INVERSION IN CIRCLES

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We prove the classical fact that inversions in circles in the plane send circles to circles (with lines understood as circles of infinite radius). Our proof uses classical pictures, but is simpler than usual because a scaling argument lets one reduce to the essential cases.

Inversion in the unit circle U means the self-map of $\mathbb{R}^2 \cup \{\infty\}$ given by $\chi : (r, \theta) \mapsto (1/r, \theta)$ in polar coordinates; often we abbreviate $\chi(a)$ to a'. It fixes U pointwise and swaps the center with ∞. Inversion about circles of any radius, centered at any point of $\mathbb{R}^2$, are defined similarly (formally: by conjugating χ by translations and scalings). To prove things about inversions in general circles it is usually enough to prove them for χ.

Lemma 1. *Conjugation by χ inverts every scaling map λ. In particular, $\chi = \lambda \circ \chi \circ \lambda$.* □

Lemma 2. *If $a, b \in \mathbb{R}^2$ and a', b' are their images under inversion about some circle with center $c \neq a, b$, then $\triangle cab$ and $\triangle cb'a'$ are similar.*

Proof. Translations and scalings send circles to circles (and centers to centers) and respect triangle similarity. So, by conjugating by them it is enough to prove this for χ. Rearranging $r_a r_{a'} = 1 = r_b r_{b'}$ gives $r_{a'}/r_{b'} = r_b/r_a$. The triangles have equal angles at 0, so they are similar. □

Theorem 3. *Inversion in any circle permutes the circles not passing through its center.*

Proof. As before, it is enough to prove the theorem for χ. Suppose C is a circle not containing 0. Consider the points of C closest to and furthest from 0. First we suppose χ swaps them, so we call them a, a'. We claim that χ preserves C. Suppose $b \in C$ and consider the attached picture. The triangle similarities let us define

$$B := \angle 0ab = \angle 0b'a' \qquad s := \angle 0a'b = \angle 0b'a$$

(mnemonic: Big and small). As an angle inscribed in a semicircle, $\angle aba'$ is a right angle. Considering angles at a shows $B+(\pi/2-s) = \pi$,

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ie $B - s = \pi/2$. Examining angles at b' shows that $\angle ab'a'$ is a right angle (the dotted right angle symbol). So $b' \in C$.

For the general case we use a scaling trick. Choose a scaling map λ so that the points of $\lambda(C)$ nearest to and furthest from 0 are exchanged by χ. By the special case, χ preserves $\lambda(C)$, so $\chi(C) = \lambda(\chi(\lambda(C))) = \lambda(\lambda(C))$. This is a circle that misses 0. □

Theorem 4. *Inversion in a circle exchanges the circles meeting the center with the lines (taken to include ∞) missing the center.*

Proof. As before, it is enough to prove the theorem for χ, and we begin with a special case. Choose $p \in U$, define L and C as the line and circle tangent to U at p, with C also required to contain 0. We claim that χ exchanges C and L. Suppose $a \neq 0$ lies in C, and write a' for where the ray it generates meets L. We must verify that $r_a r_{a'} = 1$. As an angle inscribed in a semicircle, $\angle 0ap$ is a right angle. Therefore $\triangle 0ap$ and $\triangle 0pa'$ are similar triangles. Considering lengths of edges incident at 0 gives $r_a/1 = 1/r_{a'}$.

For the general case we use the same scaling trick. Let X denote either a circle containing 0 or a line missing 0. Let λ be the scaling map with $\lambda(X)$ tangent to U. The special case shows that $\chi(\lambda(C))$ is a line missing 0, or a circle containing 0, depending on what X was. so $\chi(C) = \lambda(\chi(\lambda(C)))$ is also one. □

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