VARIATIONS ON GLAUBERMAN'S ZJ THEOREM

DANIEL ALLCOCK

Abstract. We give a new proof of Glauberman's ZJ Theorem, in a form that clarifies the choices involved and offers more choices than classical treatments. In particular, we introduce two new ZJ-type subgroups of a p-group S, that contain $ZJ_r(S)$ and $ZJ_o(S)$ respectively and are often strictly larger.

Glauberman's ZJ Theorem is a basic technical tool in finite group theory. It plays a major role in the classification of simple groups having abelian or dihedral Sylow 2-subgroups. There are several versions of the theorem, depending on how one defines the Thompson subgroup. We develop the theorem in a way that clarifies the choices involved, and offers more choices than classical treatments.

Writing S for a p-group, the following are new. First, we give an "axiomatic" version of the ZJ Theorem, theorem 1.1. Second, for $p > 2$ we construct ZJ-type groups $ZJ_{\mathrm{lex}}(S)$ and $ZJ_{\mathrm{olex}}(S)$, which contain $ZJ_r(S)$ resp. $ZJ_o(S)$ and can easily be larger. Third, we establish the "normalizers grow" property of the Thompson-Glauberman replacement process, and a consequence involving the Glauberman-Solomon group $D^*(S)$; see theorems 3.1(v) and 5.4.

1. Introduction

Suppose p is a prime, S is a p-group, $\mathfrak{Ab}(S)$ is the set of abelian subgroups of S, and $\mathcal{A} \subseteq \mathfrak{Ab}(S)$. We set

$$J_{\mathcal{A}} := \langle A : A \in \mathcal{A} \rangle \qquad I_{\mathcal{A}} := \cap_{A \in \mathcal{A}} A \qquad (I_{\mathcal{A}} = 1 \text{ if } \mathcal{A} = \emptyset).$$

$J_{\mathcal{A}}$ is a sort of generalized Thompson subgroup, and $I_{\mathcal{A}}$ lies in its center. For $P \leq S$ we define $\mathcal{A}|_P$ as $\{A \in \mathcal{A} : A \leq P\}$ and $I_{\mathcal{A}|P}$ as $I_{\mathcal{A}|_P}$. For notation, and the definition of a p-stable action, see section 2.

Theorem 1.1 ("Axiomatic" ZJ Theorem)**.** *Suppose p is a prime, S is a p-group and G is a finite group satisfying*

(a) *S is a Sylow p-subgroup of G.*

Date: July 11, 2020.

2010 *Mathematics Subject Classification.* 20D25 (20D15, 20D20).

Supported by Simons Foundation Collaboration Grant 429818.

(b) $C_G(O_p(G)) \leq O_p(G)$.
(c) *G acts p-stably on every normal p-subgroup of G.*

Then $I_{\mathcal{A}} \trianglelefteq G$ if $\mathcal{A} \subseteq \mathfrak{Ab}(S)$ has the following properties:

invariance (in S, for G): *$\forall P \trianglelefteq S$, $I_{\mathcal{A}|P}$ is $N_G(P)$-invariant.*

replacement (in S): *For every $B \trianglelefteq S$ with class ≤ 2, if there exist members of $\mathcal{A}$ that contain $[B, B]$ but not B, then B normalizes one of them.*

Furthermore, $I_{\mathcal{A}}$ is characteristic in G if it is characteristic in S.

Part of the point of the ZJ Theorem is to specify a subgroup of S which will be characteristic in suitable G, without referring to G. We will say that a subgroup of S has the *Glauberman property* (for S) if it is characteristic in any finite group G satisfying (a)–(c). Replacing $N_G(P)$ with $\mathrm{Aut}(P)$ in the definition of invariance, and quoting the theorem, lets us omit mention of G:

Corollary 1.2 (ZJ Theorem). *Suppose p is a prime, S is a p-group, and $\mathcal{A} \subseteq \mathfrak{Ab}(S)$ satifies* **replacement** *(in S) and also*

full invariance (in S): *$\forall P \trianglelefteq S$, $I_{\mathcal{A}|P}$ is $Aut(P)$-invariant.*

Then $I_{\mathcal{A}}$ has the Glauberman property. □

These results allow $p = 2$, but in this case no $\mathcal{A}$ satisfying the conditions is known. Also, every $\mathcal{A}$ we consider has the following property, much stronger even than full invariance:

completeness (in S): $\mathcal{A}$ contains every subgroup of S that is isomorphic to a member of $\mathcal{A}$.

