears in exactly one of the 11 blocks 1–7 and 3′–6′. The row in that block that applies to E is the *most specific one that matches* the isometry type of E; more-specific rows are listed before less-specific ones. Eg, $1_{\mathrm{II}}^{+2}2_{\mathrm{II}}^{+4}$ is treated by case 2b not 2d, and $1_{\pm1}^{+1}2_0^{-4}$ is treated by 6a not 6b. (Note: $1_{\pm1}^{+1}2_0^{-4} \cong 1_{\mp3}^{-1}2_0^{+4}$ by sign walking.)

(2) The boxed rows in the table are the most important, because they exactly account for the maximal groups; they are the forms in (1.2). The remaining rows give alternate descriptions of some of these groups, and also some non-maximal groups.

(3) The "related lattice" $E_{\text{even}}^{\text{char}}$ in case 7b is defined as $(E^{\text{char}})_{\text{even}}$ or $(E_{\text{even}})^{\text{char}}$. Part of the proof is checking that these coincide.

Remark 4.8 (The most interesting case). If $\mathrm{O}(E)$ is maximal, then it preserves at most two elementary lattices in V, unless $E \cong 1_{\mathrm{II}}^{+2}2_{\mathrm{II}}^{+2}$, 1_0^{+4} or 2_0^{+4}. These are all a single example, with $\mathrm{O}(E)$ preserving exactly

	E	$\mathrm{O}(E)$ maximal?	related lattices	E equals
1.	$1^{\mathrm{odd}}_{\cdots}2^{\mathrm{odd}}_{\cdots}$	yes	none	
2a.	$1^{+2}_{\mathrm{II}}2^{+2}_{\mathrm{II}}$	yes	$E^{\mathrm{ex}} \cong 1^{+4}_{0}$	$(E^{\mathrm{ex}})_{\mathrm{even}}$
			$E_{\mathrm{ex}} \cong 2^{+4}_{0}$	$(E_{\mathrm{ex}})^{\mathrm{char}}$
2b.	$1^{+2}_{\mathrm{II}}2^{\cdots}_{\mathrm{II}}$	yes	$E_{\mathrm{ex}} \cong 2^{\cdots}_{0}$	$(E_{\mathrm{ex}})^{\mathrm{char}}$
2c.	$1^{\cdots}_{\mathrm{II}}2^{+2}_{\mathrm{II}}$	yes	$E^{\mathrm{ex}} \cong 1^{\cdots}_{0}$	$(E^{\mathrm{ex}})_{\mathrm{even}}$
2d.	$1^{\cdots}_{\mathrm{II}}2^{\cdots}_{\mathrm{II}}$	yes	none	
3a.	$(1^{+3}_{\pm1} \text{ or } 1^{-3}_{\pm3})2^{\cdots}_{\mathrm{II}}$	yes	$E_{\mathrm{ex}} \cong (1^{+1}_{\pm1} \text{ or } 1^{-1}_{\pm3})2^{\mathrm{even}}_{0}$	$(E_{\mathrm{ex}})^{\mathrm{char}}$
3b.	$1^{\mathrm{odd}}_{\cdots}2^{\cdots}_{\mathrm{II}}$	yes	none	
4.	$1^{\pm d}_{t=2 \text{ or } -2}2^{\cdots}_{\mathrm{II}}$	yes	$E_{\mathrm{even}} \cong 1^{+(d-2)}_{\mathrm{II}}2^{\mathrm{even}}_{\pm t}$	$(E_{\mathrm{even}})^{\mathrm{char}}$
5a.	1^{+4}_{0}	yes	$E_{\mathrm{even}} \cong 1^{+2}_{\mathrm{II}}2^{+2}_{\mathrm{II}}$	$(E_{\mathrm{even}})^{\mathrm{ex}}$
			$E_{\mathrm{ex}} \cong 2^{+4}_{0}$	$(E_{\mathrm{ex}})^{\mathrm{ex}}$
5b.	1^{even}_{0}	yes	$E_{\mathrm{even}} \cong 1^{\cdots}_{\mathrm{II}}2^{+2}_{\mathrm{II}}$	$(E_{\mathrm{even}})^{\mathrm{ex}}$
5c.	$1^{\mathrm{even}}_{0 \text{ or } 4}2^{\cdots}_{\mathrm{II}}$	no	$E_{\mathrm{even}} \cong 1^{\cdots}_{\mathrm{II}}2^{\cdots}_{\mathrm{II}}$	
6a.	$1^{\pm1}_{t}2^{+\mathrm{even}}_{u}$, $\pm = (\frac{t+u}{2})$	yes	$E^{\mathrm{char}} \cong 1^{\pm3}_{t+u}2^{\cdots}_{\mathrm{II}}$	$(E^{\mathrm{char}})_{\mathrm{ex}}$
6b.	$1^{\mathrm{odd}}_{\mathrm{I}}2^{\mathrm{even}}_{\mathrm{I}}$	no	$E^{\mathrm{char}} \cong 1^{\mathrm{odd}}_{\mathrm{I}}2^{\cdots}_{\mathrm{II}}$	
7a.	$1^{\mathrm{even}}_{t}2^{\mathrm{even}}_{u}$, $t+u=\pm2$	no	$E^{\mathrm{char}} \cong 1^{\mathrm{even}}_{\mathrm{I}}2^{\cdots}_{\mathrm{II}}$	
			$E_{\mathrm{even}} \cong 1^{\cdots}_{\mathrm{II}}2^{\mathrm{even}}_{\mathrm{I}}$	
7b.	$1^{\mathrm{even}}_{t}2^{\mathrm{even}}_{u}$, $t+u \neq \pm2$	no	$E^{\mathrm{char}}_{\mathrm{even}} \cong 1^{\cdots}_{\mathrm{II}}2^{\cdots}_{\mathrm{II}}$	
	obtained from above by rescaled duality:			
3a′.	$1^{\cdots}_{\mathrm{II}}(2^{+3}_{\pm1} \text{ or } 2^{-3}_{\pm3})$	yes	$E^{\mathrm{ex}} \cong 1^{\mathrm{even}}_{0}(2^{+1}_{\pm1} \text{ or } 2^{-1}_{\pm3})$	$(E^{\mathrm{ex}})_{\mathrm{even}}$
3b′.	$1^{\cdots}_{\mathrm{II}}2^{\mathrm{odd}}_{\cdots}$	yes	none	
4′.	$1^{\cdots}_{\mathrm{II}}2^{\pm d}_{t=2 \text{ or } -2}$	yes	$E^{\mathrm{char}} \cong 1^{\mathrm{even}}_{\pm t}2^{+(d-2)}_{\mathrm{II}}$	$(E^{\mathrm{char}})_{\mathrm{even}}$
5a′.	2^{+4}_{0}	yes	$E^{\mathrm{char}} \cong 1^{+2}_{\mathrm{II}}2^{+2}_{\mathrm{II}}$	$(E^{\mathrm{char}})_{\mathrm{ex}}$
			$E^{\mathrm{ex}} \cong 1^{+4}_{0}$	$(E^{\mathrm{ex}})_{\mathrm{ex}}$
5b′.	2^{even}_{0}	yes	$E^{\mathrm{char}} \cong 1^{+2}_{\mathrm{II}}2^{\cdots}_{\mathrm{II}}$	$(E^{\mathrm{char}})_{\mathrm{ex}}$
5c′.	$1^{\cdots}_{\mathrm{II}}2^{\mathrm{even}}_{0 \text{ or } 4}$	no	$E^{\mathrm{char}} \cong 1^{\cdots}_{\mathrm{II}}2^{\cdots}_{\mathrm{II}}$	
6a′.	$1^{+\mathrm{even}}_{u}2^{\pm1}_{t}$, $\pm = (\frac{u+t}{2})$	yes	$E_{\mathrm{even}} \cong 1^{\cdots}_{\mathrm{II}}2^{\pm3}_{u+t}$	$(E_{\mathrm{even}})^{\mathrm{ex}}$
6b′.	$1^{\mathrm{even}}_{\mathrm{I}}2^{\mathrm{odd}}_{\mathrm{I}}$	no	$E_{\mathrm{even}} \cong 1^{\cdots}_{\mathrm{II}}2^{\mathrm{odd}}_{\mathrm{I}}$	

