Math 390C (Geometry in Group Theory)
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Homework 3, due Friday Feb 28, 2020

Everyone should solve and turn in problems marked "everyone". Undergrads should also solve and turn in problems marked "undergrads".

The remaining problem is optional, but I found it incredibly enlightening, so I highly recommend it.

Suppose G is a group described by generators and relations

$$\langle x_1, \ldots, x_m : 1 = W_1(x_1, \ldots, x_m) = \cdots = W_n(x_1, \ldots, x_m) \rangle$$

Recall that its *Cayley graph* means the graph with vertex set G, and a directed edge labeled 'x_i' from each $g \in G$ to gx_i. If x_i has order 2 then usually one draws a single unoriented edge rather than arrows going both directions. Usually one uses colors for the edge labels.

Problem 1. Construct a continuous family of right-angled hexagons in the hyperbolic plane. What is the dimension of the space of all such hexagons up to isometry? (I don't need any foundational dimension-theory stuff. Just figure out how many parameters naturally describe a right-angled hexagon.) **UNDERGRADS**

Problem 2. Draw Cayley graphs for **EVERYONE**

$$\begin{aligned} G &= \langle x, y, z : 1 = x^3 = y^3 = z^3 = xyz \rangle \\ H &= \langle x, y : 1 = x^3 = y^4 = (xy)^4 \rangle \\ I &= \langle x, y : x^2 = y^3 \rangle \\ J &= \langle a, b : aba = bab \rangle \end{aligned}$$

Hint 1: there is an element of $Z(I)$ visible; quotient by it first, build the Cayley graph for the quotient, and then use that to figure out that of I itself.

Hint 2: Find an isomorphism $J \cong I$. This gives directions for changing your graph for I to one for J.

Problem 3. Draw pictures of (or physically build) the following 3-dimensional hyperbolic Coxeter polytopes: **EVERYONE**

$$\bullet - 4 - \bullet - 4 - \bullet - - - - \bullet$$

$$\bullet \cdots\cdots \bullet - 4 - \bullet - 4 - \bullet - 6 - \bullet$$

The dots $\cdots$ mean that those faces of the polytope should be ultraparallel.

Hint: these are nontrivial! It might help to figure out the infinite-volume polytopes defined by only a subset of (the hyperplanes containing) the walls.

Problem 4. Let G be the group

$$\langle x, y, z : x^y = \bar{x}, y^z = \bar{y}, z^x = \bar{z} \rangle$$

Prove that it has a finite index subgroup isomorphic to $\mathbb{Z}^3$, and acts freely on $\mathbb{R}^3$. (This means no element of G fixes any point of $\mathbb{R}^3$, except that the identity element fixes every point.) Describe a fundamental domain for G.

Problem 5. $\mathrm{SU}(1,1)$ is conjugate to $\mathrm{SL}_2\mathbb{R}$ in $\mathrm{SL}_2\mathbb{C}$. ($\mathrm{SU}(p,q)$ is the subgroup of $\mathrm{SL}_{p+q}\mathbb{C}$ that preserves a Hermitian inner product of signature (p,q). The standard example of such an inner product is

$$\langle x \,|\, y \rangle = x_1\bar{y}_1 + \cdots + x_p\bar{y}_p - x_{p+1}\bar{y}_{p+1} - \cdots - x_{p+q}\bar{y}_{p+q}$$

(Hint: What locus in $\mathbb{C}P^1$ do the norm 0 vectors describe?)