Examples 1.3. The following subsets of $\mathfrak{Ab}(S)$ are obviously complete in S. The first three are classical and the rest are new. We will abbreviate $I_{\mathcal{A}_{\dots}(S)}$ and $J_{\mathcal{A}_{\dots}(S)}$ to $I_{\dots}(S)$ and $J_{\dots}(S)$

$$\mathcal{A}_o(S) = \{A \in \mathfrak{Ab}(S) : |A| \geq |A'| \text{ for all } A' \in \mathfrak{Ab}(S)\}$$
$$\mathcal{A}_r(S) = \{A \in \mathfrak{Ab}(S) : \mathrm{rank}(A) = \mathrm{rank}(S)\}$$
$$\mathcal{A}_e(S) = \{A \in \mathcal{A}_r(S) : A \text{ is elementary abelian}\}$$
$$\mathcal{A}_{\mathrm{lex}}(S) = \{A \in \mathfrak{Ab}(S) : A \geq_{\mathrm{lex}} A' \text{ for all } A' \in \mathfrak{Ab}(S)\}$$
$$\mathcal{A}_{\mathrm{olex}}(S) = \{A \in \mathcal{A}_o(S) : A \geq_{\mathrm{lex}} A' \text{ for all } A' \in \mathcal{A}_o(S)\}$$
$$\mathcal{A}_{O,E,\zeta}(S) = \{A \in \mathfrak{Ab}(S) : |A| = p^O, \text{ exponent}(A) \leq p^E \text{ and } A \geq_{\mathrm{lex}} \zeta\}$$

In the last case, $O, E \in \mathbb{Z}$ and $\zeta = (\zeta_1, \zeta_2, \dots)$ is a sequence of integers. For sequences $\zeta = (\zeta_1, \zeta_2, \dots)$ and $\zeta' = (\zeta'_1, \zeta'_2, \dots)$, $\zeta \geq_{\mathrm{lex}} \zeta'$ refers to the usual lexicographic order. When a group A appears on one side of $\geq_{\mathrm{lex}}$, the comparison refers to the sequence

$$(\omega_1(A), \omega_2(A), \dots) := \big(|\Omega_1(A)|, |\Omega_2(A)|, \dots\big)$$

If A, A' are abelian groups of the same order then we think of $A >_{\text{lex}} A'$ as "A is closer to being elementary abelian than A' is."

Theorem 1.4. *Suppose p is an odd prime, S is a p-group, and $D \trianglelefteq S$. Then $\mathcal{A}_o(D)$, $\mathcal{A}_r(D)$, $\mathcal{A}_e(D)$, $\mathcal{A}_{\text{lex}}(D)$, $\mathcal{A}_{\text{olex}}(D)$ and $\mathcal{A}_{O,E,\zeta}(D)$ (for any fixed O, E, ζ) have the replacement property in S.*

Corollary 1.5. *Every one of $I_o(S)$, $I_r(S)$, $I_e(S)$, $I_{\text{lex}}(S)$, $I_{\text{olex}}(S)$ and $I_{O,E,\zeta}(S)$ has the Glauberman property for S.* □

Theorem 1.4 is a wrapper around Glauberman's replacement theorem, extended to cover the last three cases (theorem 3.1). Corollary 1.5 contains the classical forms of the ZJ Theorem. Namely: $ZJ_o(S)$ and $\Omega ZJ_e(S)$ have the Glauberman property. This follows from

$$I_o(S) = ZJ_o(S) \qquad I_e(S) = \Omega ZJ_e(S)$$

The first equality uses $I_{\mathcal{A}} = Z(J_{\mathcal{A}}) = C_S(J_{\mathcal{A}})$ when every member of $\mathcal{A}$ is maximal in $\mathfrak{Ab}(S)$ under inclusion (lemma 2.1). The second is similar. $\mathcal{A}_r(S)$ gives nothing new: theorem 5.3 shows

$$I_r(S) = I_e(S) = \Omega ZJ_r(S) = \Omega ZJ_e(S)$$

We chose the new families $\mathcal{A}_{\text{lex}}(S)$ and $\mathcal{A}_{\text{olex}}(S)$ to be "small", so that $J_{\dots}(S)$ would also be "small" and $I_{\dots}(S)$ would be "large". In particular,

$$I_{\text{lex}}(S) = ZJ_{\text{lex}}(S) = C_S(J_{\text{lex}}(S)) \geq ZJ_r(S)$$
$$I_{\text{olex}}(S) = ZJ_{\text{olex}}(S) = C_S(J_{\text{olex}}(S)) \geq ZJ_o(S)$$

The equalities use lemma 2.1. The containments follow from $\mathcal{A}_{\text{lex}}(S) \subseteq \mathcal{A}_r(S)$ and $\mathcal{A}_{\text{olex}}(S) \subseteq \mathcal{A}_o(S)$, and can easily be strict (examples 5.1 and 5.2). The containment $ZJ_{\text{lex}}(S) \geq ZJ_r(S)$ is interesting because no ZJ Theorem is known for ZJ_r (or even expected, to my knowledge). All the members of $\mathcal{A}_{\text{lex}}(S)$ resp. $\mathcal{A}_{\text{olex}}(S)$ are isomorphic to each other.