TABLE 4.1. For each elementary 2-adic lattice E, this says whether $\mathrm{O}(E)$ is maximal, and provides additional information. See theorem 4.6.

three elementary lattices, one of each type. Temporarily using the symbols to refer to these lattices rather than their isometry classes,

$$1_{\mathrm{II}}^{+2}2_{\mathrm{II}}^{+2} = (1_0^{+4})_{\mathrm{even}} = (2_0^{+4})^{\mathrm{char}}$$
$$1_0^{+4} = (1_{\mathrm{II}}^{+2}2_{\mathrm{II}}^{+2})^{\mathrm{ex}} = (2_0^{+4})^{\mathrm{ex}} \qquad 2_0^{+4} = (1_{\mathrm{II}}^{+2}2_{\mathrm{II}}^{+2})_{\mathrm{ex}} = (1_0^{+4})_{\mathrm{ex}}$$

Proof of theorems 1.1 and 1.2, given theorem 4.6. We proved the odd p case of theorem 1.1 in theorem 4.2(3). When $p = 2$, the proof amounts to checking that the lattices listed in (1.1) are the same as those for which the 3rd column of table 4.1 says "yes". This is straightforward except for the fact that the lattices $(1_{\pm1}^{+1}\text{ or }1_{\pm3}^{-1})2_0^{\cdots}$, from (1.1), are the same as the lattices $1_t^{\pm1}2_0^{+\mathrm{even}}$ with $\pm = (\frac{t+u}{2})$, from row 6a of the table. This requires some sign walking and oddity fusion.

