Math 390C (Geometry in Group Theory)
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Homework 1, due Friday Jan 31, 2020

Everyone should solve and turn in problems marked "everyone". Undergrads should also solve and turn in problems marked "undergrads". The remaining problems are optional; everyone should try them, and solve them if they seem rewarding. And grad students should try the "undergrad" problems in a similar way. Students not enrolled should also try the problems and write up or talk to me about any they find interesting or troubling.

Problem 1. Classify the maximal subgroups M of A_5 up to conjugacy. (Hint: use Sylow's theorem in M.) Describe the subgroups as stabilizers of objects expressed in terms of a 5-point set Ω. For example, "the stabilizer of an unordered pair of distinct points in Ω". These objects are what I called the elements of the geometry (of A_5). **EVERYONE**

If F is a field, then projective n-space $\mathbb{P}^n F$ means the set of lines through the origin in the vector space F^{n+1}. $\mathbb{P}^1 F$ is called the projective line and $\mathbb{P}^2 F$ is called the projective plane.

Problem 2 (Fractional linear transformations). Let F be a field. Then the elements of $\mathbb{P}^1 F$ are sometimes written as follows. The line spanned by $(n, d) \neq (0, 0)$ is indicated by $n/d \in F$. The line spanned by $(n, 0)$ is indicated by the special symbol ∞. **EVERYONE**

Prove that $\mathrm{GL}_2(F)$ acts on $\mathbb{P}^1 F$ by what are called *linear fractional transformations*, namely

linear fractional transformations

$$z \mapsto \frac{az + b}{cz + d}$$

for $a, b, c, d \in F$ with $ad - bc \neq 0$. Here the variable z varies over $F \cup \{\infty\}$, and the formula is interpreted as ∞ or 0 if the denominator is 0 or ∞.

Problem 3. Build a physical model of a dodecahedron with the faces labeled $0, 1, 2, 3, 4, 5, \infty$ (opposite faces given the same label), such that orientation-preserving isometries of the dodecahedron act by fractional linear transformations of $\mathbb{P}^1 \mathbb{F}_5$. **UNDERGRADS**

[You know ahead of time that such a model must exist, because (1) $\mathrm{PSL}_2\,\mathbb{F}_5$ is nonsolvable with order 60, (2) there is only one such group, namely A_5, and (3) A_5 has only one way to act transitively on 6 points.]

Recall that a group G acting transitively on a set X acts *primitively* if there is no G-invariant partition of X other than the obvious ones (the partition into singletons and the "partition" into a single set). We primitively saw in class that this is another way to say that the point stabilizer is a maximal subgroup. In this case, X is one family of elements of the geometry (of G).

Problem 4. Prove that the isometry group of Euclidean n-space acts primitively on the points of $\mathbb{R}^n$. (Adapt the argument I hand-waved in class for the hyperbolic plane. Also, the problem as stated is wrong; fix it.)

Problem 5. The projective plane over $\mathbb{F}_2$ has 7 points and 7 lines, corresponding to the nonzero vectors in $\mathbb{F}_2^3$ and its dual space. $\mathrm{GL}_3(2)$ acts transitively on the points, hence has order divisble by 7, hence contains a cyclic group of order 7. Therefore it is possible to draw a picture in the plane where the points are represented by 7 white dots and the lines by 7 black dots, dots are joined just if the corresponding point and line are incident, and the whole picture is invariant under an order 7 rotation. Do it. **UNDERGRADS**

Problem 6 (Icosahedron exists, "soft" argument). Begin with a regular octahedron, and label the edges with arrows, such that each face has 3 arrows going around it cyclically. Imagine 12 ants, one starting at the tail of each edge, walking along the edges at speed 1 until reaching the end. Taking their convex hull is gives a 1-parameter family of polyhedra. Use the Intermediate Value Theorem to show that one member of this family has 20 equilateral triangles for its faces. Use this and the fact that all vertices are the same distance from the origin to show that any face can be sent to any other by an isometry. (Argument from Coxeter's *Regular Polytopes*.)

A *lattice* L means a free $\mathbb{Z}$-module equipped with a symmetric biliner form $L \times L \to \mathbb{Q}$. Often we define L as a subset of $\mathbb{R}^n$, in which case we mean the standard inner product unless otherwise stated. Usually we call the pairing the dot product and write $\cdot$. The *norm* of a vector x means $x \cdot x$ (NOT $\sqrt{x \cdot x}$). We call L *integral* if all inner products are integers. An *isometry* of a lattice L (or vector space equipped with a symmetric bilinear form) means an invertible self-map that preserves inner products. $\mathrm{Aut}(L)$ means the group of all isometries of L. lattice norm integral isometry

Problem 7. Suppose L is an integral lattice and $r \in L$. The *reflection* R_r in r means the linear map on the enclosing vector space **EVERYONE** reflection

$$R_r(x) = x - 2\,\frac{x \cdot r}{r^2} r$$

Check that this preserves inner products, negates r and fixes $r^\perp$ pointwise. If $r^2 \in \{\pm 1, \pm 2\}$ then $R_r \in \mathrm{Aut}(L)$. This gives a way to build lots of isometries without having to write down matrices; for example the next couple of problems.

We write $\langle \dots \rangle$ for the group generated by … $\langle \dots \rangle$

Problem 8. Suppose L is an lattice and $r, s \in L$ have inner product ± 1. Then they are equivalent under $\langle R_r, R_s \rangle$.

Problem 9. The E_8 lattice is one of the most remarkable objects in mathematics. It means

$$\left\{ (x_1, \dots, x_8) \in (\tfrac{1}{2}\mathbb{Z})^8 \;\middle|\; x_i - x_j \in \mathbb{Z} \text{ for all } i, j, \text{ and } \sum x_i \in 2\mathbb{Z} \right\}$$

Prove it is integral, find the norm 2 vectors, and prove that they are equivalent under lattice isometries.