There is no reason to expect $\mathcal{A}_{O,E,\zeta}(S)$ to be interesting; we include it mainly to give a sense of what is possible using replacement.

Corollary 1.5 uses the $D = S$ case of theorem 1.4. Since one can take D to be any normal subgroup there, this suggests trying to apply theorem 1.1 to some suitable $\mathcal{A} \subseteq \mathfrak{Ab}(D)$ with $D \trianglelefteq S$. In this way we can recover some recent results of Kızmaz. Recall that $D \trianglelefteq S$ is called *strongly closed* (in S, with respect to $G \geq S$), if the only elements of S which are G-conjugate into D are the elements of D. If this holds and $\mathcal{A} \subseteq \mathfrak{Ab}(D)$ is complete (in D), then it is not hard to see that $\mathcal{A}$ satisfies invariance (in S, rel G). In fact strong closure is stronger than necessary for this argument.

Therefore theorem 1.1 implies the following "axiomatic" version of [6, theorem B]. Corollary 1.7 below takes $D = \Omega_i(S)$, and is our analogue of [6, corollary C].

Theorem 1.6. *Suppose p is a prime, S is a p-group, G is a finite group satisfying* (a)–(c) *of theorem 1.1, and $D \trianglelefteq S$ is strongly closed in S with respect to G. Then $I_{\mathcal{A}} \trianglelefteq G$ for any $\mathcal{A} \subseteq \mathfrak{Ab}(D)$ which is complete (in D) and satisfies replacement (in S).* $\square$

Corollary 1.7. *Suppose p is an odd prime, S is a p-group, $i \geq 1$, and $\Omega_i(S)$ has exponent $\leq p^i$ (for example, suppose S has class $< p$). Then all of $ZJ_o\Omega_i(S)$, $\Omega ZJ_e\Omega_i(S)$, $ZJ_{\text{lex}}\Omega_i(S)$, $ZJ_{\text{olex}}\Omega_i(S)$ and $I_{O,E,\zeta}(\Omega_i(S))$ (for any O, E, ζ) have the Glauberman property for S.* $\square$

I am grateful to Bernd Stellmacher and M. Yasir Kızmaz for helpful correspondence.

2. Background and Notation

We mostly follow the conventions of [4]. In particular, all groups considered are finite. Let G be one. If $w, x \in G$, then w^x means $x^{-1}wx$ and $[w, x]$ means $w^{-1}x^{-1}wx$. Brackets nest to the left, so $[x_1, \ldots, x_n]$ means $[[x_1, \ldots, x_{n-1}], x_n]$ when $n > 2$. If some terms in a commutator are groups, then we mean the group generated by the corresponding commutators of elements of those groups.

Suppose p is a prime. If S is a p-group then $\Omega_i(S)$ means the subgroup generated by elements of order $\leq p^i$. When $i = 1$ we often write just $\Omega(S)$. The rank of an abelian group means the size of the smallest set of generators. The rank of a nonabelian group means the maximum of the ranks of its abelian subgroups. We will only use this notion for p-groups. We sometimes suppress parentheses, eg writing $\Omega ZJ_e(S)$ for $\Omega(Z(J_e(S)))$.

The largest normal p-subgroup of G is denoted $O_p(G)$. Now suppose G acts on a p-group P. We define $O_p(G \curvearrowright P) \trianglelefteq G$ as the preimage of $O_p(G/C_G(P))$ under the natural map $G \to G/C_G(P)$. This notation is nonstandard but natural; it can be pronounced "O_p of G's action on P". We say that $x \in G$ acts quadratically if $[P, x, x] = 1$. The action of G on P is called p-stable if every element of G that acts quadratically lies in $O_p(G \curvearrowright P)$. There is a simple "global" condition that guarantees this: that no subquotient of G is isomorphic to $\mathrm{SL}_2(p)$. A proof of this can be extracted from that of [2, lemma 6.3]. One main case of interest is when p is odd and G has abelian or dihedral Sylow 2-subgroups. Having quaternionic Sylow 2-subgroups, $\mathrm{SL}_2(p)$ cannot arise as a subquotient.

We use the following elementary lemma several times.