Theorem 1.2 relies on the 4th column of table 4.1. As mentioned in the remark, the lattices (1.2) in the statement of the theorem are the boxed cases in the table, namely 1, 2a–2d, 3a–3b, 3a′–3b′ and 4. Every maximal subgroup G of $\mathrm{O}(V)$ has the form $\mathrm{O}(L)$ for some elementary lattice L. If L appears in one of the boxed cases in the table, then take $E = L$. Otherwise, by the maximality of $\mathrm{O}(L)$, L appears in one of the rows 5a–5b, 6a, 4′, 5a′–5b′ and 6a′. One can choose an E appearing in a boxed row, from among L's related lattices. This proves that $G = \mathrm{O}(E)$ for some E from (1.2). That there is a unique E for which this holds requires checking the related lattices in the boxed rows. Namely, no such lattice has isometry type appearing in (1.2). Theorem 1.2's assertions about the $\mathrm{O}(E)$-invariant elementary lattices amount to copying the related lattices from the table. □

Proof of theorem 4.6. Lemma 3.5 shows that every elementary lattice adjacent to E either properly contains or is properly contained in E. Those that are also $\mathrm{O}(E)$-invariant are classified in lemmas 4.4 and 4.5:

$$\begin{array}{ll} E^{\mathrm{char}} & \text{when } E \cong 1_{\cdots}^{\cdots}2_{\mathrm{I}}^{\mathrm{even}} \\ E^{\mathrm{ex}} & \text{when } E \cong 1_{\mathrm{II}}^{\cdots}\big(2_{\mathrm{II}}^{+2},\, 2_{\pm1}^{+3},\, 2_{\pm3}^{-3},\, 2_0^{+4} \text{ or } 2_4^{-4}\big) \\ E_{\mathrm{even}} & \text{when } E \cong 1_{\mathrm{I}}^{\mathrm{even}}2_{\cdots}^{\cdots} \\ E_{\mathrm{ex}} & \text{when } E \cong \big(1_{\mathrm{II}}^{+2},\, 1_{\pm1}^{+3},\, 1_{\pm3}^{-3},\, 1_0^{+4} \text{ or } 1_4^{-4}\big)2_{\mathrm{II}}^{\cdots} \end{array} \tag{4.9}$$

We start with a copy of table 4.1 that is blank except for the "E" column. We begin by filling in an auxiliary column omitted from the printed table. It uses (4.9) to list the $\mathrm{O}(E)$-invariant elementary lattices adjacent to any elementary lattice E, and then (4.3), (4.5), (4.6), (4.7) and (4.8) to work out their isometry types. We work out several special cases as examples:

Cases 7a–7b: The difference between these cases only emerges at the very end of the proof, so we treat them together; suppose $E \cong$

$1_{\mathrm{I}}^{\mathrm{even}}2_{\mathrm{I}}^{\mathrm{even}}$. By (4.9), the $\mathrm{O}(E)$-invariant neighbors of E are E^{char} and E_{even}. Referring to (4.7) and (4.3) respectively, they have the forms $1_{\mathrm{I}}^{\mathrm{even}}2_{\mathrm{II}}^{\cdots}$ and $1_{\mathrm{II}}^{\cdots}2_{\mathrm{I}}^{\mathrm{even}}$. So the auxiliary column reads "$E^{\mathrm{char}} \cong 1_{\mathrm{I}}^{\mathrm{even}}2_{\mathrm{II}}^{\cdots}$ and $E_{\mathrm{even}} \cong 1_{\mathrm{II}}^{\cdots}2_{\mathrm{I}}^{\mathrm{even}}$."

Case 5c: We are assuming $E \cong 1_{0\,\mathrm{or}\,4}^{\mathrm{even}}2_{\mathrm{II}}^{\cdots}$ but $E \ncong 1_0^{\cdots}$. By (4.9), the $\mathrm{O}(E)$-invariant neighbors of E are E_{even} and E_{ex}, the latter only appearing in the special case $E \cong (1_0^{+4}\,\mathrm{or}\,1_4^{-4})2_{\mathrm{II}}^{\cdots}$. By (4.3), $E_{\mathrm{even}} \cong 1_{\mathrm{II}}^{\cdots}2_{\mathrm{II}}^{\cdots}$. In the special case, (4.5) gives $E_{\mathrm{ex}} \cong (2_0^{+4}\,\mathrm{or}\,2_0^{-4})2_{\mathrm{II}}^{\cdots} \cong 2_0^{\cdots}$. So the auxiliary column reads "$E_{\mathrm{even}} \cong 1_{\mathrm{II}}^{\cdots}2_{\mathrm{II}}^{\cdots}$ and sometimes $E_{\mathrm{ex}} \cong 2_0^{\cdots}$".

Case 5c′ is obtained from this by duality. All other cases are straightforward. It develops that the contents of the auxiliary column are the same as what we will eventually record under "related lattices", except in cases 5c, 5c′ and 7b. So, to read the auxiliary column, the reader may refer to the above examples (in these cases), or to the data printed under "related lattices" (otherwise). We have not yet verified any of the theorem's claims about the related lattices.

First we use this information to justify the no's in the "$\mathrm{O}(E)$ maximal?" column. We will treat cases 5c, 6b, 7a and 7b directly, simplest first. The remaining cases 5c′ and 6b′ follow by rescaled duality.

