Math 390C (Geometry in Group Theory)
Daniel Allcock
Homework 2, due Friday Feb 7, 2020

Everyone should solve and turn in problems marked "everyone". Undergrads should also solve and turn in problems marked "undergrads" **PLUS ONE MORE OF THEIR CHOICE**. The remaining problems are optional; everyone should try them, and solve them if they seem rewarding. And grad students should try the "undergrad" problems in a similar way. Students not enrolled should also try the problems and write up or talk to me about any they find interesting or troubling.

Problem 1. In class we discussed how the rotation group A_5 of the dodecahedron has preimage "$2A_5$" of order 120 in the 3-sphere. And we defined the 600-cell as the convex hull (in $\mathbb{R}^4$) of those 120 points. For each $\theta \in [0, \pi]$, write S_θ for the "spherical shell at angle θ from 1". Formally, this means the set of $g \in S^3$ such that the angle between 1 and g (regarded as vectors in $\mathbb{R}^4$) is θ. Each shell is a 2-sphere, except for S_0 and S_π which degenerate to points.

Find the numbers of points of $2A_5$ in each S_θ. Conjugation in $2A_5$ permutes the points in each shell. Find the sizes of the orbits.

Problem 2. This problem demonstrates a sometimes-seemingly-magical way to prove transitivity of a group action without having to get grubby and actually write down explicit transformations. **EVERYONE**

Recall the E_8 lattice from hw1. You showed there that its isometry group W acts transitively on the norm 2 vectors ("roots"). Prove that the root stabilizer acts transitively on the set of roots orthogonal to it.

Do this as follows: Let V ("visible") be the group generated by permutations of coordinates and the transformations that negate any even number of coordinates. Check that $V \leq W$. Under V, there are two orbits on roots. Choose a root, say r, and find the sizes of the orbits of the stabilizer V_r on the set of roots orthogonal to r. Then do the same with a root s from the other V-orbit. The W-stabilizer W_t of any root t contains a conjugate of V_r and a conjugate of V_s. Use this and the orbits sizes to show that W_t is transitive on the roots orthogonal to t.

Problem 3. This problem is mostly preparation for the next problem. The D_n lattice means the subset of $\mathbb{Z}^n$ with even coordinate sum. We equip it with the standard inner product. The very definition makes clear that $\mathrm{Aut}(D_n)$ contains the signed permutations of coordinates. When $n \neq 4$, prove that there are no more automorphisms.

Problem 4. We work with the E_8 lattice and continue using the notation above. Prove $|W| = 696,729,600$. (Hint: this is an iterated orbit-stabilizer problem.) **UNDERGRADS**

Problem 5. This problem demonstrates another "magical" way to prove transitivity without getting your hands grubby. We continue with E_8. Prove that W acts transitively on the set of norm 4 lattice vectors. **EVERYONE**

This is not hard to do directly by finding the V-orbits and seeing that the reflections in specific roots mix these orbits together. But I want you to do this the magical way. First count the number of norm 4 vectors. Then consider the specific example $v = (2,0,0,0,0,0,0,0)$. Prove that the W-stabilizer of v is the same as its V-stabilizer, which has order $2^6 7!$. (Hint: consider the set of norm 4 lattice vectors having the same image as v in $E_8/2E_8$.) Conclude that v's orbit consists of all the norm 4 vectors.

Problem 6. This problem develops a way to search for maximal subgroups of a finite group G, more systematic than you probably used for A_5 on hw1.

Prove: every finite group M has a normal subgroup which is a direct product of some number of copies of a single simple group.

Taking M to be a maximal subgroup of G, conclude that every maximal subgroup G can be described as the normalizer of a subgroup of this special form.

(For A_5 this gives that every maximal subgroup arises as the normalizer of a 2, or a 2×2, or a 3 or a 5. All of these normalizers are indeed maximal except for the first.)

Problem 7 (Action of $\mathrm{PSL}_2(\mathbb{F})$ on $\mathbb{P}^1\mathbb{F}$). Let $\mathbb{F}$ be a field. Obviously $\mathrm{PSL}_2(\mathbb{F})$ acts on $\mathbb{P}^1\mathbb{F}$. Prove that it acts 2-transitively (that is, transitively on ordered pairs of distinct points). Prove that it might or might not act 3-transitively, depending on $\mathbb{F}$. Find a necessary and sufficient condition on $\mathbb{F}$ for it to act 3-transitively. **EVERYONE**

Problem 8 (Tutte's 8-cage). "Tutte's 8-cage", to give it its 19th-century name, is a graph Γ you can download from the course webpage. Prove it is the incidence graph of the "duads" and "synthemes" from a set Ω of size 6. Here a duad means a 2-point set, and a syntheme means a partition of Ω into three duads. A duad is incident to a syntheme just if it is one of those three duads. (One can also think of duads and synthemes as elements of S_6, with cycle shapes 21111 and 222.)

Let G be the automorphism group of Γ (ignoring vertex color). Obviously it contains S_6. Prove that G is larger, of order 1440. (Hint: iterated orbit-stabilizer.) Prove that A_6 is normal in G, so there is a natural map $G \to \mathrm{Aut}(A_6)$. Prove that the G-centralizer of A_6 is trivial, so this map is injective (hint: anything that commutes with a subgroup $T \cong 2 \times 2$ of A_6 must preserve the fixed-point set of T in Γ). Find a way to express Γ entirely in terms of the subgroups 2×2 of A_6, so that $\mathrm{Aut}(A_6)$ embeds in G. Conclude $G = \mathrm{Aut}(A_6)$.