Lemma 2.1. *Suppose S is a p-group, $\mathcal{A} \subseteq \mathfrak{Ab}(S)$, and every member of $\mathcal{A}$ is maximal in $\mathfrak{Ab}(S)$ under inclusion. Then $I_{\mathcal{A}} = Z(J_{\mathcal{A}}) = C_S(J_{\mathcal{A}})$.*

Proof. The inclusions $I_{\mathcal{A}} \leq Z(J_{\mathcal{A}}) \leq C_S(J_{\mathcal{A}})$ are obvious. Now suppose $x \in C_S(J_{\mathcal{A}})$. For any $A \in \mathcal{A}$, $\langle A, x\rangle$ is abelian, so the maximality of A forces $x \in A$. Letting A vary over $\mathcal{A}$ gives $x \in I_{\mathcal{A}}$. $\square$

3. Replacement

Theorem 3.1 (Glauberman Replacement)**.** *Suppose p is a prime, S is a p-group and $B \trianglelefteq S$. If $p = 2$ then assume B is abelian. Suppose $A \leq S$ is abelian and contains $[B, B]$.*

Then either B normalizes A, or there exists $b \in N_B(N_S(A)) - N_B(A)$. For any such b, $A^* := (A \cap A^b)[A, b] \leq AA^b$ enjoys the properties

(i) $|A^*| = |A|$.
(ii) *A^* is abelian and contains $[B, B]$.*
(iii) *$A^* \cap B$ strictly contains $A \cap B$ and is a proper subgroup of B.*
(iv) *A^* and A normalize each other.*
(v) *$N_S(A^*)$ contains b and strictly contains $N_S(A)$.*
(vi) *If $p > 2$ then* $\mathrm{exponent}(A^*) \leq \mathrm{exponent}(A)$*. In particular, A^* is elementary abelian if A is.*
(vii) *If $p > 2$ then $\omega_i(A^*) \geq \omega_i(A)$ for all $i \geq 1$. In particular, $A^* \geq_{\mathrm{lex}} A$ and* $\mathrm{rank}(A^*) \geq \mathrm{rank}(A)$*.*

Glauberman's original result [2, theorem 4.1][4, theorem 8.2.7] includes (i)–(iii), which are enough for the ZJ_o theorem. Isaacs simplified the proof by replacing some of the counting arguments with structural ones [5]. He took B abelian, as in Thompson's replacement theorem, but with some work his arguments can be adapted. Course notes of Gagola [1] include a proof along these lines, citing long-ago unpublished work by (separately) Isaacs, Passman and Goldschmidt. This includes (vi) and removed Glauberman's hypothesis that $\mathrm{class}(B) \leq 2$. Kızmaz [6] independently adapted Isaacs' arguments from [5] and proved his own generalization of the ZJ Theorem (on which our theorem 1.6 is modeled). This includes (vii), although he only stated the $i = 1$ case. He also clarified the overall argument by isolating the commutator calculations in his lemma 2.1, from which our lemma 3.2 grew.

To my knowlege, (v) is new. It is curious because it says that $N_S(A)$ is a measure of how well-positioned A is with respect to B, yet $N_S(A)$ is independent of B. An interesting consequence is that if $A \in \mathfrak{Ab}(G)$ has largest possible normalizer, among all abelian subgroups of S with order $|A|$, then A automatically centralizes the ZJ-like group $D^*(S)$

introduced by Glauberman and Solomon [3]. We postpone the details until theorem 5.4, to avoid breaking the flow of ideas.

Lemma 3.2. *Suppose a group A acts on a group B and centralizes $[B,B]$. Then the commutator subgroup of $[B,A]$ is central in B.*

Furthermore, if A is abelian and $b \in B$ satisfies $[b,A,A,A]=1$, then the commutator subgroup of $[b,A]$ is an elementary abelian 2-group.

Proof. Because A centralizes $[B,B]$, so does $[B,A]$. Two special cases of this are $[B,[B,A],[B,A]] = 1 = [[B,A],B,[B,A]]$. Now the three subgroups lemma proves our first claim: $[[B,A],[B,A],B]=1$.

The commutator subgroup of $[b,A]$ is abelian because it is central. It is generated by the $[[b,x],[b,y]]$ with x,y varying over A. So it suffices to show that each has order ≤ 2. We fix x,y and abbreviate:

$$b_x = [b,x] \quad b_y = [b,y]$$

$$b_{xx} = [b,x,x] \quad b_{xy} = [b,x,y] \quad b_{yx} = [b,y,x] \quad b_{yy} = [b,y,y]$$

By hypothesis, x and y centralize the last four of these.

We will use the following identities, that hold in any group:

$$u^v = u[u,v] \qquad [u,vw] = [u,w][u,v]^w \qquad [uv,w] = [u,w]^v[v,w]$$

Because A centralizes $[B,B]$ we have

$$[b_{xy},b] = [b_{xy}^y, b^y] = [b_{xy}, bb_y] = [b_{xy},b_y][b_{xy},b]^{b_y \leftarrow \text{discard}}$$

We may discard the indicated conjugation because $[B,A]$ centralizes $[B,B]$. Canceling the $[b_{xy},b]$ terms leaves $1 = [b_{xy},b_y]$. Similarly,

$$\begin{aligned}[b_x,b] &= [b_x^y, b^y] = [b_x b_{xy}, bb_y] = [b_x b_{xy}, b_y][b_x b_{xy}, b]^{b_y \leftarrow \text{discard}} \\ &= [b_x,b_y]^{b_{xy} \leftarrow \text{discard}}[b_{xy},b_y] \cdot [b_x,b]^{b_{xy} \leftarrow \text{discard}}[b_{xy},b]\end{aligned}$$

We discard conjugations as before, and we just saw that the second commutator is trivial. The first commutator is central, so we may cancel the $[b_x,b]$ terms. This leaves $(\star)$ $1 = [b_x,b_y][b_{xy},b]$.