Case 6b: We are assuming E has the form $1_{\mathrm{I}}^{\mathrm{odd}}2_{\mathrm{I}}^{\mathrm{even}}$, but not the form $1_t^{\pm 1}2_u^{+\mathrm{even}}$ with $\pm = (\frac{t+u}{2})$. In this case, the auxiliary column (recorded under "related lattices") informs us that $E^{\mathrm{char}} \cong 1_{\mathrm{I}}^{\mathrm{odd}}2_{\mathrm{II}}^{\cdots}$ is the only $\mathrm{O}(E)$-invariant neighbor of E. Using whichever of cases 3a and 3b applies to E^{char}, we can read off the $\mathrm{O}(E^{\mathrm{char}})$-invariant neighbors of E^{char} from that row's auxiliary column (again, recorded under "related lattices"). Only in case 3a does such a neighbor exist, when it is unique, namely $(E^{\mathrm{char}})_{\mathrm{ex}} \cong (1_{\pm 1}^{+1}\,\mathrm{or}\,1_{\pm 3}^{-1})2_0^{\mathrm{even}}$. If the sign on the second constituent is $+$, then $(E^{\mathrm{char}})_{\mathrm{ex}} \ncong E$ by assumption on E. If the sign is $-$, then sign walking yields $(E^{\mathrm{char}})_{\mathrm{ex}} \cong (1_{\mp 3}^{-1}\,\mathrm{or}\,1_{\mp 1}^{+1})2_0^{+\mathrm{even}}$, so again $(E^{\mathrm{char}})_{\mathrm{ex}} \ncong E$. It follows that $\mathrm{O}(E^{\mathrm{char}})$ preserves no lattice isometric to E. In particular, it strictly contains $\mathrm{O}(E)$.

Case 5c: We are assuming $E \cong 1_{0\,\mathrm{or}\,4}^{\mathrm{even}}2_{\mathrm{II}}^{\cdots}$ but $E \ncong 1_0^{\cdots}$. The auxiliary column is worked out above: $E_{\mathrm{even}} \cong 1_{\mathrm{II}}^{\cdots}2_{\mathrm{II}}^{\cdots}$ and sometimes $E_{\mathrm{ex}} \cong 2_0^{\cdots}$. We claim $\mathrm{O}(E_{\mathrm{even}})$ does not preserve E. To prove this, we consult whichever of 2a–2d applies to E_{even}. In these cases, the auxiliary column, although not printed, is identical to the related lattices column. This informs us that every $\mathrm{O}(E_{\mathrm{even}})$-invariant neighbor of E_{even} has one of the forms $1_0^{\cdots}$ or $2_0^{\cdots}$. Since E has neither of these forms, it is not $\mathrm{O}(E_{\mathrm{even}})$-invariant. Therefore $\mathrm{O}(E)$ is not maximal. (Although not needed, similar reasoning also shows $\mathrm{O}(E) \subset \mathrm{O}(E_{\mathrm{ex}})$ when E_{ex} is present.)

Cases 7a and 7b: We are assuming $E \cong 1_{\mathrm{I}}^{\mathrm{even}} 2_{\mathrm{I}}^{\mathrm{even}}$; its auxiliary column is worked out above: $E^{\mathrm{char}} \cong 1_{\mathrm{I}}^{\mathrm{even}} 2_{\mathrm{II}}^{\cdots}$ and $E_{\mathrm{even}} \cong 1_{\mathrm{II}}^{\cdots} 2_{\mathrm{I}}^{\mathrm{even}}$. One of cases 4 or 5a–5c applies to E^{char}. If case 5c applies, then we just saw that $\mathrm{O}(E^{\mathrm{char}})$ is not maximal, so $\mathrm{O}(E)$ cannot be either. In the other cases, the $\mathrm{O}(E^{\mathrm{char}})$-invariant neighbors of E^{char} can be read from the related lattices column. No such neighbor has the form $1_{\mathrm{I}}^{\mathrm{even}} 2_{\mathrm{I}}^{\mathrm{even}}$, so $\mathrm{O}(E^{\mathrm{char}})$ does not preserve E, so $\mathrm{O}(E)$ is not maximal. Similar reasoning also shows $\mathrm{O}(E) \subset \mathrm{O}(E_{\mathrm{even}})$.

We have justified all the no's. Next suppose that the auxiliary column reads "none", ie E has no $\mathrm{O}(E)$-invariant elementary neighbors. Then lemma 3.4 shows that w_E is the only $\mathrm{O}(E)$-invariant point of $\mathcal{E}$. So $\mathrm{O}(E)$ is the full $\mathrm{O}(V)$-stabilizer of every $\mathrm{O}(E)$-invariant vertex, hence maximal among lattice stabilizers. This accounts for four "yes" entries in the table.

A similar but more complicated argument justifies the remaining "yes" entries. Except in cases 2a, 5a and 5a′, E has exactly one $\mathrm{O}(E)$-invariant elementary neighbor L. One can check that E may be recovered from L by the operation in the last column. It follows that $\mathrm{O}(E) = \mathrm{O}(L)$. (Note: from data so far compiled, one can read off that $\mathrm{O}(L)$ leaves invariant a lattice isometric to E. But we need the stronger result that $\mathrm{O}(L)$ leaves E itself invariant. We checked each case, using the definitions of the operations $_{\mathrm{even}}$, $_{\mathrm{ex}}$, $^{\mathrm{char}}$ and $^{\mathrm{ex}}$.) Furthermore, in these cases, one can read from the table that L has a unique $\mathrm{O}(L)$-invariant neighbor. This can only be E, and it follows that the fixed-point set of $\mathrm{O}(E)$ in $\mathcal{E}$ is the segment $\overline{w_E w_L}$. Since $\mathrm{O}(E)$ is the full stabilizer of each endpoint of this segment, it is maximal.