Next, we have $[b,xy] = [b,y][b,x]^y = b_y b_x^y = b_y b_x b_{xy}$. Exchanging x and y doesn't change the left side, so $b_y b_x b_{xy} = b_x b_y b_{yx}$. Moving two terms to the right yields $b_{xy} = [b_x,b_y]b_{yx}$. Bracketing by b, and using the centrality of $[b_x,b_y]$, gives $[b_{xy},b] = [b_{yx},b]$. By $(\star)$ and its analogue with x and y swapped, this implies $[b_x,b_y] = [b_y,b_x]$. That is, $[b_x,b_y]^2 = 1$. □

Proof of theorem 3.1. Suppose B does not normalize A. Since $N_B(A)$ is proper in B, it is proper in its own normalizer $N_B(N_B(A))$. Because $N_S(A)$ normalizes A and B, it also normalizes $N_B(A)$ and $N_B(N_B(A))$, hence acts on $N_B(N_B(A))/N_B(A) \neq 1$. So some $b \in N_B(N_B(A)) - N_B(A)$ is $N_S(A)$-invariant modulo $N_B(A)$, ie $[b, N_S(A)] \leq N_B(A)$. This inclusion also says that b normalizes $N_S(A)$. So $b \in N_B(N_S(A)) - N_B(A)$, as claimed.

Now set $N := N_S(A)$ and suppose $b \in N_B(N) - N_B(A)$ is arbitrary. From $[B,B] \leq A$ we have $A \cap B \trianglelefteq B$, hence $A \cap B \leq A \cap A^b$.

From $A \trianglelefteq N \trianglelefteq \langle N, b\rangle$ follows $A^b \trianglelefteq N$. So A, A^b normalize each other. Setting $H = AA^b \trianglelefteq N$, it follows that $[H,H] \leq A \cap A^b \leq Z(H)$. In particular, H has class ≤ 2. The identity $(aa'^{-1})(a')^b = a[a',b]$, for any $a, a' \in A$, shows that H is also equal to $A[A,b]$.

Using bars for images in $H/(A \cap A^b)$, obviously we have $\bar{A} \cdot \overline{[A,b]} = \bar{H}$. On the other hand, $[A,b]$ lies in $H \cap B$, and $\overline{H \cap B}$ meets $\bar{A}$ trivially because $A \cap B \leq A \cap A^b$. So $\overline{[A,b]}$ meets every coset of $\bar{A}$ in $\bar{H}$, yet lies in $\overline{H \cap B}$, which contains at most one point of each coset. Therefore $\overline{[A,b]}$ and $\overline{H \cap B}$ coincide and form a complement to $\bar{A}$ in $\bar{H}$. So

$$A^* = (A \cap A^b)[A,b] = (A \cap A^b)(H \cap B)$$

is a complement to A in H, modulo $A \cap A^b$. Since A^b is another such complement, we have $A^*/(A \cap A^b) \cong A^b/(A \cap A^b)$ and therefore $|A^*| = |A|$, proving (i).

(ii) First, $[B,B] \leq A \cap B \leq A \cap A^b \leq A^*$. Now we prove A^* abelian. Because $A \cap A^b$ is central in H it is enough to prove $[A,b]$ abelian. If $p = 2$ this follows from the hypothesis that B is abelian. So take p odd. We may apply lemma 3.2 because $[B,B] \leq A$ and $[b,A,A,A] \leq [H,A,A] \leq [Z(H),A] = 1$. The lemma shows that the commutator subgroup of $[A,b]$ is a 2-group, hence trivial.

(iii) We already saw $A \cap B \leq A \cap A^b \leq A^*$. The strict containment $A \cap B < A^* \cap B$ comes from the fact that b does not normalize A. Namely, A omits $b^{-1}ab$ for some $a \in A$, so it also omits $a^{-1}b^{-1}ab = [a,b] \in A^* \cap B$. And $A^* \cap B$ is strictly smaller than B because it lies in N and therefore omits b.

(iv) Both A, A^* contain $A \cap A^b$, hence $[H,H]$, so are normal in H.

(v) N normalizes $A^* = (A \cap A^b)(H \cap B)$ because it normalizes all four terms on the right. And b normalizes $A^* = (A \cap A^b)[A,b]$ because

$$[A \cap A^b, b] \leq [A,b] \quad \text{and} \quad [[A,b],b] \leq [B,B] \leq A \cap A^b.$$

Because $b \notin N$ it follows that $N_S(A^*)$ is strictly larger than N.

(vi) H has class ≤ 2, so the identities

$$(xy)^e = x^e y^e [x,y]^{e(e-1)/2} \quad \text{and} \quad [x,y]^e = [x,y^e]$$

hold for all $x, y \in H$. Together with the oddness of p, they show that exponent$(\Omega_i(H)) \leq p^i$ for every i. In particular, AA^b has exponent bounded by that of A, so its subgroup A^* does too.

(vii) We fix i and write A_i for $\Omega_i(A)$, which is normal in H. As we saw in the proof of (vi), $A_iA_i^b \leq \Omega_i(H)$ has exponent $\leq p^i$, so

$$A_i^* := (A_i \cap A_i^b)[A_i, b] \leq A_iA_i^b$$

does too. Obviously A^* contains A_i^*, so $\omega_i(A^*) \geq \omega_i(A_i^*)$. Therefore $\omega_i(A^*) \geq \omega_i(A)$ will follow from $|A_i^*| \geq |A_i|$. Quotienting $A_iA_i^* = A_iA_i^b$ by A_i gives $A_i^*/(A_i^* \cap A_i) \cong A_i^b/(A_i^b \cap A_i)$. Because A_i^* contains $A_i^b \cap A_i$ this implies $|A_i^*| \geq |A_i|$, as desired. (One can go further and prove $|A_i^*| = |A_i|$.) $\square$

Proof of theorem 1.4. Write $\mathcal{A}$ for any one of $\mathcal{A}_o(D), \ldots, \mathcal{A}_{O,E,\zeta}(D)$. Supposing $B \trianglelefteq S$, and that some $U \in \mathcal{A}$ contains $[B, B]$ but not B, we will show that B normalizes some $A \in \mathcal{A}$ that also contains $[B, B]$ but not B. Among all members of $\mathcal{A}$ that contain $[B, B]$ but not B, and lie in $\langle U^S \rangle$, choose A with $|A \cap B|$ maximal. Supposing that B does not normalize A, we will derive a contradiction.

Let b and A^* be as in theorem 3.1. In particular, A^* is abelian and lies in $AA^b \leq \langle U^S \rangle \leq D$. So $A^* \in \mathfrak{Ab}(D)$. A^* contains $[B, B]$ but not B by (ii) and (iii). (iii) also implies $|A^* \cap B| > |A \cap B|$, so the maximality in our choice of A forces $A^* \notin \mathcal{A}$.

This is a contradiction because $A^* \in \mathcal{A}$ by other parts of theorem 3.1. For $\mathcal{A}_o$ we use $|A^*| = |A|$. For $\mathcal{A}_r$ we use $\mathrm{rank}(A^*) \geq \mathrm{rank}(A)$. For $\mathcal{A}_e$ we use both of these properties. For $\mathcal{A}_{\mathrm{lex}}$ we use $A^* \geq_{\mathrm{lex}} A$. For $\mathcal{A}_{\mathrm{olex}}$ we use this and $|A^*| = |A|$. For $\mathcal{A}_{O,E,\zeta}$ we use $|A^*| = |A|$, $\mathrm{exponent}(A^*) \leq \mathrm{exponent}(A)$ and $A^* \geq_{\mathrm{lex}} A$. $\square$

In fact we have proven that $\mathcal{A}$ satisfies a strengthening of the replacement axiom, got by removing "with class ≤ 2" from the statement of the axiom. We stated the axiom the way we did because only the class ≤ 2 case is needed to prove the ZJ Theorem.

4. Proof of the ZJ Theorem

Proof of theorem 1.1. We write I for $I_{\mathcal{A}}$. We prove $I \trianglelefteq G$ by induction starting with $1 \trianglelefteq G$, by establishing the following inductive step:

$$\text{if } \exists\, W \trianglelefteq G \text{ with } W < I, \text{ then } \exists\, B \trianglelefteq G \text{ with } W < B \leq I.$$

Fix such a W. Since S preserves I, it acts on I/W. We define $X \leq I$ as the preimage of the fixed-point subgroup. So X is normal in S, and is strictly larger than W because $I/W \neq 1$. To complete the proof we will show that $B := \langle X^G \rangle \trianglelefteq G$ lies in I. Supposing to the contrary, we will derive a contradiction.

Step 1: $B \leq O_p(G)$. Being a subgroup of I, X is abelian. Together with $X \trianglelefteq S$ this gives $[O_p(G), X, X] \leq [X, X] = 1$. Because G acts p-stably on $O_p(G)$, X lies in $O_p(G \curvearrowright O_p(G))$. This equals $O_p(G)$ because $C_G(O_p(G))$ is a p-group by hypothesis. Since X lies in $O_p(G)$, so does $B = \langle X^G \rangle$.