The idea is the same in the special cases 2a, 5a and 5a′. Consider the three isometry classes $1_{\mathrm{II}}^{+2} 2_{\mathrm{II}}^{+2}$, 1_0^{+4} and 2_0^{+4}. If E represents one of these classes, then it has exactly two $\mathrm{O}(E)$-invariant elementary neighbors, which represent the other two classes. Furthermore, E can be recovered from either of them via the operations listed in the last column. It follows that all three lattices have the same isometry group, which we call G to avoid breaking the symmetry. One also checks that these neighbors of E are neighbors of each other, so that they and E form the vertices of a 2-simplex of $\mathcal{E}$. The two G-invariant neighbors, of any vertex of this triangle, can only be the other two vertices. It follows that this triangle is the entire fixed-point set of G. Since G is the full stabilizer of each vertex of it, G is maximal.

To finish the proof in each case with $\mathrm{O}(E)$ maximal, we must find all $\mathrm{O}(E)$-invariant elementary lattices in V, record them under "related lattices", and record how to recover E from each of them. The

$\mathrm{O}(E)$-invariant elementary lattices are the same as the $\mathrm{O}(E)$-invariant vertices of $\mathcal{E}$, and we just worked out $\mathrm{O}(E)$'s fixed-point set. In particular, the $\mathrm{O}(E)$-invariant vertices are E and the $\mathrm{O}(E)$-invariant neighbors of E. The latter are exactly the contents of the auxiliary column, which explains why we copy its contents to "related lattices". While proving maximality, we already checked that the last column describes how to recover E from any $\mathrm{O}(E)$-invariant neighbor.

To finish the proof in each case with $\mathrm{O}(E)$ non-maximal, we must record under "related lattices" at least one $\mathrm{O}(E)$-invariant lattice whose isometry group is maximal. By rescaled duality, it is enough to do this in cases 5c, 6b and 7a–7b. We recall the $\mathrm{O}(E)$-invariant neighbors of E, and in cases 5c and 6b we explain why each such neighbor has maximal orthogonal group:

Case	$\mathrm{O}(E)$-invariant neighbor L	reasoning about $\mathrm{O}(L)$
5c	$E_{\mathrm{even}} \cong 1_{\mathrm{II}}^{\cdots}2_{\mathrm{II}}^{\cdots}$	maximal by one of 2a–2d
	$E_{\mathrm{ex}} \cong 2_0^{\cdots}$ (if present)	maximal by one of 5a′–5b′
6b	$E^{\mathrm{char}} \cong 1_{\mathrm{I}}^{\mathrm{odd}}2_{\mathrm{II}}^{\cdots}$	maximal by one of 3a–3b
7a–7b	$E^{\mathrm{char}} \cong 1_{\mathrm{I}}^{\mathrm{even}}2_{\mathrm{II}}^{\cdots}$	see below
	$E_{\mathrm{even}} \cong 1_{\mathrm{II}}^{\cdots}2_{\mathrm{I}}^{\mathrm{even}}$	

This justifies their "related lattices" in table 4.1. In cases 7a–7b we are assuming $E \cong 1_t^{\mathrm{even}}2_u^{\mathrm{even}}$. Because the dimensions of the constituents are even, so are t and u. By (4.3),

$$E_{\mathrm{even}} \cong \left\{\begin{matrix} 1_{\mathrm{II}}^{\cdots} \oplus 2_{\pm t}^{\cdots 2} \oplus 2_u^{\cdots} & \text{if } t = 2 \text{ or } -2 \\ 1_{\mathrm{II}}^{\cdots} \oplus 2_{\mathrm{II}}^{\cdots 2} \oplus 2_u^{\cdots} & \text{if } t = 0 \text{ or } 4 \end{matrix}\right\} \cong 1_{\mathrm{II}}^{\cdots}2_v^{\cdots}$$

where $v \equiv t + u \bmod 4$. Similarly, $E^{\mathrm{char}} \cong 1_w^{\cdots}2_{\mathrm{II}}^{\cdots}$ with $w \equiv t + u \bmod 4$. In case 7a we assumed $t + u \in \{\pm 2\}$, so $\mathrm{O}(E^{\mathrm{char}})$ and $\mathrm{O}(E_{\mathrm{even}})$ are maximal by cases 4 and 4′. This justifies the "related lattices" in row 7a. In case 7b we have $t + u \in \{0, 4\}$, so E^{char} resp. E_{even} falls into one of the cases 5a–5c resp. 5a′–5c′. It can happen that $\mathrm{O}(E^{\mathrm{char}})$ and/or $\mathrm{O}(E_{\mathrm{even}})$ is maximal, eg if $E \cong 1_0^{+2}2_0^{+2}$. But neither is maximal if 5c and 5c′ apply. On the other hand, in case 7b one can check that $(E^{\mathrm{char}})_{\mathrm{even}}$ and $(E_{\mathrm{even}})^{\mathrm{char}}$ are the same lattice, which we take as the definition of $E_{\mathrm{even}}^{\mathrm{char}}$. This has the form $1_{\mathrm{II}}^{\cdots}2_{\mathrm{II}}^{\cdots}$, whose orthogonal group is maximal by one of the cases 2a–2d. □

5. The global case

Our goal is to prove theorem 1.4. Philosophically, it is a direct application of strong approximation. But for some elementary lattices E,

$\Theta(E)$ is not large enough. In those cases we find a reflection in $\mathrm{O}(E)$. We recommend that the reader skip to the "generic case" in the proof of lemma 5.4, and then to lemma 5.5 and the proof of theorem 1.4. Lemmas 5.1–5.3, and most of the proof of lemma 5.4, adapt these simple ideas to some troublesome cases.