Step 2: $[B, B] \leq W \leq Z(B)$. By the definition of X, S acts trivially on X/W. In particular $O_p(G)$ does. Conjugation shows that $O_p(G)$ acts trivially (mod W) on every G-conjugate of X, hence trivially on B/W. That is, $[B, O_p(G)] \leq W$. By step 1 this implies $[B, B] \leq W$. And $W \leq Z(B)$ because B is generated by abelian groups that contain W.

Step 3: Set $H = O_p(G \curvearrowright B)$ and $P = H \cap S$. Then some $A \in \mathcal{A}|_P$ fails to contain B. Because we are supposing $B \not\leq I$, some $A \in \mathcal{A}$ fails to contain B. It does contain $[B, B]$, because step 2 showed $[B, B]$ lies in W, which lies in I, hence A. Step 2 also showed that $B \trianglelefteq S$ has class ≤ 2. By the replacement property, some member of $\mathcal{A}$ contains $[B, B]$ but not B, and is also normalized by B. We lose nothing by using it in place of A, because it has all the properties of A established so far. That is, we may suppose B normalizes A. By $[B, A, A] \leq [A, A] = 1$ and the p-stability of G's action on B, we have $A \leq H$. Together with $A \leq S$ this gives $A \leq P$, hence $A \in \mathcal{A}|_P$.

Step 4: $B \leq I_{\mathcal{A}|P}$. By the Frattini argument and the definition of H,

$$G = HN_G(P) = C_G(B)PN_G(P) \leq C_G(X)N_G(P).$$

So the G-conjugates of X are the same as the $N_G(P)$-conjugates. It therefore suffices to show that every $N_G(P)$-conjugate of X lies in $I_{\mathcal{A}|P}$. This follows from

$$X \leq I = \bigcap_{A' \in \mathcal{A}} A' \underset{\substack{\uparrow \\ \text{by } \mathcal{A} \supseteq \mathcal{A}|_P \neq \emptyset}}{\leq} \bigcap_{A' \in \mathcal{A}|_P} A' = I_{\mathcal{A}|P} \underset{\substack{\uparrow \\ \text{by invariance}}}{\trianglelefteq} N_G(P).$$

The contradiction. By $B \leq I_{\mathcal{A}|P}$, every member of $\mathcal{A}|_P$ contains B. But in step 3 we found one which does not.

The final claim follows from the Frattini argument: $\operatorname{Aut} G$ is generated by inner automorphisms and automorphisms that preserve S. $\square$

Our method of "growing" the normal subgroup from W to B derives from Stellmacher's construction [7, theorem 9.4.4][8] of a different subgroup of S that also has the Glauberman property.

5. Etc

Here we collect some results and examples we mentioned in passing. First, simple examples show our ZJ-like groups ZJ_{lex} and ZJ_{olex} can be strictly larger than ZJ_r and ZJ_o, as we claimed in the introduction.

Example 5.1 ($ZJ_{\text{lex}} > ZJ_r$). The group

$$S = \bigl\langle x, y, u \bigm| 1 = [x, y] = x^9 = y^3 = u^3, x^u = xy, y^u = yx^3 \bigr\rangle$$

is a semidirect product $(\mathbb{Z}/9 \times \mathbb{Z}/3) \rtimes \mathbb{Z}/3$. For any a in $A := \langle x, y\rangle$ but outside $\langle x^3\rangle$, $C_S(a) = A$. It follows that $Z(S)$ can be no larger than $\langle x^3\rangle$. It follows that A is the unique abelian subgroup of S with order 27, because its intersection with any other such subgroup would be central in S and have order 9. In particular, $\operatorname{rank} S = 2$, $\mathcal{A}_{\text{lex}}(S) = \{A\}$ and $ZJ_{\text{lex}}(S) = J_{\text{lex}}(S) = A$. But $\mathcal{A}_r(S)$ also contains $\langle x^3, u\rangle$, so $J_r(S) = S$ and $ZJ_r(S) = \langle x^3\rangle$.

Example 5.2 ($ZJ_{\text{olex}} > ZJ_o$). Consider the Heisenberg group

$$S = \bigl\langle a, b, c \bigm| c = [a, b],\ 1 = [c, a] = [c, b] = a^9 = b^9 = c^9 \bigr\rangle$$

$\mathcal{A}_o(S)$ consists of the preimages of the 13 order 9 subgroups of $S/\langle c\rangle$. So $J_o(S) = S$ and $ZJ_o(S) = \langle c\rangle$. One member of $\mathcal{A}_o(S)$ is isomorphic to $(\mathbb{Z}/3)^2 \times \mathbb{Z}/9$, namely $A := \langle a^3, b^3, c\rangle$. The rest are isomorphic to $(\mathbb{Z}/9)^2$. So $\mathcal{A}_{\text{olex}}(S) = \{A\}$ and $J_{\text{olex}}(S) = ZJ_{\text{olex}}(S) = A$.