Lemma 5.1. *Each indefinite elementary $\mathbb{Z}$-lattice E of rank $n \geq 3$ is the unique lattice in its genus.*

Proof. We use Thm. 19 from Sec. 9.7 of [5, Ch. 15], or equivalently the corollary to 3.7 of [4, Ch. 11]. It says that if an indefinite lattice E is not unique in its genus, then there is a prime p for which E_p has a Jordan decomposition with every constituent being 1-dimensional. But then E would not be elementary. □

Lemma 5.2. *Suppose L is a $\mathbb{Z}$-lattice and $N \in \mathbb{Z}$. If L_p has a root of norm N for every place p, then so does some lattice in the genus of L. In particular, if L is unique in its genus, then L has a norm N root.*

Proof. This is a minor variation on the standard argument that if all L_p represent N, then some lattice in L's genus does too. Let r_p be a norm N root in L_p, and let $K_p = r_p^\perp \subseteq L_p$. Because $L \otimes \mathbb{Q}$ is a quadratic space over $\mathbb{Q}$, its p-adic invariants satisfy the "product rule"

$$\sigma_2(L) \equiv \sigma_\infty(L) + \sum_{\mathrm{odd}\,p} \varepsilon_p(L) \qquad \mathrm{mod}\ 8$$

and similarly with $\langle N \rangle$ in place of L. (See [5, Sec. 7.7 of Ch. 15] for this formulation of the product rule, including the invariants σ_p and ε_p. See [4, Lem. 1.1 of Ch. 6] for the classical formulation.) Because the invariants are additive under direct sum, subtraction yields $\sigma_2(K_2) \equiv \sigma_\infty(K_\infty) + \sum_{\mathrm{odd}\,p} \varepsilon_p(K_p)$ mod 8. This guarantees the existence of a $\mathbb{Z}$-lattice K whose localizations are the K_p. (See [5, Sec. 7.7 of Ch. 15]. In the classical formulation, combine Thm. 1.3 of Ch. 6 and Thm. 1.2 of Ch. 9 from [4]).

Write r for a generator of the $\mathbb{Z}$-lattice $\langle N \rangle$. We will enlarge $K \oplus \langle r \rangle$ to get a lattice in the genus of L. It is already isomorphic to L at every place except 2: because $\frac{1}{2} \in \mathbb{Z}_p^\times$, the root r_p generates a summand of L_p, so $L_p = K_p \oplus \langle r_p \rangle$. If r_2 generates a summand of L_2, then the same holds at 2, so $K \oplus \langle r \rangle$ lies in the genus of L. Otherwise, $K_2 \oplus \langle r_2 \rangle$ must have index 2 in L_2, because r_2 being a root of L_2 gives $L_2 \cdot r_2 \subseteq \frac{1}{2}\mathbb{Z}_2 r_2^2$. Define J as the corresponding index 2 enlargement of $K \oplus \langle r \rangle$. By construction, $J_2 \cong L_2$. An index 2 enlargement does not change any other localization of $K \oplus \langle r \rangle$. Therefore J lies in the genus of L, as desired.

To finish the proof, we show that r is a root of J. First, because J contains $K \oplus \langle r \rangle$ with index ≤ 2, the projection of J to $\mathbb{Q}r$ lies in $\frac{1}{2}\mathbb{Z}r$. Together with the primitivity of r, which follows from that of r_2, this shows that r is a root. □

Lemma 5.3. *If E_2 is an elementary 2-adic lattice of rank ≥ 3, then there exists a power of 2, say N_2, such that E_2 has roots of all norms $N_2 \cdot u$ with $u \in \mathbb{Z}_2^\times$.*

Proof. One checks directly that 1_{II}^{+2} and 1_{II}^{-2} have roots of all norms $2 \cdot (\text{odd})$. Also, if a, b, c are odd, then $\langle a, b, 2c \rangle$ has roots of every odd norm, for example $(1, 0 \,\text{or}\, 2, 0 \,\text{or}\, 1)$. These roots are also roots of $\langle a, b, 2c \rangle' \cong \langle 2a, 2b, c \rangle$, with doubled norms. From these examples it follows that if E admits a summand

$$1_{\mathrm{II}}^{\pm 2}, \quad 2_{\mathrm{II}}^{\pm 2}, \quad \langle a, b, 2c \rangle, \quad \text{or } \langle 2a, 2b, c \rangle$$

then we may take $N_2 = 2$, 4, 1 or 2 respectively. One can check that every elementary lattice of rank ≥ 3 admits such a summand. □

If V is a nondegenerate quadratic space over $\mathbb{Q}$, then for each prime p, section 3 defines the complex $\mathcal{E}$ of elementary lattices in V_p. To avoid confusion we will write $\mathcal{E}_p$. If E is an elementary lattice in V, and p is understood, then we will abbreviate the vertex $w_{E_p} \in \mathcal{E}_p$ to w_E.

Lemma 5.4. *Suppose V is an indefinite quadratic space over $\mathbb{Q}$ of dimension ≥ 3 and E is an elementary lattice in V. If p is a prime and $w_E \in \mathcal{E}_p$ has an $\mathrm{O}(E)$-invariant neighbor, then $p = 2$ and that neighbor is also $\mathrm{O}(E_2)$-invariant.*

Proof. Lemma 3.5 shows that the neighbors of w_E correspond to elementary lattices properly containing or properly contained in E_p. So it is enough to prove that if such a superlattice resp. sublattice is $\mathrm{O}(E)$-invariant, then it is $\mathrm{O}(E_p)$-invariant and $p = 2$. We give details for the sublattice case, and indicate the minor changes needed for superlattices.

We begin with "the generic case": if $p = 2$ then we assume E_2 does not appear in lemma 4.4(2), while if p is odd then we assume that the unimodular constituent of E_p is not $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The proof is easy and does not use anything else from this section. An $\mathrm{O}(E)$-invariant elementary proper sublattice of E_p is obviously $\Theta(E)$-invariant, hence $\Theta(E_p)$-invariant by strong approximation. If p is odd, then no such sublattice exists by theorem 4.2(1). If $p = 2$ then lemma 4.4 shows that the only possibility for such a sublattice is $(E_2)_{\text{even}}$, which is obviously $\mathrm{O}(E_2)$-invariant.

Now for the non-generic case. First suppose $p = 2$, so E_2 appears in lemma 4.4(2), ie $E_2 \cong (1^{+2}_{\mathrm{II}}, 1^{+3}_{\pm 1}, 1^{-3}_{\pm 3}, 1^{+4}_0 \text{ or } 1^{-4}_4)2^{\cdots}_{\mathrm{II}}$. We will show that E has a root whose norm N is twice an odd number. Assuming this for the moment, we write R for this root's reflection, and use the argument of the previous paragraph with $\langle \Theta(E), R\rangle$ in place of $\Theta(E)$, and lemma 4.4(2) in place of the other parts of that lemma. This shows: the only $\mathrm{O}(E)$-invariant elementary proper sublattices of E_2 are $(E_2)_{\mathrm{ex}}$ and maybe $(E_2)_{\mathrm{even}}$. These are obviously $\mathrm{O}(E_2)$-invariant.

To construct the required root, we first define $N = 2\prod_q N_q$ with q varying over the odd primes, where

$$N_q = \begin{cases} 1 & \text{if } E_q\text{'s unimodular constituent has rank} \geq 2 \\ q & \text{if } E_q\text{'s non-unimodular constituent has rank} \geq 2 \end{cases} \tag{5.1}$$

(At least one case applies; if both do then choose either.) By lemmas 5.1 and 5.2, it is enough to show that every completion of E has a norm N root. E_2 does because it admits a summand $1^{\pm 2}_{\mathrm{II}}$, which has roots of all norms $2 \cdot (\text{odd})$. Now fix an odd prime q. We will use the fact that every nondegenerate $\mathbb{F}_q$ inner product space of rank ≥ 2 is universal: it represents every nonzero element of $\mathbb{F}_q$. If $N_q = 1$ then we apply this to $U/qU = E_q/qE_q^*$, where U is the unimodular constituent. It follows that U represents every element of $\mathbb{Z}_q^\times$, hence has a norm N vector. Since $q \nmid N$, this vector generates a summand of U and hence is a root of E_q. On the other hand, if $N_q = q$, then we apply a scaled version of this argument to $\frac{1}{q}V/V = \Delta(E_q)$, where V is the non-unimodular constituent. For $p = 2$, this finishes the proof that E has a norm N vector. We already showed that the $\mathrm{O}(E_2)$-invariance of every $\mathrm{O}(E)$-invariant neighbor of w_E follows.

When p is odd in the non-generic case, the unimodular constituent of E_p is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The proof is modeled on the $p = 2$ case. We define $N = N_2 \prod_q N_q$, where the N_q are as before except we specify $N_p = 1$, and N_2 comes from applying lemma 5.3 to E_2. In particular, $p \nmid N$. E has a norm N root by the same argument. The same argument concerning $\langle \Theta(E), R\rangle$, but using theorem 4.2(1) in place of lemma 4.4(2), shows the absence of $\mathrm{O}(E)$-invariant elementary proper sublattices of E_p.

Finally, we indicate how to adapt the argument to apply to $\mathrm{O}(E)$-invariant elementary superlattices of E. If $p = 2$, then the generic case is that E_2 does not appear in lemma 4.5(2). If p is odd, then the generic case is that E's non-unimodular constituent is not $\left(\begin{smallmatrix} 0 & p \\ p & 0 \end{smallmatrix}\right)$. In either of these cases the proof goes through as before, except that we quote lemma 4.5 rather than 4.4 when $p = 2$, and quote theorem 4.2(2) rather than 4.2(1) when p is odd. In the non-generic case when $p = 2$, the argument is the same except that we take $N = 4\prod N_q$, use the presence of a summand $2_{\mathrm{II}}^{\pm 2}$ in place of a $1_{\mathrm{II}}^{\pm 2}$ summand, and quote lemma 4.5(2) rather than 4.4(2). In the non-generic case when p is odd, take $N_p = p$ rather than $N_p = 1$, work with the non-unimodular constituent rather than the unimodular one, and quote Theorem 4.2(2) rather than 4.2(1). □

Lemma 5.5. *Suppose E and F are elementary lattices in the same indefinite rational quadratic space of dimension ≥ 3. Suppose their completions coincide at all odd primes, and represent neighboring vertices of $\mathcal{E}_2$ at the prime 2. Then*

$$\mathrm{O}(E) \subseteq \mathrm{O}(F) \iff \mathrm{O}(E_2) \subseteq \mathrm{O}(F_2). \tag{5.2}$$

Furthermore, the same holds with $\subseteq$ replaced by $\supseteq$ or by $=$.