In the introduction we mentioned $\Omega ZJ_r = \Omega ZJ_e$. This is part of:

Theorem 5.3. *Suppose p is a prime and S is p-group. Then*

$$I_e(S) = I_r(S) = \Omega ZJ_r(S) = \Omega ZJ_e(S) = \Omega C_S(J_r(S)) = \Omega C_S(J_e(S))$$

Proof. First, $I_r(S) \le I_e(S)$ because $\mathcal{A}_e(S) \subseteq \mathcal{A}_r(S)$. And $I_e(S) \le I_r(S)$ because every member of $\mathcal{A}_r(S)$ contains a member of $\mathcal{A}_e(S)$, hence $I_e(S)$. We have proven the first equality. For the others it is enough to establish the inclusions:

$$\begin{tikzcd}
I_e(S) \arrow[r, equal] \arrow[d, hook, "\text{obvious}"] & I_r(S) \arrow[r, hook] & \Omega ZJ_r(S) \arrow[r, hook, "\text{obvious}"] & \Omega C_S(J_r(S)) \arrow[d, hook, "\text{by } J_e(S)\le J_r(S)"] & \\
\Omega ZJ_e(S) \arrow[rrr, hook, "\text{obvious}"] & & & \Omega C_S(J_e(S)) \arrow[r, hook] & I_e(S)
\end{tikzcd}$$

The unlabeled inclusion in the top row is obvious, except for the fact that $I_r(S)$ has exponent $\le p$, which holds by $I_r(S) = I_e(S)$. The unlabeled inclusion in the bottom row is standard, with proof similar to that of lemma 2.1. $\square$

Just before lemma 3.2, we mentioned that the "normalizers grow" property of the Thompson-Glauberman replacement process implies that abelian subgroups of S with "large" normalizers automatically centralize $D^*(S)$. Here $D^*(S)$ is the characteristic subgroup introduced by Glauberman and Solomon [3], who gave a lovely simple proof that it has the Glauberman property. Following Bender, $D^*(S)$ may be defined as the largest normal subgroup of S with the property that it centralizes every abelian subgroup of S that it normalizes. (It is easy to see that this exists. And considering how it acts, on abelian normal subgroups of itself, leads to a proof that $D^*(S)$ is abelian.)

Theorem 5.4. *Suppose p is a prime, S is a p-group, and $A \in \mathfrak{Ab}(S)$. Define $\mathcal{A}$ as the set of all $A^* \in \mathfrak{Ab}(S)$ satisfying*

$$\begin{aligned} &\text{if } p = 2: && |A^*| = |A|; \\ &\text{if } p > 2: && |A^*| = |A|, \text{ exponent}(A^*) \leq \text{exponent}(A), \text{ and} \\ &&& \omega_i(A^*) \geq \omega_i(A) \text{ for all } i. \end{aligned}$$

Then A centralizes $D^*(S)$, if $N_S(A)$ is maximal among all $N_S(A^*)$ with $A^* \in \mathcal{A}$.

Proof. We apply the replacement theorem with B equal to the abelian group $D^*(S) \trianglelefteq S$. Arguing as for theorem 1.4 shows that $D^*(S)$ normalizes A. By (Bender's) definition, $D^*(S)$ therefore centralizes A. $\square$

References

[1] Gagola, S., unpublished course notes. Year unknown.

[2] Glauberman, G., A characteristic subgroup of a p-stable group, *Canadian J. Math.* **20** (1968) 1101–1135.

[3] Glauberman, G. and Solomon, R., A new characteristic subgroup of a p-stable group, *J. Alg.* **368** (2012) 231–236.

[4] Gorenstein, D., *Finite Groups*, Chelsea, 1980.

[5] Isaacs, I. M. An alternative proof of the Thompson replacement theorem. *J. Algebra* **15** (1970), 149–150.

[6] Kızmaz, M. Y., An extension of the Glauberman ZJ-theorem, preprint 2019, *arxiv:1903.08286v3*.

[7] Kurzweil, H. and Stellmacher, B., *The Theory of Finite Groups: an Introduction*, Springer-Verlag, 2004.

[8] Stellmacher, B., An analogue to Glauberman's ZJ-theorem, *Proc. A.M.S.* **109** (1990) 925–929. Errata in *Proc. A.M.S.* **114** (1992) 588.

Department of Mathematics, University of Texas, Austin
Email address: `allcock@math.utexas.edu`
URL: `http://www.math.utexas.edu/~allcock`