Proof. We only prove "⇐", because "⇒" is part of the previous lemma. So suppose $\mathrm{O}(E_2)$ preserves F_2. As a subgroup of $\mathrm{O}(E_2)$, $\mathrm{O}(E)$ does too. So it preserves the unique $\mathbb{Z}$-lattice in V whose 2-adic completion is F_2 and whose other completions coincide with those of E. That is, $\mathrm{O}(E)$ preserves F.

The statement (5.2) with $\subseteq$ replaced by $\supseteq$ follows from symmetry in E and F. The statement with $\subseteq$ replaced by $=$ is a formal consequence of the $\subseteq$ and $\supseteq$ statements. □

Proof of theorem 1.4. By lemma 5.4, every $\mathrm{O}(L)$-invariant elementary lattice in V coincides with L at all odd primes. Therefore completion-at-2 is a bijection from the set of such lattices to the set of $\mathrm{O}(L)$-invariant 2-adic elementary lattices in V_2. So, whenever we have a 2-adic elementary lattice in mind, the "corresponding $\mathbb{Z}$-lattice" will mean the one whose 2-adic completion is that lattice, and whose other completions coincide with those of L.

First we suppose $\mathrm{O}(L_2)$ is not maximal. In this case, the theorem's only claim is that $\mathrm{O}(L)$ is also non-maximal. By non-maximality, $\mathrm{O}(L_2)$ appears in one of the rows 5c, 6b, 7a–7b, 5c′ and 6b′ of table 4.1. We claim there exists a neighboring 2-adic elementary lattice M_2 with strictly larger orthogonal group. We recall that the "related lattices" column of table 4.1, in the case that $\mathrm{O}(L_2)$ is non-maximal, lists one or more $\mathrm{O}(L_2)$-invariant 2-adic lattices with maximal orthogonal group. Except for case 7b, these are neighbors of L_2, so we may take M_2 to be one of them. In case 7b, one can take $M_2 = (L_2)_{\mathrm{even}}$ or $(L_2)^{\mathrm{char}}$; the proof of theorem 4.6 showed that both their orthogonal groups contain $\mathrm{O}(L_2)$ properly. Take M to be the corresponding $\mathbb{Z}$-lattice. From $\mathrm{O}(L_2) \subset \mathrm{O}(M_2)$, lemma 5.4 gives $\mathrm{O}(L) \subset \mathrm{O}(M)$. In particular, $\mathrm{O}(L)$ is not maximal.

Now we assume $\mathrm{O}(L_2)$ is maximal. We consider the fixed points of $\mathrm{O}(L)$ in $\mathcal{E}_2$, and quote lemma 3.4 to deduce the following. Every $\mathrm{O}(L)$-invariant elementary 2-adic lattice in V_2 is the final term in a sequence $L_2^{(0)}, \ldots, L_2^{(k)}$ of such lattices, with $L_2^{(0)} = L_2$ and each consecutive pair being neighbors. We write $L^{(0)}, \ldots, L^{(k)}$ for the corresponding $\mathbb{Z}$-lattices, and prove by induction that $\mathrm{O}(L^{(i)}) = \mathrm{O}(L)$ and $\mathrm{O}(L_2^{(i)}) = \mathrm{O}(L_2)$. The base case is trivial, so suppose $i > 0$. By construction, $L_2^{(i)}$ is invariant under $\mathrm{O}(L)$, hence under $\mathrm{O}(L^{(i-1)})$ by induction. From $\mathrm{O}(L^{(i-1)}) \subseteq \mathrm{O}(L_2^{(i)})$, lemma 5.5 deduces $\mathrm{O}(L_2^{(i-1)}) \subseteq \mathrm{O}(L_2^{(i)})$. The left side is maximal, because it coincides with $\mathrm{O}(L_2)$ by induction. So the inclusion is an equality. From $\mathrm{O}(L_2^{(i)}) = \mathrm{O}(L_2^{(i-1)})$ follow both inductive claims. First, $\mathrm{O}(L_2^{(i)}) = \mathrm{O}(L_2)$ by $\mathrm{O}(L_2^{(i-1)}) = \mathrm{O}(L_2)$. Second, $\mathrm{O}(L^{(i)}) = \mathrm{O}(L^{(i-1)}) = \mathrm{O}(L)$ by lemma 5.5 and $\mathrm{O}(L^{(i-1)}) = \mathrm{O}(L)$.

The previous paragraph implies two things. First, every $\mathrm{O}(L)$-invariant elementary lattice in V has the same orthogonal group as L. So $\mathrm{O}(L)$ is maximal. Second, the $\mathrm{O}(L)$- and $\mathrm{O}(L_2)$-invariant elementary lattices in V_2 coincide. The rest of the theorem follows by applying theorem 1.2 to L_2. □

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